Olympiad Corner

Below are the problems of the 2011 Canadian Math Olympiad, which was held on March 23, 2011.

Problem 1. Consider 70-digit numbers $n$, with the property that each of the digits 1, 2, 3, ..., 7 appears in the decimal expansion of $n$ ten times (and 8, 9 and 0 do not appear). Show that no number of this form can divide another number of this form.

Problem 2. Let $ABCD$ be a cyclic quadrilateral whose opposite sides are not parallel, $X$ the intersection of $AB$ and $CD$, and $Y$ the intersection of $AD$ and $BC$. Let the angle bisection of $\angle AXD$ intersect $AD$, $BC$ at $E$, $F$ respectively and let the angle bisection of $\angle AYB$ intersect $AB$, $CD$ at $G$, $H$ respectively. Prove that $EFGH$ is a parallelogram.

Problem 3. Amy has divided a square up into finitely many white and red rectangles, each with sides parallel to the sides of the square. Within each red rectangle, she writes down its length. Among each pair of adjacent rectangles, each with sides parallel to the sides of the square, there are the numbers $a$ and $b$. Within each white rectangle, she writes down its area. Within each pair of adjacent rectangles, each with sides parallel to the sides of the square, there are the numbers $a$ and $b$. She asks you find the sum of all the numbers in all the white rectangles.

Harmonic Series (I)

Leung Tat-Wing

A series of the form

$$\frac{1}{m} + \frac{1}{m + d} + \frac{1}{m + 2d} + \cdots,$$

where $m$, $d$ are numbers such that the denominators are never zero, is called a harmonic series. For example, the series

$$H(n) = H(1,n) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$$

is a harmonic series, or more generally

$$H(m,n) = \frac{1}{m} + \frac{1}{m + 1} + \frac{1}{m + 2} + \cdots + \frac{1}{n}$$

is also a harmonic series. Below we always assume $1 \leq m < n$. There are many interesting properties concerning this kind of series.

Example 1: $H(1,n)$ is unbounded, i.e. for any positive number $A$, we can find $n$ big enough, so that $H(1,n) \geq A$.

Solution: For any positive integer $n$, note that $\frac{1}{r+1} + \frac{1}{r+2} + \cdots + \frac{1}{2r} \geq \frac{1}{2}$, which can be proved by induction. Hence we can take enough pieces of these fractions to make $H(1,n)$ as large as possible.

Example 2: $H(m,n)$ is never an integer.

Solution: (i) For the special case $m=1$, let $s$ be such that $2^s \leq n < 2^{s+1}$. We then multiply $H(1,n)$ by $2^{s+1}Q$, where $Q$ is the product of all odd integers in $[1,n]$. All terms in $H(1,n)$ will become an integer except the term $2^s$ will become an integer divided by 2 (a half integer). This implies $H(1,n)$ is not an integer.

(ii) Alternatively, for the case $m=1$, let $p$ be the greatest prime number not exceeding $n$. By Bertrand’s postulate there is a prime $q$ with $p < q < 2p$. Therefore we have $n < 2p$. If $H(1,n)$ is an integer, then

$$n!H(n) = \sum_{i=1}^{n} \frac{n!}{i}$$

is an integer divisible by $p$. However the term $n!/p$ (an addend) is not divisible by $p$ but all other addends are.

(iii) We deal with the case $m > 1$. Suppose $2^r | k$ but $2^{r+1}$ does not divide $k$ (write this as $2^r || k$), then we call $a$ the “parity order” of $k$. Now observe $2^r$, $2^{r+1}$, $2^{r+2}$, ..., all have the same parity order. Between these numbers, there are $2^{r-1}$, $4^{r-1}$, $8^{r-1}$, ..., all have greater parity orders. Hence, between any two numbers of the same parity order, there is one with greater parity order. This implies among $m, m+1, ..., n$, there is a unique integer with the greatest parity order, say $q$ of parity order $\mu$. Now multiply

$$\frac{1}{m} + \frac{1}{m + 1} + \cdots + \frac{1}{n}$$

by $2^\mu L$, where $L$ is the product of all odd integers in $[m,n]$. Then $2^\mu L H(m,n)$ is an odd number. Hence

$$H(m,n) = \frac{2r+1}{2^\mu L} = \frac{q}{p},$$

where $p$ is even and $q$ is odd and so is not an integer.

