Mathematical Excalibur

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Olympiad Corner

Below are the problems of the 28th Italian Math Olympiad.

Problem 1. Let ABC be a triangle with right angle at A. Choose points D, E, F on sides BA, CA, AB respectively so that $AFDE$ is a square. Denote by x the side-length of this square. Prove that

$$\frac{1}{x}=\frac{1}{AB}+\frac{1}{AC}.$$

Problem 2. Determine all positive integers that are 300 times the sum of their digits.

Problem 3. Let n be an integer greater than or equal to 2. There are n persons in a line, and each of these persons is either a villain (and this means that he/she always lies) or a knight (and this means he/she always tells the truth). Apart from the first person in the line, every person indicates one of those before him and declares either "this person is a villain" or "this person is a knight". It is known that the number of villains is greater than the number of knights. Prove that, watching the declarations, it is possible to determine, for each of the n persons, whether he/she is a villain or a knight.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***March 10, 2013***.

For individual subscription for the next five issues for the 09-10 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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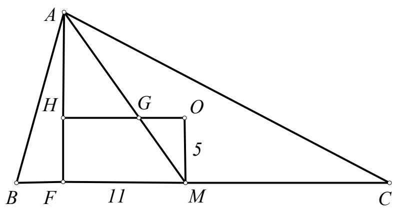
Putnam Exam

Kin Y. Li

The title of our article is an abbreviated name for the famous *William Lowell Putnam Mathematical Competition*. It started in the year 1938. Thousands of students in many US and Canadian universities participate in this competition annually. The top five scorers each year are designated as Putnam Fellows. These Putnam Fellows include the Physics Nobel Laureates Richard Feynman, Kenneth Wilson, the Fields' Medalists John Milnor, David Mumford, Dan Quillen and many other famous celebrities.

Although it is a math competition for undergraduate students, some of the problems may be solved by secondary school students interested in math olympiads. Below we will provide some examples.

Example 1 (*1997 Putnam Exam*) A rectangle $HOMF$ has sides HO=11 and OM=5. A triangle ABC has H as the intersection of the altitudes, O the center of the circumscribed circle, M the midpoint of BC and F the foot of the altitude from A. What is the length of BC?



Solution. Recall the centroid G of ΔABC is on the Euler line OH (see *Math Excalibur*, *vol.* 3, *no.* 1, *p.* 1) and AG/GM = 2. As FH, $MO \perp OH$ and $\angle AGH = \angle MGO$, so $\Delta AHG \sim \Delta MOG$. Hence AH = $2OM$ = 10. Then $OC^2 = OA^2 = AH^2 + OH^2 = 221$ and $BC = 2MC = 2(OC^2 - OM^2)^{1/2} = 28$.

Example 2 (*1991 Putnam Exam*) Suppose p is an odd prime. Prove that

$$\sum_{j=0}^{p}\binom{p}{j}\binom{p+j}{j}\equiv 2^p+1 \pmod{p^2}.$$

Solution. Let W be the left side of the equation. Since $\binom{p+j}{j}=\binom{p+j}{p}$, W is the coefficient of x^p in the polynomial

$$\sum_{j=0}^{p}\binom{p}{j}\sum_{k=0}^{p+j}\binom{p+j}{k}x^k$$
$$=\sum_{j=0}^{p}\binom{p}{j}(1+x)^{p+j}$$
$$=(1+x)^p\sum_{j=0}^{p}\binom{p}{j}(1+x)^j$$
$$=(1+x)^p(2+x)^p.$$

Expanding $(1+x)^p(2+x)^p$, we see

$$W=\sum_{k=0}^{p}\binom{p}{k}\binom{p}{p-k}2^k.$$

For $0 < k < p$, p divides $p!$, but not $k!(p-k)!$. So p divides $\binom{p}{k}=\binom{p}{p-k}$.

In (mod p^2) of W, we may ignore the terms with $0 < k < p$ to get

$$W\equiv\binom{p}{0}\binom{p}{p}2^0+\binom{p}{p}\binom{p}{0}2^p=1+2^p \pmod{p^2}.$$

Example 3 (*2000 Putnam Exam*) Let B be a set of more than $2^{n+1}/n$ distinct points with coordinates of the form (±1, ±1, …, ±1) in n-dimensional space with $n\geq3$. Show that there are three distinct points in B which are the vertices of an equilateral triangle.

