Olympiad Corner

Below are the problems of the 2011 IMO Team Selection Contest from Estonia.

Problem 1. Two circles lie completely outside each other. Let \( A \) be the point of intersection of internal common tangents of the circles and let \( K \) be the projection of this point onto one of their external common tangents. The tangents, different from the common tangent, to the circles through point \( K \) meet the circles at \( M_1 \) and \( M_2 \). Prove that the line \( AK \) bisects angle \( M_1KM_2 \).

Problem 2. Let \( n \) be a positive integer. Prove that for each factor \( m \) of the number \( 1+2+\cdots+n \) such that \( m \geq n \), the set \( \{1,2,\ldots,n\} \) can be partitioned into disjoint subsets, the sum of the elements of each being equal to \( m \).

Problem 3. Does there exist an operation \( * \) on the set of all integers such that the following conditions hold simultaneously:

1. for all integers \( x, y \) and \( z \),
   \[ (x*y)*z = x*(y*z); \]
2. for all integers \( x \) and \( y \),
   \[ x*(y*x*y) = y*(x*y*x); \]

(continued on page 4)

Casey’s Theorem

Kin Y. Li

We recall Ptolemy’s theorem, which asserts that for four noncollinear points \( A, B, C, D \) on a plane, we have

\[ AB \cdot CD + AD \cdot BC = AC \cdot BD \]

if and only if \( ABCD \) is a cyclic quadrilateral (cf vol. 2, no. 4 of Mathematical Excalibur). In this article, we study a generalization of this theorem known as

\[ t_{12} \cdot t_{14} + t_{23} \cdot t_{24} = t_{13} \cdot t_{34}, \]

(*)

where \( t_{ab} \) denote the length of an external common tangent of circle \( C_a \) and \( C_b \).

To prove this, consider the following figure.

Let line \( AB \) be an external common tangent to \( C_1, C_2 \) intersecting \( C_1 \) at \( Q_1 \), \( C_2 \) at \( Q_2 \). Let line \( P_1Q_1 \) intersect \( C \) at \( S \). Let \( r_1, r_2 \) be the respective radii of \( C_1, C_2 \). Then the isosceles triangles \( P_1O_1Q_1 \) and \( P_2O_2Q_2 \) are similar. So \( O_2Q_2 \parallel OS \). Since \( O_1Q_1 \perp AB \), so \( OS \perp AB \), hence \( S \) is the midpoint of arc \( AB \). Similarly, line \( P_2Q_2 \) passes through \( S \). Now \( \Delta SQ_2Q_1 = \Delta P_1Q_2A = \frac{1}{2} \Delta P_1O_1Q_1 = \frac{1}{2} \Delta P_2O_2Q_2 = \Delta SP_2P_1 \). Then \( \Delta SQ_1Q_2 \) and \( \Delta SP_2P_1 \) are similar. So

The converse of Casey’s theorem and its extension are also true. However, the proofs are harder, longer and used inversion in some cases. For the curious readers, a proof of the converse can be found in Roger A. Johnson’s book *Advanced Euclidean Geometry*, published by Dover.

Next we will present some geometry problems that can be solved by Casey’s theorem and its converse.
Example 1. (2009 China Hong Kong Math Olympiad) Let \( \triangle ABC \) be a right-angled triangle with \( \angle C = 90^\circ \). CD is the altitude from C to AB, with D on AB, w is the circumcircle of \( \triangle BCD \). v is a circle situated in \( \triangle ACD \), it is tangent to the segments AD and AC at M and N respectively, and is also tangent to circle w.

(i) Show that \( BD \cdot CN + BC \cdot DM = CD \cdot BM \).

(ii) Show that \( BM = BC \).

