Olympiad Corner

Below are the problems of the 2011 Asia Pacific Math Olympiad, which was held in March 2011.

Problem 1. Let \(a, \ b, \ c\) be positive integers. Prove that it is impossible to have all of the three numbers \(a^2+b+c, \ b^2+c+a, \ c^2+a+b\) to be perfect squares.

Problem 2. Five points \(A_1, A_2, A_3, A_4, A_5\) lie on a plane in such a way that no three among them lie on a same straight line. Determine the maximum possible value that the minimum value for the angles \(\angle A_i A_j A_k\) can take where \(i, j, k\) are distinct integers between 1 and 5.

Problem 3. Let \(ABC\) be an acute triangle with \(\angle ABC = 30^\circ\). The internal and external angle bisectors of \(\angle ABC\) meet the line \(AC\) at \(B_1\) and \(B_2\), respectively, and the internal and external angle bisectors of \(\angle ACB\) meet the line \(AB\) at \(C_1\) and \(C_2\), respectively. Suppose that the circles with diameters \(B_1 B_2\) and \(C_1 C_2\) meet inside the triangle \(ABC\) at point \(P\). Prove that \(\angle BPC = 90^\circ\).

(continued on page 4)

Harmonic Series (II)

Leung Tat-Wing

As usual, for integers \(a, \ b, \ n\) (with \(n > 0\)), we write \(a \equiv b \pmod{n}\) to mean \(a - b\) is divisible by \(n\). If \(b \neq 0\) and \(n\) are relatively prime (i.e. they have no common prime divisor), then \(0, \ b, \ 2b, \ldots, (n-1)b\) are distinct \((\pmod{n})\) because for \(0 \leq s < r < n\), \(rb \equiv sb \pmod{n}\) implies \((r-s)b = kn\). Since \(b, \ n\) have no common prime divisor, this means \(b\) divides \(k\). Then \(0 < (k/b)n = r-s < n\), contradicting \(b \leq k\). Hence, there is a unique \(r\) among 1, …, \(n\) such that \(rb = 1 \pmod{n}\). We will denote this \(r\) by \(b^{-1}\pmod{n}\).

Next, we will introduce Wolstenholme’s theorem, which is an important relation concerning harmonic series.

Theorem (Wolstenholme): For a prime \(p \geq 5\),
\[
H(p-1) = 1 + \frac{1}{2} + \ldots + \frac{1}{p-1} = a \pmod{p^2}.
\]

(More precisely, for a prime \(p \geq 5\), if \(H(p-1) = 1 + \frac{1}{2} + \ldots + \frac{1}{p-1} = a \pmod{p}\), then \(p^2 \mid a\).)

Example We have
\[
H(10) = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{10} = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} = 1 + \frac{1}{10} = 7381.
\]

First proof We have
\[
H(p-1) = 1 + \frac{1}{2} + \ldots + \frac{1}{p-1} = \sum_{n=1}^{p-1} \frac{1}{n} = \frac{(p-1)(p-2)(p-3)\ldots(1)}{p-1}.
\]

So we need to prove
\[
\sum_{n=1}^{p-1} \frac{1}{n(p-1)} = 0 \pmod{p^2}.
\]

Now \(\sum_{n=1}^{p-1} \frac{1}{n(p-1)} = \sum_{n=1}^{p-1} \frac{1}{n^2} \pmod{p}\).

Since every \(1/n^2\) is congruent to exactly one of the numbers \(1^2, 2^2, \ldots, (p-1)^2\) \((\pmod{p})\) and \(1/n^2\) are all distinct for \(n = 1, 2, \ldots, (p-1)/2\), we have when \(p \geq 5\),
\[
\sum_{n=1}^{(p-1)/2} \frac{1}{n^2} = \frac{(p^2-1)p}{24} = 0 \pmod{p^2}.
\]

Wolstenholme’s theorem follows.

Second proof (using polynomials \(\pmod{p}\)) We use a theorem of Lagrange, which says if \(f(x) = c_0 + c_1x + \ldots + c_\lambda x^\lambda\) is a polynomial of degree \(\lambda\), with integer coefficients, and if \(f(x) \equiv 0 \pmod{p}\) has more than \(n\) solutions, where \(p\) is prime, then every coefficient of \(f(x)\) is divisible by \(p\). The proof is not hard. It can be done basically by induction and the division algorithm \(\pmod{p}\). The statement is false if \(p\) is not prime. For instance, \(x^2 - 1 \equiv 0 \pmod{8}\) has 4 solutions. Here is the other proof.