Solution. Let S be the set of all points $(x_1, x_2, \ldots, x_n)$ with all $x_i = \pm1$. For each P in B, let S_P be the set of all points in S which differ from P in exactly one coordinate. Each S_P contains n points. So the union of all S_P's over all P in B (counting points repeated as many times as they appeared in the union) must contain more than 2^{n+1} points. Since this is more than twice 2^n, by the pigeonhole principle, there must exist a point T appeared in at least three of the sets S_P, S_Q, S_R, where P, Q, R are distinct points in B. Then any two of P, Q, R have exactly two different coordinates. Then ΔPQR is equilateral with sides $2^{3/2}$.

Example 4 (*1947 Putnam Exam*) Given $P(z) = z^2 + az + b$, a quadratic polynomial for the complex variable z with complex coefficients a and b. Suppose that $|P(z)| = 1$ for every z such that $|z| = 1$. Prove that $a = b = 0$.

Solution. Let $\omega \neq 1$ be a cube root of unity. Let $\alpha = P(1)$, $\beta = \omega P(\omega)$ and $\gamma = \omega^2 P(\omega^2)$. We have $|\alpha| = |\beta| = |\gamma| = 1$ and $\alpha+\beta+\gamma = 3+a(1+\omega^2+\omega^4)+b(1+\omega+\omega^2) = 3$. Hence, $|\alpha+\beta+\gamma| = |\alpha| + |\beta| + |\gamma|$. By the equality case of the triangle inequality, we get $\alpha = \beta = \gamma = 1$. Then $P(1) = 1$, $P(\omega) = \omega^2$ and $P(\omega^2) = \omega = \omega^4$. Since P is of degree 2 and $P(z)-z^2 = 0$ has three distinct roots 1, ω and ω^2, we get $P(z) = z^2$ for all complex number z.

Example 5 (*1981 Putnam Exam*) Prove that there are infinitely many positive integers n with the property that if p is a prime divisor of n^2+3 then p is also a divisor of k^2+3 for some integer k with $k^2 < n$.

Solution. First we look at the sequence m^2+3 with $m \geq 0$. The terms are 3, 4, 7, 12, 28, 39, 52, 67, 84, …. We can observe that 3×4, 4×7, 7×12,… are also in the sequence. This suggests multiplying $(m^2+3)[(m+1)^2+3]$. By completing square of the result, we see

$$(m^2+3)[(m+1)^2+3] = (m^2+m+3)^2+3.$$

Let $n = (m^2+m+2)(m^2+m+3)+3$. Using the identity above twice, we see $n^2+3 = (m^2+3)[(m+1)^2+3][(m^2+m+2)^2+1]$. So if p is a prime divisor of n^2+3, then p is also a divisor of either m^2+3 or $(m+1)^2+3$ or $(m^2+m+2)^2+1$ and m^2, $(m+1)^2$, $(m^2+m+2)^2 < n$. Letting m = 1,2,3,…, we get infinitely many such n.

Example 6 (*1980 Putnam Exam*) Let $A_1, A_2, \ldots, A_{1066}$ be subsets of a finite set X such that $|A_i| > ½|X| \geq 5$ for $1 \leq i \leq 1066$. Prove there exists ten elements $x_1, x_2, \ldots, x_{10}$ of X such that every A_i contains at least one of $x_1, x_2, \ldots, x_{10}$. (Here $|S|$ means the number of elements in the set S.)

Solution. Let $X = \{x_1, x_2, \ldots, x_m\}$ with $m = |X|$ and n_k be the number of i such that x_k is in A_i. We may arrange the x_k's so that n_k is decreasing. For $1 \leq i \leq 1066$ and $1 \leq k \leq m$, let $f(i,k)=1$ if x_k is in A_i and $f(i,k)=0$ otherwise. Then

$$n_1|X| \geq \sum_{k=1}^{m} n_k = \sum_{k=1}^{m}\sum_{i=1}^{1066} f(i,k) = \sum_{j=1}^{1066}\sum_{k=1}^{m} f(i,k)$$
$$= \sum_{i=1}^{1066}|A_i| \geq \sum_{i=1}^{1066}\frac{1}{2}m = 533|X|.$$

Then n_1 is greater than 533, i.e. x_1 is in more than 533 A_i's.