Solution. (i) Think of B, C, D as circles with radius 0 externally tangent to w. Then \( t_{BD} = BD, t_{CN} = CN, t_{BC} = BC, t_{DM} = DM, t_{CD} = CD \) and \( t_{BN} = BM \). By Casey’s theorem, (*) yields

\[ BD \cdot CN + BC \cdot DM = CD \cdot BM. \]

(ii) Let circles v and w meet at P. Then \( \angle BPC = 90^\circ \). Let O and O’ be centers of circles v and w. Then O, P, O’ are collinear. So

\[ \angle PNC + \angle PCN = \angle (\angle PNO + \angle POC) = \frac{1}{2} (360^\circ - \angle O'NC - \angle OCN) = 90^\circ. \]

So \( \angle NPC = 90^\circ \). Hence, B, P, N are collinear. By the power-of-a-point theorem, BM^2 = BP \cdot BN. Also \( \angle C > 90^\circ \) and CP \perp BN imply BC^2 = BP \cdot BN. Therefore, BM = BC.

Example 2. (Feuerbach’s Theorem) Let D, E, F be the midpoints of sides AB, BC, CA of \( \triangle ABC \) respectively.

(i) Prove that the inscribed circle S of \( \triangle ABC \) is tangent to the (nine-point) circle N through D, E, F.

(ii) Prove that the described circle T on side BC is also tangent to N.

Solution. (1) We consider D, E, F as circles of radius 0. Let A’, B’, C’ be the points of tangency of S to sides BC, CA, AB respectively.

First we recall that the two tangent segments from a point to a circle have the same length. Let \( AB’ = x = CA’, \ BC’ = y = A’B’, \ CA’ = z = B’C \) and \( s = (a+b+c)/2 \), where \( a = BC, \ b = CA, \ c = AB \). From \( x+y = BA = c, \ z+y = CB = a \) and \( x+z = AC = b \), we get \( x = (c+b-a)/2, \ y = s-b, \ z = s-c \). By the midpoint theorem, \( t_{DE} = DE = \frac{1}{2}BA = c/2 \) and

\[ t_{FS} = FC’ = |FB’ - BC’| = |(c/2) - y| = |(c-2s(b-s))/2| = |b-a|/2. \]

Similarly, \( t_{DS} = |a-c|/2 \). Without loss of generality, we may assume \( a \leq b \leq c \). Then

\[ t_{DE} \cdot t_{FS} + t_{EF} \cdot t_{DS} = \frac{c}{4} + a(c-b)/4 = \frac{b-c}{4}. \]

By the converse of Casey’s theorem, we get S is tangent to the circle N through D, E, F.

(2) Let I’ be the center of T, let PQ, R be the points of tangency of T to lines BC, AB, CA respectively. As in (1), \( t_{DE} = c/2 \).

To find t_{FT}, we need to know BQ. First note \( AQ = AR, BP = BQ \) and CR = CP. So \( 2AQ = AQ + AR = AB + BP + CP + AC = 2s \). So \( AQ = s/2 \). Next \( BQ = AB - AS = -c \). Hence, \( t_{EF} = FQ = FB + BQ = (c/2) + (s-c) = (b+a)/2 \). Similarly, \( t_{EF} = (a+c)/2 \). Now \( t_{FP} = DP = DB - BP = DB - BQ = (a/2) - (s-c) = (c-b)/2 \). Then

\[ t_{FT} \cdot t_{EF} + t_{EF} \cdot t_{DS} = \frac{b+c}{2} + a(c-b)/4 = \frac{c(b+a)}{4}. \]

By the converse of Casey’s theorem, we get T is tangent to the circle N through D, E, F.

Example 3. (2011 IMO) Let ABC be an acute triangle with circumcircle \( \Gamma \). Let L be a tangent line to \( \Gamma \), and let \( L_a, L_b, L_c \) be the line obtained by reflecting L in the lines BC, CA, AB respectively. Show that the circumcircle of the triangle determined by the lines \( L_a, L_b, L_c \) is tangent to the circle \( \Gamma \).

Solution. (Due to CHOW Chi Hong, 2011 Hong Kong IMO team member)

Below for brevity, we will write \( \angle A, \angle B, \angle C \) to denote \( \angle CAB, \angle ABC, \angle BCA \) respectively.