From Fermat’s Little theorem, \(x^{p-1} \equiv 1 \pmod{p}\) has 1, 2, …, \(p-1\) as solutions. Thus \(x^{p-1} - 1 = (x-1)(x^{p-2} + \ldots + x + 1)\) \((\pmod{p})\). Let
\[
(x-1)(x^{p-2} + \ldots + x + 1) \equiv (x-p+1)(x^{p-1} - 1) \equiv 0 \pmod{p^2}.
\]

By Wilson’s theorem, \(p-1 \equiv -1 \pmod{p}\). Thus
\[
0 \equiv sp^{p-2} - s_2x^2 + \ldots - spx^p + sp^{-1} \pmod{p}.
\]

The formula is true for every integer \(x\). By Lagrange’s theorem, \(p\) divides each of \(s_1, s_2, \ldots, s_{p-2}\). Putting \(x = p\) in (*), we get \(p-1 \equiv sp^{-1} - sp^{p-1} - \ldots - s_{p-2}p + sp^{-1}\). C canceling out \((p-1)!\) and dividing both sides by \(p\), we get
\[
0 \equiv sp^{p-2} - s_1sp^{p-3} + \ldots + sp_{p-2}p - sp_{p-2}.
\]

As \(p \geq 5\), each of the terms is congruent to 0 \((\pmod{p^2})\). Hence, we have \(s_{p-2} \equiv 0 \pmod{p^2}\). Finally,
\[
s_{p-2} = (p-1)! \left(\frac{1}{2} + \ldots + \frac{1}{p-1}\right) = (p-1)! \frac{a}{b}\,
\]

This proves Wolstenholme’s theorem.
Using Wolstenholme’s theorem and setting $x = kp$ in (*), we get

$$(kp-1)(kp-2)\cdots(kp-p+1)$$

$$= (kp)^{p-2} - s_p(kp)^{p-3} + \cdots + s_{p-3}(kp)^2 - s_{p-2}kp + s_{p-1},$$

$$≡ (p-1)! \pmod{p^3}. $$

Upon dividing by $(p-1)!$, we have

$$\binom{kp-1}{p-1} = 1 \pmod{p^3}, \quad k = 1, 2, \ldots$$

This result may in fact be taken as the statement of Wolstenholme’s theorem.

Here are a few further remarks. Wolstenholme’s theorem on the congruence of harmonic series is related to the Bernoulli numbers. Here are a few further remarks. Wolstenholme’s theorem on the congruence of harmonic series is related to the Bernoulli numbers. For instance, we have

$$\frac{(p^2-1)(p^2-2)\cdots(p^2-(p-1))}{(p-1)!} - 0 \pmod{p^3} \quad (1)$$

Now let

$$f(x) = (x-1)(x-2)\cdots(x-(p-1))$$

$$= x^{p-2} + s_p x^{p-3} + \cdots + s_{p-2} x + s_{p-1}. \quad (2)$$

Thus the first congruence relation is the same as $f(p^2) - (p-1)! \equiv 0 \pmod{p^3}$. Therefore it suffices to show that $s_{p-2}p^2 \equiv 0 \pmod{p^3}$ or $s_{p-2} \equiv 0 \pmod{p^3}$, which is exactly Wolstenholme’s theorem.

**Example 11 (Putnam 1996):** Let $p$ be a prime number greater than 3 and $k = [2p/3]$. Show that

$$\binom{p}{1} + \binom{p}{2} + \cdots + \binom{p}{k} = 0 \pmod{p^3}$$

For example,

$$\binom{7}{1} + \binom{7}{2} + \binom{7}{3} + \binom{7}{4} = 98 = 0 \pmod{7^3}.$$

**Solution** Recall

$$\binom{p}{i} = \frac{p(p-1)\cdots(p-i+1)}{i! \cdots \cdot 1}.$$ 

This is a multiple of $p$ if $1 \leq i \leq p-1$. Modulo $p$, the right side after divided by $p$ is congruent to

$$\frac{(-1)\cdots(-(i-1))}{i!} = (-1)^{i-1} \frac{1}{i}.$$ 

Hence, to prove the congruence, it suffices to show

$$1 - \frac{1}{2} + \frac{1}{3} - \cdots - (-1)^{i-1} \frac{1}{k} = 0 \pmod{p^3}.$$ 

Now observe that

$$-\frac{1}{2i} = \frac{1}{2i} + \frac{1}{p-i} \pmod{p}.$$ 

This allows us to replace the sum by

$$1 + \frac{1}{2} + \cdots + \frac{1}{p-i} = 0 \pmod{p},$$

which is Wolstenholme’s theorem.