Example 3 (APMO 1997): Given that

$$S = 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{201}$$

where the denominators contain partial sum of the sequence of reciprocals of triangular numbers. Prove that $S > 1001$.

Solution: Let $T_n$ be the $n$th triangular number. Then $T_n = \frac{n(n+1)}{2}$ and hence

$$\frac{1}{T_1} + \frac{1}{T_2} + \cdots + \frac{1}{T_n} = \frac{2}{2\cdot3} + \frac{2}{3\cdot4} + \cdots + \frac{2}{(n+1)\cdot n}$$

$$= 2(\frac{1}{2\cdot3} + \frac{1}{3\cdot4} + \cdots + \frac{1}{n\cdot(n+1)}) = 2(\frac{1}{n+1} - \frac{1}{n+2}) = \frac{2n}{n+1}$$

Since $1993006 \rightarrow 1996 - 1997/2$, we get

$$S = \frac{1}{2} \left( \frac{2}{1} + \frac{3}{2} + \cdots + \frac{2007}{2006} \right)$$

$$> \frac{1}{2} \left( 1996 + \frac{1}{2} + \frac{13}{3} + \cdots + \frac{1}{1024} \right)$$

Hence, $S > (1996/6)/2001$ using example 1 that $H(r(1,2r)) \geq 1/2$ for $r = 2, 4, 8, 16, 32, 64, 128, 256, 512$. 

(continued on page 4)
Congruence relations of harmonic series are of some interest. First, let us look at an example.

**Example 4 (IMO 1979):** Let \( p, q \) be natural numbers such that
\[
\frac{p}{q} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n}.
\]
Prove that \( p \) is divisible by 1979.

**Solution** We will prove the famous Catalan identity (due to N. Botez (1872) and later used by Catalan):
\[
\sum_{k=1}^{1977} (-1)^{k+1} \frac{1}{k!} = \prod_{k=1}^{1977} (1 + \frac{1}{k}) = \frac{1}{1979} \sum_{k=1}^{1977} (k^{1977} + (1979 - k)^{1977}) = \sum_{k=1}^{1977} (k^{1977} + 1979^{1977} - (k + 1979)^{1977}) = \sum_{k=1}^{1977} (k^{1977} + (1979 - k)^{1977} - 1979^{1977})
\]
Note that \( 1/k \pmod{p} \) (as well as many fraction mod \( p \)) makes sense if \( k \neq 0 \pmod{m} \). Also, as a generalization, we have

**Example 5:** If \( H(m,n) = q/p \) and \( m+n \) is an odd prime number, then \( m^n + n^q \). 

**Solution** Note that \( H(m,n) \) has an even number of terms and it equals
\[
\sum_{j=0}^{(m+n)/2} \frac{1}{m+j + \frac{1}{n-j}} = \sum_{j=0}^{(m+n)/2} \frac{m+n}{m+j(n-j)} = \frac{(m+n)^{m+n}}{r}
\]
where \( \gcd(s,r) = 1 \). Since \( m+n \) is prime, \( \gcd(r,m+n) = 1 \). Then \( q/p = (m+n)/r \) and \( m+n \mid q \).

The Catalan identity is also used in the following example.

**Example 6 (Roman Math Magazine, July 1998):** Let
\[
A = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{1007-2012} = \frac{1007}{2011-2012}
\]
and
\[
B = \frac{1}{1007-2012} + \frac{1}{1008-2011} + \cdots + \frac{1}{2012-1007} = \frac{1}{2011-2012}
\]
Evaluate \( A/B \).

**Solution**
\[
A = \sum_{k=1}^{1007} \frac{1}{(2k-1)2k} = \sum_{k=1}^{1007} \left( \frac{1}{2k-1} - \frac{1}{2k} \right)
\]
\[
= \frac{1}{2} \sum_{k=1}^{1007} \left( \frac{1}{1007} + \frac{1}{1008} + \cdots + \frac{1}{2012} \right)
\]
\[
= \frac{1}{2} \sum_{k=1}^{1007} \frac{3019}{1008-2011} + \frac{3019}{12012-1007} = \frac{3019B}{2}
\]
Hence
\[
A/B = \frac{3019}{2}.
\]

**Example 7:** Given any proper fraction \( m/n \), where \( m, n \) are positive integers satisfying \( 0 < m < n \), then prove it is the sum of fractions of the form
\[
\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_k},
\]
where \( x_1, x_2, \ldots, x_k \) are distinct positive integers.