Lemma. In the figure below, L is a tangent line to \( \Gamma \), T is the point of tangency. Let \( h_a, h_b, h_c \) be the length of the altitudes from A, B, C to L respectively. Then

\[ \sqrt{h_a} \sin \angle A + \sqrt{h_b} \sin \angle B = \sqrt{h_c} \sin \angle C. \]

Proof. By Ptolemy’s theorem and sine law,

\[ AT \cdot BC + BT \cdot CA = CT \cdot BC \] (or \( AT \sin \angle A + BT \sin \angle B = CT \sin \angle C \).

Let \( \theta \) be the angle between lines AT and L as shown. Then \( AT = h_a / \sin \theta = h_a (2k/AT) \), where k is the circumradius of \( \triangle ABC \). Solving for AT (then using similar argument for BT and CT), we get

\[ AT = \sqrt{2kh_a}, \quad BT = \sqrt{2kh_b}, \quad CT = \sqrt{2kh_c}. \]

Substituting these into (*), the result follows. This finishes the proof of the lemma.

For the problem, let \( L_a \cap L = A', \ L_b \cap L = B', \ L_c \cap L = C' \), \( L_a \cap L_b = A', \ L_b \cap L_c = C', \ L_c \cap L_a = A' \), \( L_a \cap L_b = B' \). Next

\[ \angle A'C'B' = \angle A'B'A' - \angle C'A'B' = 2 \angle CBA - (180^\circ - 2 \angle CAB) = 180^\circ - 2 \angle C. \]

Similarly, \( \angle A''B''C'' = 180^\circ - \angle B \) and \( \angle B''A''C'' = 180^\circ - 2 \angle A \). (***)

Consider \( \Delta A'C'B' \). Now \( A'B'B' \) bisects \( \angle A'B'A' \) and \( C'B'B' \) bisects \( \angle A'C'B' \). So B is the excenter of \( \angle A'C'B' \) opposite C’. Hence \( B'B'' \) bisects \( \angle A''B''C'' \). Similarly, \( A''A \) bisects \( \angle B''A'' \) and \( C''C \) bisects \( \angle B''C'' \). Therefore, they intersect at the incentre I of \( \Delta A'B'C' \).

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Problem 392. Let $S(x)$ denote the sum of the digits of the positive integer $x$ in base 10. Determine whether there exist distinct positive integers $a, b, c$ such that $S(a+b)<5, S(b+c)<5, S(c+a)<5$, but $S(a+b+c)>50$ or not.

Problem 393. Let $p$ be a prime number and $p = 1$ (mod 4). Prove that there exist integers $x$ and $y$ such that $x^2 - py^2 = -1$.

Solution. Mathematics Group (Carmel Alison Lam Foundation Secondary School) and William PENG.

Let $x = f(c)$, $x_1 = c$ and $x_{n+2} = 4x_{n-3} - 3x_{n+1}$ for $n = 0, 1, 2, \ldots$

From the given conditions, we can check by math induction that $x_n = f(x_{n+1}) \geq 0$ for $n = 0, 1, 2, \ldots$. Since $x^2 + 3x - 4 = (x-1)(x+4)$, we see $x_n = \alpha^n - \beta^n$ for some real $\alpha$ and $\beta$. Taking $n = 0$ and 1, we get $f(c) = \alpha + \beta$ and $c = \alpha - \beta$. Then $\beta = f((\alpha - \beta)/5) \neq 5$.

If $\beta > 0$, then $x_{n+1} = x_n + (\alpha - \beta)\beta - k \rightarrow -\infty$ as $k \rightarrow \infty$, a contradiction. Similarly, if $\beta < 0$, then $x_{n+1} = x_n + (\alpha - \beta)\beta \rightarrow -\infty$ as $k \rightarrow \infty$, yet another contradiction.

Other commended solvers: CHAN Yin Hong (St. Paul’s Co-educational College) and YEUNG Sai Wing (Hong Kong Baptist University, Math, Year 1).