We can also give a more detailed proof as follow. Let

$$H(n) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

and

$$P(n) = 1 + \frac{1}{2} + \cdots + (-1)^{n-1} \frac{1}{n}.$$ 

Then the problem is reduced to showing that for any $p > 3$, $p$ divides the numerator of $P([2p/3])$. First we note that $p$ divides the numerator of $H(p-1)$ because

$$2H(p-1) = 1 + \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{p-1} + 1$$

$$= (1 + \frac{1}{p-1}) + (\frac{1}{2} + \frac{1}{p-2}) + \cdots + (\frac{1}{p-1} + 1)$$

$$= \frac{p}{p-1} + \frac{p}{2(p-2)} + \cdots + \frac{p}{p-1} = 0 \pmod{p}.$$ 

Next we have two cases.

**Case 1** ($p = 3n+1$) Then $[2p/3] = 2n$. So we must show $p$ divides the numerator of $P(2n)$. Now

$$H(3n) - P(2n)$$

$$= \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2n} + \frac{1}{2n+1} + \frac{2}{2n+2} + \cdots + \frac{1}{3n}$$

$$= (1 + \frac{1}{2} + \cdots + \frac{1}{n}) + (\frac{1}{2n+1} + \frac{1}{2n+2} + \cdots + \frac{1}{3n})$$

$$= (1 + \frac{1}{p-1} + (2 + \frac{1}{p-2}) + \cdots + \frac{1}{n} + \frac{1}{p-n})$$

$$= \frac{p}{p-1} + \frac{p}{2(p-2)} + \cdots + \frac{p}{n(p-n)}.$$ 

So $p$ divides the numerators of both $H(3n)$ and $H(3n) - P(2n)$, hence also the numerator of $P(2n)$.

**Case 2** ($p = 3n+2$) Then $[2p/3] = 2n+1$. So we must show $p$ divides the numerator of $P(2n+1)$. Now

$$H(3n+1) - P(2n+1)$$

$$= 2(1 + \frac{1}{2} + \cdots + \frac{1}{2n}) + (\frac{1}{2n+1} + \frac{1}{2n+2} + \cdots + \frac{1}{3n+1})$$

$$= (1 + \frac{1}{2} + \cdots + \frac{1}{n}) + (\frac{1}{2n+1} + \frac{1}{2n+2} + \cdots + \frac{1}{3n+1})$$

$$= (1 + \frac{1}{p-1} + (2 + \frac{1}{p-2}) + \cdots + \frac{1}{n} + \frac{1}{p-n})$$

$$= \frac{p}{p-1} + \frac{p}{2(p-2)} + \cdots + \frac{p}{n(p-n)}.$$ 

So, $p$ divides the numerator of $H(3n+1) - P(2n+1)$, and hence $P(2n+1)$.

**Example 12:** Let $p \geq 5$ be a prime, show that if

$$1 + \frac{1}{2} + \cdots + \frac{1}{a} = \frac{a}{p-b},$$

then $p^n | ap - b$.

(continued on page 4)
Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver’s name, home (or email) address and school affiliation. Please send submissions to Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong. The deadline for sending solutions is June 25, 2011.

Problem 371. Let \( a_1, a_2, a_3, \ldots \) be a sequence of nonnegative rational numbers such that \( a_n + a_m = a_{nm} \) for all positive integers \( m, n \). Prove that there exist two terms that are equal.

Problem 372. (Proposed by Terence ZHU) For all \( a, b, c > 0 \) and \( abc = 1 \), prove that

\[
\frac{1}{a(a+1)} + \frac{1}{b(b+1)} + \frac{1}{c(c+1)} + \frac{1}{a(b+1)} + \frac{1}{b(c+1)} + \frac{1}{c(a+1)} \geq \frac{3}{4}.
\]

Problem 373. Let \( x \) and \( y \) be the sums of positive integers \( x_1 + x_2 + \ldots + x_9 \) and \( y_1 + y_2 + \ldots + y_9 \) respectively. Prove that there exists a 50-element subset \( S \) of \( \{1, 2, 3, \ldots, 99\} \) such that the sum of all \( x_a \) with \( a \in S \) is at least \( x/2 \) and the sum of all \( y_a \) with \( a \in S \) is at least \( y/2 \).