Next let $B_1, B_2, \ldots, B_r$ be those A_i's not containing x_1 and $Y = \{x_2, x_3, \ldots, x_m\}$. Then $r = 1066-n_1 \leq 532$ and each $|B_i| > ½|X| > ½|Y|$. Repeating the reasoning above, we will get $n_2 > r/2$. Let $C_1, C_2, \ldots, C_s$ be those A_i's not containing x_1, x_2 and $Z = \{x_3, x_4, \ldots, x_m\}$. Then $s = r-n_2 < r/2$, i.e. $s \leq 265$. After 532 and 265, repeating the reasoning, we will get 132, 65, 32, 15, 7, 3, 1. Then at most 1 set is left not containing $x_1, x_2, \ldots, x_9$. Finally, we may need to use x_{10} to take care of the last possible set.

Example 7 (*1970 Putnam Exam*) A quadrilateral which can be inscribed in a circle is said to be <u>*inscribable*</u> or <u>*cyclic*</u>. A quadrilateral which can be circumscribed to a circle is said to be <u>*circumscribable*</u>. If a circumscribable quadrilateral of sides a, b, c, d has area $A = \sqrt{abcd}$, then prove that it is also inscribable.

Solution. Since the two tangent segments from a point (outside a circle) to the circle are equal and the quadrilateral is circumscribable, we have $a+c = b+d$. Let k be the length of a diagonal and α, β be opposite angles of the quadrilateral so that

$$a^2+b^2-2ab\cos\alpha = k^2 = c^2+d^2-2cd\cos\beta.$$

Subtracting $(a-b)^2 = (c-d)^2$, we get

$$2ab(1-\cos\alpha) = 2cd(1-\cos\beta). \quad (*)$$

Now $2\sqrt{abcd} = 2A = ab\sin\alpha + cd\sin\beta$. Squaring and using (*) twice, we get

$$\begin{aligned} 4abcd &= a^2b^2(1-\cos^2\alpha)+2abcd\sin\alpha\sin\beta \\ &\quad +c^2d^2(1-\cos^2\beta) \\ &= abcd(1+\cos\alpha)(1-\cos\beta) \\ &\quad +2abcd\sin\alpha\sin\beta \\ &\quad +abcd(1+\cos\beta)(1-\cos\alpha). \end{aligned}$$

Simplifying this, we get $4 = 2-2\cos(\alpha+\beta)$, i.e. $\alpha+\beta = 180°$. Therefore the quadrilateral is cyclic.

Example 8 (*1964 Putnam Exam*) Show that the unit disk in the plane cannot be partitioned into two disjoint congruent subsets.

Solution. Let D be the unit disk, O be its center and $d(X,Y)$ denote the distance between X and Y in D. <u>*Assume*</u> D can be partitioned into two disjoint congruent subsets A and B. Without loss of generality, suppose O is in A. For each X in A, let X^* be the corresponding point in B. Then O^* is in B. For all X, Y in A, $d(X,Y) = d(X^*,Y^*)$.

Since $d(O,X) \leq 1$ for all X in A and the set $B = \{X^* : X \text{ in } A\}$, so $d(O^*,Z) \leq 1$ for all Z in B. Let R and S be the endpoints of the diameter perpendicular to line OO^*. Then $d(O^*,R) = d(O^*,S) > 1$. Hence, R and S are in A. Now $d(R^*,S^*) = d(R,S) = 2$, so R^*S^* is a diameter. Since O is the midpoint of diameter RS in A, O^* must be the midpoint of the diameter R^*S^*. Then $O^* = O$, which contradicts A, B are disjoint.

Example 9 (*1950 Putnam Exam*) In each of N houses on a straight street are one or more boys. At what point should all the boys meet so that the sum of the distances that they walk is as small as possible?