**Solution** We use the “greedy method”. Let \( x_1 \) be the positive integer such that
\[
\frac{1}{x_1} \leq \frac{m}{n} < \frac{1}{x_1-1},
\]
i.e., \( x_1 \) is the least integer greater than or equal to \( n/m \). If \( 1/x_1 = m/n \), then the problem is done. Otherwise
\[
\frac{m}{n} \cdot x_1 = m \cdot x_1/n = m_1
\]
where \( m_1 \) is another positive integer such that
\[
1 \leq m_1 < \frac{1}{x_1}/n.
\]
The procedure can be repeated until \( m > m_1 > m_2 > \cdots > m_k > 0 \) and
\[
\frac{m}{n} \cdot x_k = m \cdot x_k/n = \frac{1}{x_k} = \frac{1}{1} + \frac{1}{1} + \cdots + \frac{1}{n+1},
\]
where \( 1 \leq k \leq m \). (Note: writing
\[
\frac{1}{n+1} = \frac{1}{1} + \frac{1}{1} + \cdots + \frac{1}{n+1}
\]
we observe actually there are infinitely many ways of writing any proper fractions as sum of fractions of this kind. These fractions are called unit fractions or Egyptian fractions.)

**Example 8:** Remove those terms in
\[
1 + \frac{1}{2} + \cdots + \frac{1}{n} + \cdots
\]
such that its denominator in decimal expansion contains the digit “9”, then prove that the sequence is bounded.

**Solution** The integers without the digit 9 in the interval \([10^{m-1}, 10^{m-1}]\) are \(m\)-digit numbers. The first digit from the left cannot be the digits “0” and “9”, (9 choices), the other digits cannot contain “9”, hence nine choices 0, 1, 2, 3, 4, 5, 6, 7 and 8. Altogether there are \(8 \cdot 9^{m-1}\) such integers. The sum of their reciprocals is less than
\[
\frac{8 \cdot 9^{m-1}}{10^{m-1}} = 8 \left( \frac{9}{10} \right)^{m-1}.
\]
Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver’s name, home (or email) address and school affiliation. Please send submissions to Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong. The deadline for sending solutions is May 29, 2011.

Problem 366. Let \( n \) be a positive integer in base 10. For \( i = 1, 2, \ldots, 9 \), let \( a(i) \) be the number of digits of \( n \) that equal \( i \). Prove that

\[
2^{a(1)}3^{a(2)} \cdots 9^{a(8)}10^{a(9)} \leq n + 1
\]

determine all equality cases.

Problem 367. For \( n = 1, 2, 3, \ldots \), let \( x_n \) and \( y_n \) be positive real numbers such that

\[
x_{n+2} = x_n + x_{n+1}^2
\]

and

\[
y_{n+2} = y_n + y_{n+1}^2.
\]

If \( x_1, x_2, y_1, y_2 \) are all greater than 1, then prove that there exists a positive integer \( N \) such that for all \( n > N \), we have \( x_n > y_n \).

Problem 368. Let \( C \) be a circle, \( A_1, A_2, \ldots, A_n \) be distinct points inside \( C \) and \( B_1, B_2, \ldots, B_n \) be distinct points on \( C \) such that no two of the segments \( A_1B_1, A_2B_2, \ldots, A_nB_n \) intersect. A grasshopper can jump from \( A_j \) to \( A_k \), if the line segment \( A_jA_k \) does not intersect any line segment \( A_lB_t \). Prove that after a certain number of jumps, the grasshopper can jump from any \( A_u \) to any \( A_v \).

Problem 369. \( ABC \) is a triangle with \( BC > CA > AB \). \( D \) is a point on side \( BC \) and \( E \) is a point on ray \( BA \) beyond \( A \) so that \( BD = BE = CA \). Let \( P \) be a point on side \( AC \) such that \( E, B, D, P \) are concyclic. Let \( Q \) be the intersection point of ray \( BP \) and the circumcircle of \( \triangle ABC \) different from \( B \). Prove that \( AQ + CQ = BP \).