Problem 374. \( O \) is the circumcenter of \( \triangle ABC \) and \( T \) is the circumcenter of \( \triangle AOC \). Let \( M \) be the midpoint of side \( AC \). On sides \( AB \) and \( BC \), there are points \( D \) and \( E \) respectively such that \( \angle BDM = \angle BEM = \angle ABC \). Prove that \( BT \parallel DE \).

Problem 375. Find (with proof) all odd integers \( n > 1 \) such that if \( a, b \) are divisors of \( n \) and are relatively prime, then \( a + b - 1 \) is also a divisor of \( n \).

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Solutions

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Problem 365. Let \( n \) be a positive integer in base 10. For \( i = 1, 2, \ldots, 9 \), let \( a(i) \) be the number of digits of \( n \) that equal \( i \). Prove that

\[
2^{a(1)}3^{a(2)} \cdots 9^{a(9)}10^{a(0)} \leq n + 1
\]
determine all equality cases.

Solution. LAU Chun Ting (St. Paul’s Co-educational College, Form 2).

Let \( f(n) = 2^{a(1)}3^{a(2)} \cdots 9^{a(8)}10^{a(9)} \). If \( n \) is a number with one digit, then \( f(n) = n + 1 \). Suppose all numbers \( A \) with \( k \) digits satisfy the given inequality \( f(A) \leq A + 1 \). For any \( (k+1) \) digit number, it is of the form \( 10^k + B \), where \( A \) is a \( k \) digit number and \( 0 \leq B \leq 9 \). We have

\[
f(10^kB) = (B + 1)f(A) \leq (B + 1)(A + 1) = (B + 1)A + B + 1 \leq 10A + B + 1.
\]

Equality holds if and only if \( f(A) = A + 1 \) and \( B = 9 \). By induction, the inequality holds for all positive integers \( n \) and equality holds if and only if all but the leftmost digits of \( n \) are 9’s.

Other commented solvers: CHAN Long Tin (Diocesan Boys’ School), LEE Tak Wing (Carmel Alison Lam Foundation Secondary School), GORDON MAN Siu Hang (CCC Ming Yin College) and YUNG Fai.

Solution 367. For \( n = 1, 2, 3, \ldots \), let \( x_0 \) and \( y_0 \) be positive real numbers such that

\[
x_{n+2} = x_n + x_{n+1}^2
\]
and

\[
y_{n+2} = y_n + y_{n+1}^2.
\]
If \( x_1, x_2, y_1, y_2 \) are all greater than 1, then prove that there exists a positive integer \( N \) such that for all \( n > N \), we have \( x_n > y_n \).

Solution. LAU Chun Ting (St. Paul’s Co-educational College, Form 2) and Gordon MAN Siu Hang (CCC Ming Yin College).

Since \( x_1, x_2, y_1, y_2 \) are all greater than 1, by induction, we can get \( x_{n+1} > x_n^2 \) and \( y_{n+1} > y_n^2 \) for \( n \geq 2 \). Then \( x_{n+2} = x_n + x_{n+1}^2 > x_n^2 + x_{n+1} > x_4 \) and \( y_{n+2} = y_n + y_{n+1}^2 > y_n^2 + y_{n+1} > 3y_n^2 < y_n^3 \) for all \( n \geq 4 \).

Hence, log \( x_n > 4 \log x_0 \) and log \( y_n > 3 \log y_0 \). So for \( n \geq 4 \),

\[
\frac{\log x_{n+2}}{\log y_{n+2}} > \frac{4}{3} \left( \frac{\log x_n}{\log y_n} \right).
\]
As \( 4/3 > 1 \), by taking logarithm, we can solve for a positive integer \( k \) satisfying the inequality

\[
\left( \frac{4}{3} \right)^k \min \left( \frac{\log x_4}{\log y_4}, \frac{\log x_5}{\log y_5} \right) > 1.
\]
Let \( N = 2k+3 \). If \( n > N \), then either \( n = 2m+4 \) or \( n = 2m+5 \) for some integer \( m \geq k \).

Applying \( (*) \) times, we have

\[
\frac{\log x_n}{\log y_n} > \left( \frac{4}{3} \right)^m \min \left( \frac{\log x_4}{\log y_4}, \frac{\log x_5}{\log y_5} \right) > 1.
\]
This implies \( x_n > y_n \).