Problem 388. In $\triangle ABC$, $\angle BAC = 30^\circ$ and $\angle ABC = 70^\circ$. There is a point $M$ lying inside $\triangle ABC$ such that $\angle MAB = \angle MCB = 20^\circ$. Determine $\angle MBA$ (with proof).

Solution. 1. CHOW Chi Hong (Bishop Hall Jubilee Schol) and AN-anduu Problem Solving Group (Ulaanbaatar, Mongolia).

Extend $CM$ to meet the circumcircle $\Gamma$ of $\triangle ABC$ at $P$.

Then we have $\angle BPC = \angle BAC = 30^\circ$ and $\angle BPC = 180^\circ - \angle BPC = \angle BAC = 70^\circ$. So line $CM$ passes through center $O$ of $\Gamma$.

Let lines $AO$ and $BC$ meet at $D$. Then $\angle AOB = 2 \angle ACB = 160^\circ$. Now $OA = OB$ implies $\angle OAB = 10^\circ$. Then $\angle MAO = 10^\circ = \angle MAC$ and $\angle ADC = 180^\circ - 100^\circ = 80^\circ = \angle ACD$. These imply $AM$ is the perpendicular bisector of $CD$. Then $MD = MC$. This along with $OB = OC$ and $\angle BOC = 60$ imply $\triangle OCB$ and $\triangle MCD$ are equilateral, hence $BOMD$ is cyclic. Then $\angle DBM = \angle DOM = 2 \angle OAC = 40^\circ$. So $\angle MBA = \angle ABC - \angle DBM = 30^\circ$.

Solution 2. CHAN Yin Hong (St. Paul’s Co-educational College), Mathematics Group (Carmel Alison Lam Foundation Secondary School), O Kin Chit Alex (G.T.(Ellen Yeung) College) and MIHAI STOINESCU (Bischwiller, France).

Let $x = \angle MBA$. Applying the sine law to $\triangle ABC$, $\triangle ABM$, $\triangle AMC$ respectively, we get

$$AB \sin 30^\circ = \frac{\sin 80^\circ + x}{\sin 80^\circ} \sin 30^\circ = \frac{\sin 20^\circ}{AM} \frac{\sin 80^\circ}{\sin 70^\circ} \frac{\sin 30^\circ}{AM} = \frac{\sin 20^\circ}{\sin 70^\circ} \frac{\sin 30^\circ}{\sin 20^\circ} \tag{†}$$

Multiplying the last 2 equations, we get

$$\frac{\sin 20^\circ + x}{\sin x} = \frac{AB \sin 80^\circ \sin 30^\circ}{AM \sin 70^\circ \sin 20^\circ} \tag{†}$$

Multiplying

$$\frac{\sin 80^\circ}{\sin 40^\circ} = 2 \cos 40^\circ = \frac{\sin 50^\circ}{\sin 30^\circ}$$

$$\frac{\sin 40^\circ}{\sin 20^\circ} = 2 \cos 20^\circ = \sin 70^\circ \sin 30^\circ$$

we see (†) can be simplified to $\sin(20^\circ + x)/\sin x = \sin 50^\circ/\sin 30^\circ$. Since the left side is equal to $\sin 20^\circ \cos x + \sin 20^\circ$, which is strictly decreasing (hence injective) for $x$ between $0^\circ$ to $70^\circ$, we must have $x = 30^\circ$.

Comments: One can get a similar equation as (†) directly by using the trigonometric form of Ceva’s theorem.

Other commended solvers: CHEUNG Ka Wai (Munsang College (Hong Kong Island)), NG Ho Man (La Salle
College, Form 5), Bobby POON (St. Paul’s College), St. Paul’s College Mathematics Team, Aliaksei SEMCHANKAU (Secondary School No.41, Minsk, Belarus) and ZOLBABY SHADGAR (9th grader, Orchlon International School, Ulaanbaatar, Mongolia).