Solution. Think of the street is the real axis. Suppose the i-th boy's house is at x_i so that $x_1 \leq x_2 \leq \cdots \leq x_n$. Suppose they meet at x, the first and the n-th boy together must walk a distance of $x_n - x_1$ if x is in $[x_1, x_n]$ and more if x is outside $[x_1, x_n]$. This is similar for the second boy and the $(n-1)$-st boy, etc.

If n is even, say $n=2k$, then the least distance all n boys have to walk is

$$(x_n-x_1)+(x_{n-1}-x_2)+\cdots+(x_{k+1}-x_k)$$

with equality if x is in $[x_k, x_{k+1}]$. If n is odd, say $n = 2k - 1$, then the least distance they have to walk is

$$(x_n-x_1)+(x_{n-1}-x_2)+\cdots+(x_{k+1}-x_k)+0$$

with equality if $y = x_k$.

Example 10 (*1956 Putnam Exam*) The nonconstant polynomials $P(z)$ and $Q(z)$ with complex coefficients have the same set of numbers for their zeros but possibly different multiplicities. The same is true of the polynomials $P(z)+1$ and $Q(z)+1$. Prove that $P(z) \equiv Q(z)$.

Solution. Observe that if $P(z)$ has c as a zero with multiplicities $k > 0$, then the derivative $P'(z)$ has c as a zero with multiplicities $k-1$, which follows from differentiating $P(z) = (z-c)^k R(z)$ on both sides.

Now suppose $P(z)$ has degree m and $Q(z)$ has degree n. By symmetry, we may assume $m \geq n$. Let the distinct zeros of $P(z)$ be $a_1, a_2, \ldots, a_s$ and let the distinct zeros of $P(z)+1$ be $b_1, b_2, \ldots, b_t$. Clearly, $a_1, a_2, \ldots, a_s, b_1, b_2, \ldots, b_t$ are all distinct.

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***March 10, 2013.***

Problem 411. A and B play a game on a square board divided into 100×100 squares. Each of A and B has a checker. Initially A's checker is in the lower left corner square and B's checker is in the lower right corner square. They take turn to make moves. The rule is that each of them has to move his checker one square up, down, left or right within the board and A goes first. Prove that no matter how B plays, A can always move his checker to meet B's checker eventually.

Problem 412. $\triangle ABC$ is equilateral and points D, E, F are on sides BC, CA, AB respectively. If

$$\angle BAD+\angle CBE+\angle ACF = 120^\circ,$$

then prove that $\triangle BAD$, $\triangle CBE$ and $\triangle ACF$ cover $\triangle ABC$.

Problem 413. Determine (with proof) all integers $n\geq3$ such that there exists a positive integer M_n satisfying the condition for all n positive numbers a_1, a_2, …, a_n, we have

$$\frac{a_1+a_2+\cdots+a_n}{\sqrt[n]{a_1a_2\cdots a_n}}\leq M_n\left(\frac{a_2}{a_1}+\frac{a_3}{a_2}+\cdots+\frac{a_n}{a_{n-1}}+\frac{a_1}{a_n}\right).$$

Problem 414. Let p be an odd prime number and $a_1, a_2, \ldots, a_{p-1}$ be positive integers not divisible by p. Prove that there exist integers $b_1, b_2, \ldots, b_{p-1}$, each equals 1 or -1 such that

$$a_1b_1+a_2b_2+\cdots+a_{p-1}b_{p-1}$$

is divisible by p.

Problem 415. (*Due to MANOLOUDIS Apostolos, Piraeus, Greece*) Given a triangle ABC such that $\angle BAC=103^\circ$ and $\angle ABC=51^\circ$. Let M be a point inside $\triangle ABC$ such that $\angle MAC=30^\circ$ and $\angle MCA=13^\circ$. Find $\angle MBC$ with proof.

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Solutions

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Problem 406. For every integer $m>2$, let P be the product of all those positive integers that are less than m and relatively prime to m, prove that P^2-1 is divisible by m.

Solution. **Jon GLIMMS** (Vancouver, Canada), **Corneliu MĂNESCU- AVRAM** (Technological Transportation High School, Ploieşti, Romania), **WONG Ka Fai** and **YUNG Fai**.