Problem 370. On the coordinate plane, at every lattice point \((x,y)\) (these are points where \( x, y \) are integers), there is a light. At time \( t = 0 \), exactly one light is turned on. For \( n = 1, 2, 3, \ldots \), at time \( t = n \), every light at a lattice point is turned on if it is at a distance 2005 from a light that was turned on at time \( t = n - 1 \). Prove that every light at a lattice point will eventually be turned on at some time.

\[
x + y + z, \quad xy, \quad \frac{1}{x + y + z} + \frac{1}{x} + \frac{1}{y} + \frac{1}{z}
\]

are integers.

Solution. CHAN Long Tin (Diocesan Boys’ School), Hong Kong Joint School Math Society, Raymond LO (King’s College), Anna PUN Ying (HKU Math) and The 7B Math Group (Carmel Alison Lam Foundation Secondary School).

Let \( A = x + y + z \), \( B = xyz \) and \( C = 1/x + 1/y + 1/z \), then \( A, B, C \) are integers. Since \( xy + yz + zx = BC \), so \( x, y, z \) are the roots of the equation \( t^2 - At + BCt - B = 0 \). Since the coefficients are integers and the coefficient of \( t^2 \) is 1, by Gauss lemma or the rational root theorem, the roots \( x, y, z \) are integers.

Since they are positive, without loss of generality, we may assume \( x \geq y \geq z \geq 1 \). Now \( 1 \leq 1/x + 1/y + 1/z \leq 3/x \) lead to \( x = 1, 2, 3 \). For \( x = 1 \), \( 1/y + 1/z = 1 \) or 2, which yields \((y,z) = (1,1) \) or \((2,2)\). For \( x = 2 \), \( 1/y + 1/z = 1/2 \), which yields \((y,z) = (3,6) \) or \((4,4)\). For \( x = 3 \), \( 1/y + 1/z = 2/3 \), which yields \((y,z) = (3,3)\). So the solutions are \((x,y,z) = (1,1,1), (1,2,2), (2,3,6), (2,4,4), (3,3,3)\) and permutations of coordinates.

Other commended solvers: LI Pak Hin (PLK Vuekood K. T. Chong Sixth Form College) and Alice WONG Sze Nga (Diocesan Girls’ School).

Problem 363. Extend side \( CB \) of triangle \( ABC \) beyond \( B \) to a point \( D \) such that \( DB = AB \). Let \( M \) be the midpoint of side \( AC \). Let the bisector of \( \angle ABC \) intersect line \( DM \) at \( P \). Prove that \( \angle BAP = \angle ACB \).

Solution. Raymond LO (King’s College).

Construct line \( BF \parallel \) line \( CA \) with \( F \) on line \( AD \). Let \( DM \) intersect \( BF \) at \( E \).

Since \( BD = AB \), we get \( \angle BDF = \angle BAF = \frac{\angle ABC}{2} = \angle BAP = \angle CBP \). Then line \( FD \parallel \) line \( PB \). Hence, \( \triangle DFE \) is similar to \( \triangle PBE \).

Since \( BF \parallel CA \) and \( M \) is the midpoint of \( AB \).
AC, so E is the midpoint of FB, i.e. FE = BE. Then ∠DFE is congruent to ∠PBE. Hence, FD = PB.

This along with DB = BA and ∠DBF

= ∠ABP imply ∠DBF is congruent to ∠ABP. Therefore, ∠ABP = ∠DBF = ∠ACB.

Other commended solvers: U. BATZORIG (National University of Mongolia), CHAN Long Tin (Diocesan Boys’ School), Hong Kong Joint School Math Society, Abby LEE Shing Chi (SKH Lam Woo Memorial Secondary School), LI Pak Hin (PLK Viewko K. T. Chong Sixth Form College), LKL Excalibur (Madam Lau Kam Lung Secondary School of MFBFM), Anna PUN Ying (HKU Math), The 7B Math Group (Carmel Alison Lam Foundation Secondary School), Ercole SUPPA (Liceo Scientifico Statale E.Einstein, Teramo, Italy) and Alice WONG Sze Nga (Diocesan Girls’ School).

Problem 364. Eleven robbers own a treasure box. What is the least number of locks they can put on the box so that there is a way to distribute the keys of the locks to the eleven robbers with no five of them can open all the locks, but every six of them can open all the locks? The robbers agree to make enough duplicate keys of the locks for this plan to work.