Problem 389. There are 80 cities. An airline designed flights so that for each of these cities, there are flights going in both directions between that city and at least 7 other cities. Also, passengers from any city may fly to any other city by a sequence of these flights. Determine the least k such that no matter how the flights are designed subject to the conditions above, passengers from one city can fly to another city by a sequence of at most k flights.

(Source: 2004 Turkish MO.)

Solution. William PENG.

Below we denote the number of elements in a set A by |A|.

To show k≥27, take cities A1, A2,...,A29. For i=1,2,...,27, design flights between A1 and Ai+1. For the remaining 25 cities, partition them into pairwise disjoint subsets Y0,...,Y5, so |Yi|=6=|Yj| and the other |Yi|=5. Let Z0={A1,A2}∪Y0, Z6={A27,A28}∪Y6 and for 1≤sm≤8, let Zm={A3m−1,Am+1,Am+2}∪Ym. Then design flights between each pair of cities in Zm for 1≤sm≤8. In this design, from A1 to A29 requires 27 flights.

Assume k>27. Then there would exist two cities A1 and A29 the shortest connection between them would involve a sequence of 28 flights from cities A1 to Ai+1 for i=1,2,...,28. Due to the shortest condition, each of A1 and A29 has flights to 6 other cities not in B={A1,...,A28}. Each Ai in B has flights to 5 other cities not in C={A1,...,A28}.

Next for each Ai in {A1,...,A10,A13,...,A15,A16,A19,...,A22,A25,...,A29}, let Xi be the set of cities not in C that have a flight to Ai. We have |X1|≥5, |X2|≥6 and the other |X1|≥5. Now every pair of Xi’s is disjoint, otherwise we can shorten the sequence of flights between Ai and A29. However, the union of C and all the Xi’s would yield at least 29+6+2+5+8=81 cities, contradiction. So k=27.

Problem 390. Determine (with proof) all ordered triples (x, y, z) of positive integers satisfying the equation

\[ x^2 + y^2 = z^2 \]

Solution. CHEUNG Ka Wai (Munsang College (Hong Kong Island)), Ioan Viorel CODREANU (Satulung Secondary School, Maramure, Romania) and Aliaksei SEMCHANKAU (Secondary School No.41, Minsk, Belarus).

Lemma. The system \( a^2 - b^2 = c^2 \)

and \( a^2 + b^2 = w^2 \) has no solution in positive integers.

Proof. Assume there is a solution. Then consider a solution with minimal \( a^2 + b^2 \).

Due to minimality, gcd(a, b)=1. Also 2a^2 = w^2 + c^2. Considering (mod 2), we see w+c and w-c are even. Then \( d = w^2 + c^2 \),

\[ d = w^2 + c^2, \]

Then \( d = gcd(a, r, s) \). Then \( d \) divides \( a \) and \( r+s \). Since \( a^2 + b^2 = w^2 \), \( d \) divides \( b \).

If \( gcd(a, b)=1 \), we get \( d=1 \). By the theorem on Pythagorean triples, there are relatively prime positive integers \( m, n \) with \( m^2+n^2 \) such that \( \{x,y,z\} = \{m^2-n^2, 2mn \} \) and \( a=m^2+n^2 \).

Now \( b^2 = (w^2 - c^2) / 2 = 2r = 4mn(m^2-n^2) \) implies \( b \) is an even integer, say \( b=2k \).

Then \( k = mm(m+n)^2(m-n) \). As \( gcd(m, n)=1 \), we see \( m, n, m+n, m-n \) are pairwise relatively prime integers. Hence, there exist positive integers \( d, e, f, g \) such that \( d^2=m^2+n^2 \), \( e^2+m^2 n^2 \) and \( m-n=g^2 \). Then \( \frac{d^2 - e^2 - g^2}{2} \) and \( \frac{d^2 + e^2 + g^2}{2} \), contradicting \( a^2 + b^2 \) is minimal. The lemma is proved.