Other commented solvers: LEE Tak Wing (Carmel Alison Lam Foundation Secondary School) and NGUYEN Van Thien (Luong The Vinh High School, Dong Nai, Vietnam).

Problem 368. Let \( C \) be a circle, \( A_1, A_2, \ldots, A_n \) be distinct points inside \( C \) and \( B_1, B_2, \ldots, B_n \) be distinct points on \( C \) such that no two of the segments \( A_1B_1, A_2B_2, \ldots, A_nB_n \) intersect. A grasshopper can jump from \( A_i \) to \( A_j \) if the line segment \( A_iA_j \) does not intersect any line segment \( A_kB_k \) (\( \neq \)). Prove that after a certain number of jumps, the grasshopper can jump from any \( A_i \) to any \( A_j \).

Solution. William PENG.

The cases \( n = 1 \) or \( 2 \) are clear. Suppose \( n \geq 3 \). By reordering the pairs \( A_i, B_i \), we may suppose the convex hull of \( A_1, A_2, \ldots, A_n \) is the polygonal region \( M \) with vertices \( A_1, A_2, \ldots, A_k (k \leq n) \). For \( 1 \leq m \leq k \), if every \( A_mB_m \) intersects \( M \) only at \( A_m \), then the \( m \)-th case follows by removing two pairs of \( A_m, B_m \) separately and applying case \( n-1 \).

Otherwise, there exists a segment \( A_mB_m \) intersecting \( M \) at more than one point. Let it intersect the perimeter of \( M \) again at \( D_m \). Since \( A_mB_m \) do not intersect, so \( A_mD_m \)’s (being subsets of \( A_mB_m \)’s) do not intersect. In particular, \( D_m \) is not a vertex of \( M \).

Now \( A_iD_m \) divides the perimeter of \( M \) into two parts. Moving from \( A_i \) to \( D_m \) clockwise on the perimeter of \( M \), there are points \( A_j, D_j \), such that there is no \( D_m \) between them. As \( D_j \) is not a vertex, there is a vertex \( A_k \) between \( A_j \) and \( D_j \). Then \( A_mB_j \) only intersect \( M \) at \( A_k \). Also, moving from \( A_k \) to \( D_m \) anti-clockwise on the perimeter of \( M \), there is \( A_j \) such that \( A_mB_j \) only intersects \( M \) at \( A_j \). Then \( A_mB_j \) and \( A_jB_j \) do not intersect any diagonal of \( M \) with endpoints different from \( A_i \) and \( A_j \).
Removing $A_5, B_5$ and applying case $n-1$, the grasshopper can jump between any two of the points $A_1, \ldots, A_{j-1}, A_j, \ldots, A_n$. Also, removing $A_5, B_5$ and applying case $n-1$, the grasshopper can jump between any two of the points $A_1, \ldots, A_{j-1}, A_j, \ldots, A_n$. Using these two cases, we see the grasshopper can jump from any $A_n$ to any $A_i$ via $A_k$ ($i 
eq k, j$).

Other commended solvers: T. h. G.

**Problem 369.** $ABC$ is a triangle with $BC > CA > AB$. $D$ is a point on side $BC$ and $E$ is a point on ray $BA$ beyond $A$ so that $BD=BE=CA$. Let $P$ be a point on side $AC$ such that $A, B, D, P$ are concyclic. Let $Q$ be the intersection point of ray $BP$ and the circumcircle of $\triangle ABC$ different from $B$. Prove that $\angle QAC = 180^\circ - \angle ABC = \angle EPD$ and $\angle PED = \angle PBD = \angle QAC$.

Hence, $\angle QAC$ and $\angle EPD$ are similar. So we have $AQ/AC = PE/DE$ and $CQ/AC = PD/DE$. Cross-multiplying and adding these two equations, we get

$(AQ+CQ) \times DE = (PE+PD) \times AC$.

(1)

For cyclic quadrilateral $EPDB$, by the Ptolemy theorem, we have

$BP \times DE = PE \times BD + PD \times BE$

$(PE+PD) \times AC$

(2)

Comparing (1) and (2), we have $AQ+CQ=BP$.

Other commended solvers: LEE Tak Wing (Carmel Alison Lam Foundation Secondary School).