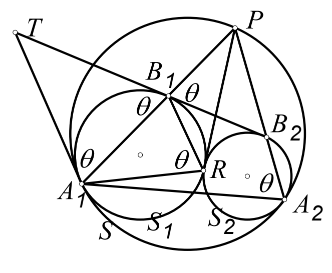
Let a in interval $[1,m)$ be relatively prime to m. By Bezout's theorem, there exists a unique a^{-1} in $[1,m)$ such that $aa^{-1}\equiv1$ (mod m). Then $\gcd(a^{-1},m) = 1$. Since $a^{-1}a \equiv 1$ (mod m), by uniqueness, $(a^{-1})^{-1} = a$.

For those factor a in the product P satisfying $a\neq a^{-1}$, a will be cancelled by a^{-1} (mod m). Thus, P is congruent modulo m to the product of those remaining factor a satisfying $a= a^{-1}$. Now $a = a^{-1}$ implies $a^2 = aa^{-1}\equiv1$ (mod m). It follows $P^2\equiv 1$ (mod m) and we are done.

Other commended solvers: **F5D** (Carmel Alison Lam Foundation Secondary School).

Problem 407. Three circles S, S_1, S_2 are given in a plane. S_1 and S_2 touch each other externally, and both of them touch S internally at A_1 and at A_2 respectively. Let P be one of the two points where the common internal tangent to S_1 and S_2 meets S. Let B_i be the intersection points of PA_i and S_i (i=1,2). Prove that line B_1B_2 is a common tangent to S_1 and S_2.

Solution. **F5D** (Carmel Alison Lam Foundation Secondary School), **William FUNG and Jacob HA** and **NGUYEN Van Thien** (Luong The Vinh High School, Dongnai Province, Vietnam)**.**



Let the tangent at A_1 to S (and S_1) and line B_1B_2 meet at T. Let R be the tangent point of S_1 and S_2. By the intersecting chord theorem, we have

$$PB_1\times PA_1 = PR^2 = PB_2\times PA_2.$$

So A_1, B_1, B_2, A_2 are concyclic. Using (1) line TA_1 is tangent to S_1, (2) line TA_1 is tangent to S, (3) A_1, B_1, B_2, A_2 concyclic and (4) vertical angles are congruent in that order, we get

$$\angle B_1RA_1 = \angle B_1A_1T = \angle PA_2A_1 = \angle PB_1B_2 = \angle TB_1A_2.$$

Then line TB_1=B_1B_2 is tangent to S_1 at B_1. Similarly, line B_1B_2 is tangent to S_2 at B_2. Therefore, line B_1B_2 is a common tangent to S_1 and S_2.

Other commended solvers: **Dusan DROBNJAK**(Mathematical Grammar School, Belgrade, Serbia), **Jon GLIMMS** (Vancouver, Canada), **MANOLOUDIS Apostolos** (4° Lyk. Korydallos, Piraeus, Greece) and **Vijaya Prasad NALLURI** (Retired Principal, AP Educational Service, Andhra Pradesh, India).

Problem 408. Let $\mathbb{Q}$ denote the set of all rational numbers. Let $f:\mathbb{Q}\to\{0,1\}$ be a function such that for all x,y in $\mathbb{Q}$ with $f(x)=f(y)$, we have $f((x+y)/2)=f(x)$. If $f(0)=0$ and $f(1)=1$, then prove that $f(x)=1$ for every rational $x > 1$.
(*Source*: 2000 *Indian Math Olympiad*)

Solution. **Ioan Viorel CODREANU,** (Secondary School Satulung, Maramures, Romania) and **Dusan DROBNJAK**(Mathematical Grammar School, Belgrade, Serbia).

We *claim* that if a,b are rational numbers and $f(a)\neq f(b)$, then for all positive integer n, we have $f(a+n(b-a))=f(b)$.

We will prove this by induction on n. The case n=1 is clear. Suppose the case $n=k$ is true. Then we have $f(a+k(b-a))=f(b)$. *Assume* $f(a+(k+1)(b-a)) \neq f(b)$. Since $f(r)$=0 or 1 for all r in $\mathbb{Q}$ and $f(a)\neq f(b)$, we get $f(a+(k+1)(b-a)) = f(a)$. Let $x = a$, $y = a+(k+1)(b-a)$, $x' = b$, $y' = a+k(b-a)$. From above, we have $f(x) = f(y)$ and $f(x') = f(y')$. By the given property of f, since

$$\frac{x+y}{2}=\frac{(k+1)b-(k-1)a}{2}=\frac{x'+y'}{2},$$

we get $f(a)=f(x)=f(x')=f(b)$, contradiction. Hence the case $n=k+1$ is true and we complete the induction.