Solution. CHAN Long Tin (Diocesan Boys’ School), Hong Kong Joint School Math Society, LI Pak Hin (PLK Viewko K. T. Chong Sixth Form College), LKL Excalibur (Madam Lau Kam Lung Secondary School of MFBFM), Raymond LO (King’s College), Emanuele NATALE (Università di Roma “Tor Vergata”, Roma, Italy), Anna PUN Ying (HKU Math), The 7B Math Group (Carmel Alison Lam Foundation Secondary School) and Alice WONG Sze Nga (Diocesan Girls’ School).

Let n be the least number of locks required. If for every group of 5 robbers, we put a new lock on the box and give a key to each of 6 other robbers only, then the plan works. Thus

\[ n \geq \binom{11}{5} = 462. \]

Conversely, in the case when there are n locks, for every group G of 5 robbers, there exists a lock L(G), which they do not have the key, but the other 6 robbers all have keys to L(G). Assume there exist G ≠ G’ such that L(G) = L(G’). Then there is a robber in G and not in G’. Since G is one of the 6 robbers not in G’, he has a key to L(G’), which is L(G), contradiction. So G ≠ G’ implies L(G) ≠ L(G’). Then the number of locks is at least as many groups of 5 robbers. So

\[ n \geq \binom{11}{5} = 462. \]

Therefore, \( n = 462. \)

Problem 365. For nonnegative real numbers a, b, c satisfying \( ab + bc + ca = 1 \), prove that

\[ \frac{1}{a + b} + \frac{1}{b + c} + \frac{1}{c + a} - \frac{1}{a + b + c} \geq 2. \]

Solution. CHAN Long Tin (Diocesan Boys’ School) and Alice WONG Sze Nga (Diocesan Girls’ School).

Since a, b, c ≥ 0 and \( ab + bc + ca = 1 \), none of the denominators can be zero. Multiplying both sides by \( a + b + c \), we need to show

\[ \frac{c}{a + b} + \frac{a}{b + c} + \frac{b}{c + a} + 2 \geq 2(a + b + c). \]

This follows from using the Cauchy-Schwarz inequality and expanding \( (c + a + b - 2)^2 \) ≥ 0 as shown below

\[ \left( \frac{c}{a + b} + \frac{a}{b + c} + \frac{b}{c + a} \right)^2 \geq 4(a + b + c)^2 - \frac{c}{a + b} \left( c + a + b - 2 \right)^2 \]

≥ 4(a + b + c)^2 - 4.

Harmonic Series (I)

(continued from page 2)

The sum of reciprocals of all such numbers is therefore less than

\[ \sum_{n=0}^{\infty} \left( \frac{9}{10} \right)^n = \frac{8}{9} \approx 80. \]

Example 9: Let \( m > 1 \) be a positive integer. Show that \( 1/m \) is the sum of the first \( m-1 \) terms in the sequence

\[ \frac{1}{j(j+1)} = \frac{1}{j} - \frac{1}{j+1}. \]

Hence, the problem is reduced to finding integers a and b such that

\[ \frac{1}{m} = \frac{1}{a} - \frac{1}{b} \]

(continued from page 1)

The red rectangles, what is the smallest possible value of x?

Problem 4. Show that there exists a positive integer N such that for all integers \( a > N \), there exists a contiguous substring of the decimal expansion of a which is divisible by 2011. (For instance, if \( a = 153204 \), then \( 15, 53, 20, 4 \) are all contiguous substrings of a. Note that 0 is divisible by 2011.)

Problem 5. Let \( d \) be a positive integer. Show that for every integer \( S \) there exists an integer \( n \) greater than 0 and a sequence \( \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \) where for any \( k, \varepsilon_k = 1 \) or \( \varepsilon_k = -1 \), such that

\[ S = \varepsilon_1(1+d)^2 + \varepsilon_2(1+2d)^2 + \varepsilon_3(1+3d)^2 + \cdots + \varepsilon_n(1+nd)^2. \]

Let \( a = m - c \), then \( b = (m^2/c) - m \). For each \( c \) satisfying \( c < m^2 \) and \( 1 \leq c \leq m \), there exists one and only one pair of \( a \) and \( b \) satisfying (\( \ast \)), and because \( a < b \), the representation is unique. Let \( d(n) \) count the number of factors of \( n \). Now consider all factors of \( m^2 \) except \( m \), there are \( d(m^2) - 1 \) of them. If \( c \) is one of them, then exactly one of \( c \) or \( m^2/c \) will be less than \( m \). Hence the number of solutions of (\( \ast \)) is \( d(m^2) - 1 \).