**Problem 370.** On the coordinate plane, at every lattice point $(x, y)$ (these are points where $x$ and $y$ are integers), there is a light. At time $t = 0$, exactly one light is turned on. For $n = 1, 2, 3, \ldots$, at time $t = n$, every light at a lattice point is turned on if it is at a distance 2005 from a light that was on at time $t = n-1$. Prove that every light at a lattice point will eventually be turned on at some time.

Solution. CHAN Long Tin (Diocesan Boys’ School), GIORGOS KALANTZIS (Democritius’s Public School, Patras, Greece) and LAU Chun Ting (St. Paul’s Co-educational College, Form 2).

Since $A, B, C, Q$ are concyclic and $EP, D, B$ are concyclic, we have

$\angle QAC = 180^\circ - \angle ABC = \angle EPD$

and

$\angle PED = \angle PBD = \angle QAC$.

(1)

By the Euclidean algorithm, we get

$\gcd(1037, 1716) = 1$. By eliminating the remainders in the calculations, we get

$84 \times 1716 - 139 \times 1037 = 1$.

(2)

Let $V_1, V_2, V_3, V_4, V_5$ be the vectors from the origin to (2005,0), (1037, 1716), (1037, −1716), (1716, 1037), (1716, −1037) respectively. We have $V_2 + V_3 = (2 \times 1037, 0)$ and $V_4 + V_5 = (2 \times 1716, 0)$. Then we can get (1,0) = 1003(84(V_2+V_3)−139(V_4+V_5)=V_1).

(3)

So, from the origin, following these vector movements, we can get to the point (1,0). Similarly, we can get to the point (0,1).

As $(a, b) = (1,0) + (0,1)$, we can get to any lattice point.

**Olympiad Corner (continued from page 1)**

**Problem 4.** Let $n$ be a fixed positive odd integer. Take $m+2$ distinct points $P_0, P_1, \ldots, P_{m+1}$ (where $m$ is a non-negative integer) on the coordinate plane in such a way that the following 3 conditions are satisfied:

1. $P_0 = (0,0), P_{m+1} = (n+1, m)$, and for each integer $i$, $1 \leq i \leq m$, both $x$- and $y$-coordinates of $P_i$ are integers lying in between 1 and $n$ (1 and $n$ inclusive).
2. $P_i P_{i+1}$ is parallel to the $x$-axis if $i$ is even, and is parallel to the $y$-axis if $i$ is odd.
3. (For each pair $i, j$ with $0 \leq i < j \leq m$, line segments $P_iP_{i+1}$ and $P_jP_{j+1}$ share at most 1 point.)

Determine the maximum possible value that $m$ can take.

**Problem 5.** Find all functions $f: \mathbb{R} \rightarrow \mathbb{R}$, where $\mathbb{R}$ is the set of all real numbers, satisfying the following 2 conditions:

1. There exists a real number $M$ such that for every real number $x$, $f(x) < M$ is satisfied.
2. For every pair of real numbers $x$ and $y$, $f(xf(x)) + yf(x) = xf(y) + f(xy)$ is satisfied.

**Harmonic Series (II) (continued from page 2)**

Solution. By Wolstenholme’s theorem,

$p^2 \left( p-1 \right) \left( \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{p-1} \right)$

So,

$\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{p-1} = \frac{p^2}{p} \times \frac{x}{y}$

where $x$, $y$ are integers with $y$ not divisible by $p$. So we have

$a \times b - 1 = \frac{p^2}{p} \times \frac{x}{y}$

which implies $ap-b = p^2 bx/y$. Finally,

$a = 2 \times 3 \times p + 1 \times 3 \times 4 \times p + \cdots + 1 \times 2 \times (p-1)$

and the numerator of the right side is of the form $mp^2(p-1)!$. Hence, it is not divisible by $p$. So $p \mid b$ and $p^4 \mid p^2 bx/y = ap-b$.

**Example 13:** Let $p$ be an odd prime, then prove that

$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{(p-1)^2} \equiv 0 \pmod{p}$

**Solution**. The proof is not hard. Indeed,

$\sum_{i=1}^{p-1} \left( -1 \right)^{i+1} \frac{1}{k^2} = -\sum_{i=1}^{p-1} \left( -1 \right)^{i+1} \frac{1}{k^2} + \frac{1}{(p-k)^2}$

$= -\sum_{i=1}^{p-1} \left( -1 \right)^{i+1} \frac{1}{k^2} + \frac{1}{(k)^2} \equiv 0 \pmod{p}$.