Now for the problem, the equation may be rearranged as \( \frac{1}{2} \left( x^2 + y^2 \right) ^2 - \frac{1}{2} \left( x^2 - y^2 \right) ^2 = 0 \). If there is a solution \( (x, y, z) \) in positive integers, then considering discriminant, we see \( x^2 + 6x^2 y^2 + y^2 = w^2 \) for some integer \( w \).

This can be written as \( (x^2 + y^2)^2 + 2(2xy)^2 = w^2 \). Also, we have \( (x^2 - y^2)^2 + (2xy)^2 = (x^2 + y^2)^2 \).

Letting \( c = |x^2 - y^2| \), \( b = 2xy = \sqrt{2} \) and \( a = x^2 + y^2 \). Then we have \( c^2 = b^2 + a^2 \) and \( c^2 + 2b^2 = w^2 \). This contradicts the lemma above. So there is no solution.

Other commended solvers: Mathematics Group (Carmel Alson Lam Foundation Secondary School)

Olympiad Corner

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Problem 4. Let \( a, b, c \) be positive real numbers such that \( 2a^2 + b^2 = 9c^2 \). Prove that

\[ \frac{2c}{a} + \frac{c}{b} \geq \sqrt{3}. \]

Problem 5. Prove that if \( n \) and \( k \) are positive integers such that \( 1 < k < n-1 \),

Then the binomial coefficient \( \binom{n}{k} \) is divisible by at least two different primes.

Problem 6. On a square board with \( m \) rows and \( n \) columns, where \( m \leq n \), some squares are colored black in such a way that no two rows are alike. Find the biggest integer \( k \) such that for every possible coloring to start with one can always color \( k \) columns entirely red in such a way that no two rows are still alike.

Casey’s Theorem

(continued from page 2)

We have \( AIB = \angle AA'C + \angle AC'A'' = \frac{1}{2} \angle (\angle B'A'C + \angle B'C'A'') = \frac{1}{2} \angle A'B'C" \) and similarly \( \angle AIB = \frac{1}{2} \angle A'B'C" \). So \( \angle AIB = 180^\circ - \angle A'B'C" = \angle IB'A" \)

\[ = 180^\circ - \angle C'AA"B = \angle A'B'C" = 90^\circ + \angle A'B'C" \]

\[ = 90^\circ + \frac{1}{2}(180^\circ - 2\angle C) \] by (***) we get

\[ \angle A'B'C" = \frac{180^\circ - \angle ABC}{2} \]

Hence, \( I \) lies on \( G \).

Let \( D \) be the foot of the perpendicular from \( I \) to \( A'B" \), then \( ID=pr \) is the inradius of \( \Delta A'B'C" \). Let \( E, F \) be the feet of the perpendiculars of \( B \) to \( A'B", B'A" \) respectively. Then \( BE = BF = h_b \).

Let \( T(X) \) be the length of tangent from \( X \) to \( G \), where \( X \) is outside of \( G \). Since \( \angle A'B"A' = \frac{1}{2} \angle A'B'C" = 90^\circ - \angle 2C \) by (**), we get

\[ T(B") = \frac{BE}{BB'} \]

\[ \frac{BE}{BB'} = \frac{BE}{ID} \]

\[ \frac{BE}{ID} = \frac{\sin(90^\circ - \angle A'B"C")}{\sin(90^\circ - \angle A'B")} \]

\[ = \frac{h_b}{r_C} \cos B \]

Let \( R \) be the circumradius of \( \Delta A'B'C" \). Then

\[ T(B") \cdot C"A" = \frac{h_b}{r_C} \cdot 2R \sin(180^\circ - 2\angle L \angle B") \]

\[ = 4R^2 \sqrt{h_b} \sin B \]

Similarly, we can get expressions for \( T(A") \cdot B"C" \) and \( T(C") \cdot A"B" \). Using the lemma, we get

\[ T(A") \cdot B"C" + T(B") \cdot C"A" = T(C") \cdot A"B" \]

By the converse of Casey’s theorem, we have the result.