Now by the claim, since $f(0)=0\neq1=f(1)$, for all positive integer n, we get $f(n) = f(1) =1$. For a rational $r > 1$, let $r-1=p/q$, where p, q are positive integers. *Assume* $f(r)\neq1$. Using the claim with $a = 1$, $b = r$, and $n = q$, we get $f(1+q(r-1)) = f(r)$. But $f(1+q(r-1)) = f(1+p) = 1$, contradiction. So, for all rational $r>1$, $f(r) = 1$.

Other commended solvers: **F5D** (Carmel Alison Lam Foundation Secondary School).

Problem 409. The population of a city is one million. Every two citizens there know another common citizen (here knowing is mutual). Prove that it is possible to choose 5000 citizens from the city such that each of the remaining citizens will know at least one of the chosen citizens.
(*Source*: 63*rd* *St. Petersburg Math Olympiad*)

Solution. **Jon GLIMMS** (Vancouver, Canada).

Let $m=10^6$ and $x_1, x_2, \ldots, x_m$ be all the citizens in the city. Let $F(x_i)$ be all the citizens (not including x_i) that x_i knows and $|F(x_i)|$ denote the number of such citizens.

If there exists a x_i with $|F(x_i)|\le 5000$, then let us choose any 5000 citizens including all members of $F(x_i)$. For any x_j not among the chosen 5000 citizens, by the given assumption, x_i and x_j know a common citizen in $F(x_i)$, who is in the chosen 5000 citizens.

Otherwise, we may assume for every x_i, $|F(x_i)| > 5000$. Now there are m^{5000} ordered 5000-tuples $(C_1, C_2, \ldots, C_{5000})$, where each C_k may be any one of the m citizens. For each x_i, let

$$S(x_i)=\{(C_1,C_2,\ldots,C_{5000})\text{: all } C_k\notin F(x_i)\}$$

Now $S(x_i)$ has less than $(m-5000)^{5000}$ members since $|F(x_i)|>5000$. Let S be the union of $S(x_1), S(x_2),\ldots,S(x_m)$. We claim that $m(m-5000)^{5000} < m^{5000}$. The claim means there exists $(C_1, C_2, \ldots, C_{5000})$ not in every $S(x_i)$. That means by choosing $C_1, C_2, \ldots, C_{5000}$, every x_i will know at least one C_k and we are done.

For the claim, using $(1+x)^n\ge 1+nx$ from the binomial theorem, we have the equivalent inequality

$$\left(\frac{m}{m-5000}\right)^{5000}=\left(1+\frac{5000}{m-5000}\right)^{5000}$$
$$>\left(1+\frac{1}{200}\right)^{200\times 25}>2^{25}>(10^3)^{2.5}>m.$$

Other commended solvers: **F5D** (Carmel Alison Lam Foundation Secondary School).

Problem 410. (*Due to Titu ZVONARU and Neculai STANCIU, Romania*) Prove that for all positive real x,y,z,

$$\sum_{cyc}(x+y)\sqrt{(x+z)(y+z)}\ge 4(xy+yz+zx)$$
$$+\frac{xy+yz+zx}{3(x^2+y^2+z^2)}((x-y)^2+(y-z)^2+(z-x)^2).$$

Here $\sum_{cyc} f(x,y,z)=f(x,y,z)+f(y,z,x)+f(z,x,y).$

Solution of Proposers.

Observe that $4(xy+yz+zx)$ is the cyclic sum of $x(y+z)+y(x+z)$. Now

$$\begin{aligned}&(x+y)\sqrt{(x+z)(y+z)}-x(y+z)-y(x+z)\\&=x\sqrt{y+z}(\sqrt{x+z}-\sqrt{y+z})\\&\quad+y\sqrt{x+z}(\sqrt{y+z}-\sqrt{x+z})\\&=(x\sqrt{y+z}-y\sqrt{x+z})(\sqrt{x+z}-\sqrt{y+z})\\&=\frac{(x^2(y+z)-y^2(x+z))}{(x\sqrt{y+z}+y\sqrt{x+z})}\frac{((x+z)-(y+z))}{(\sqrt{x+z}+\sqrt{y+z})}\\&=\frac{(xy+yz+zx)(x-y)^2}{(x\sqrt{y+z}+y\sqrt{x+z})(\sqrt{x+z}+\sqrt{y+z})}.\end{aligned}$$

By the *AM-GM* inequality, we have $(x+y)^2 \le 2(x^2+y^2)$ and $xy+yz+zx \le x^2+y^2+z^2$. Using these, we get

$$\begin{aligned}&(x\sqrt{y+z}+y\sqrt{x+z})(\sqrt{x+z}+\sqrt{y+z})\\&=(x+y)\sqrt{(x+z)(y+z)}+x(y+z)+y(x+z)\\&\le(x+y)\frac{x+y+2z}{2}+2xy+yz+zx\\&=\frac{(x+y)^2}{2}+2(xy+yz+zx)\\&\le x^2+y^2+2(x^2+y^2+z^2)\\&\le 3(x^2+y^2+z^2).\end{aligned}$$

So it follows that

$$\begin{aligned}&(x+y)\sqrt{(x+z)(y+z)}-2xy-yz-zx\\&\ge\frac{xy+yz+zx}{3(x^2+y^2+z^2)}(x-y)^2.\end{aligned}$$

Rotating x,y,z to y,z,x to z,x,y, we get two other similar inequalities. Adding the three inequalities, we will get the desired inequality. Equality holds if and only if $x=y=z$.

Other commended solvers: **Paolo PERFETTI** (Math Dept, Università degli studi di Tor Vergata Roma, via della ricerca scientifica, Roma, Italy).

Comment: The proposers mention that this is a refinement of problem 2 of the 2012 Balkan Math Olympiad.

Olympiad Corner

(continued from page 1)

Problem 4. Let $x_1, x_2, x_3, \ldots$ be the sequence defined by the following recurrence: $x_1=4$ and for $n\ge 1$,

$$x_{n+1}=x_1x_2x_3\cdots x_n+5.$$

(The first few terms of the sequence are then $x_1=4$, $x_2=4+5=9$, $x_3=4\cdot 9+5=41$, …) Find all pairs $\{a,b\}$ of positive integers such that x_ax_b is a perfect square.

Problem 5. Let $ABCD$ be a square. Find the locus of points P in the plane, different from A, B, C, D such that

$$\angle APB+\angle CPD=180^\circ.$$

Problem 6. Determine all pairs $\{a,b\}$ of positive integers with the following property: for any possible coloring of the set of all positive integers with two colors A and B, there exist either two positive integers colored by A with difference a or two positive integers colored by B with difference b.

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(continued from page 2)

Now let r_i's be the multiplicities of the a_i's as zeros of $P(z)$, then the sum of the r_i's is m. By the observation above, the multiplicity of a_i as zeros of $P'(z)$ is r_i-1 and these multiplicities sum to $m-s$. Similarly, the sum of the multiplicities of the b_i's as zeros of $P'(z)=(P+1)'(z)$ is $m-t$. So

$$(m-s)+(m-t)\le \deg P'(z)<m.$$

Hence $s+t>m$. However, $a_1, a_2,\ldots,a_s, b_1, b_2, \ldots, b_t$ are zeros of $P(z)-Q(z)$ with degree at most m. So, $P(z)\equiv Q(z)$.

The interested readers are highly encouraged to browse the following books for more problems of the Putnam Exam.

A. M. Gleason, R. E. Greenwood and L. M. Kelly, The William Lowell Putnam Mathematical Competition Problems and Solutions: 1938-1964, MAA, USA, 1980.

G. L. Alexanderson, L. F. Klosinski and L. C. Larson, The William Lowell Putnam Mathematical Competition Problems and Solutions: 1965-1984, MAA, USA, 1985.

K. S. Kedlaya, B. Poonen and R. Vakil, The William Lowell Putnam Mathematical Competition 1985-2000 Problems, Solutions and Commentary, MAA, USA, 2002.