Mathematical Excalibur

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Olympiad Corner

Below are the problems of the 2013 International Mathematical Olympiad.

Problem 1. Prove that for any pair of positive integers k and n, there exist k positive integers $m_1, m_2, \ldots, m_k$ (not necessarily different) such that

$$1+\frac{2^k-1}{n}=\left(1+\frac{1}{m_1}\right)\left(1+\frac{1}{m_2}\right)\cdots\left(1+\frac{1}{m_k}\right).$$

Problem 2. A configuration of 4027 points in the plane is called Colombian if it consists of 2013 red points and 2014 blue points, and no three of the points of the configuration are collinear. By drawing some lines, the plane is divided into several regions. An arrangement of lines is good for a Colombian configuration if the following two conditions are satisfied:

• no line passes through any point of the configuration;
• no region contains points of both colors.

Find the least value of k such that for any Colombian configuration of 4027 points, there is a good arrangement of k lines.

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Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***November 8, 2013***.

For individual subscription for the next five issues for the 09-10 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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IMO 2013 – Leader Report (I)

Leung Tat-Wing

The 54th International Mathematical Olympiad (IMO) was held in Santa Marta, Colombia from July 18th to July 28th, 2013. It took me 40 hours of flight and waiting time to travel from Hong Kong to Amsterdam, then to Panama City, and then to Barranquilla, Colombia (where the leaders stayed before they met the contestants in Santa Marta after two days of 4½-hour contests held on the mornings of 23rd and 24th of July). Tired and exhausted, I were picked up in the airport of Barranquilla and delivered to Hotel El Prada. We managed to settle down and be prepared for the next two days' Jury meetings. Our team arrived at Santa Marta, three days later, safe and intact, luckily. The next day they still had to travel two hours from Santa Marta to Barranquilla, participating in another opening ceremony, then another two hours back to Santa Marta. It was tough for them. Accommodation was fine though. Contestants stayed in a nice seaside resort hotel (Iratoma), while leaders stayed in a hotel in Barranquilla. They would join the contestants after the two day contests.

Jury meetings were chaired by Maria Losada, a long time veteran of IMO activities. She was very experienced and chaired the meetings well. Interesting to note, she kept on reminding us (leaders) that we should try to form the best possible paper, a paper that can provide intellectual challenge to contestants, that has some aesthetic sense and that allows every contestant to achieve the most. We were also supposed to work out as many possible solutions as possible. We should be able to tell whether a problem is easy, medium and/or hard. Really sometimes I did not know how the goals may be attained or even verified. She also reminded us ethically we should keep the problems with strict security, not to disclose any information to any contestant beforehand, etc. Indeed the Jury meetings were very educational.

After the two days' contests students enjoyed a break. Leaders and deputy leaders had to check the solutions of the contestants, discussed or argued with coordinators and sorted out how many points should be award to contestants. (This process is called coordination). Luckily this year many coordinators were again very experienced. Many of them are old time leaders from Europe and are experienced problem solvers. They were able to discern mistakes made by the contestants (trivial, small or big) and were able to award points accurately. Personally I recognized many of them and I think I have known many of them for at least more than 10 years. That is why little trouble was observed during the coordination process.

The awards (closing) ceremony was held near a historical site, 45 minute drive from the hotel. We were delivered to the site around 7:00 pm. Then the ceremony lasted for more than two hours. Participants were than sent back to the resort for the banquet. That night was surely hectic. The next day we started our trip home. When we arrived at Bogota, we found that the flight from Bogota to Paris was overbooked. Eventually two of us (deputy leader and a member of the team) had to take another flight from Bogota to Frankfurt, then back to Hong Kong, about 10 hours late. Air France is famous (notorious) in terms of scheduling, here is another example. All in all, we did not get delayed too much and we eventually returned home safely. Lucky! Lucky!

Talking about organization of the event, personally I have no problem with the Jury meeting and/or coordination. Accommodation was very nice. However anything concerned a coach (transportation) was simply not good enough. Say, what is the point of waiting for several hours for a bus, then

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visit an old town or take a short walk for less than an hour, and then heading back? ? I do not mean to blame the host country. Indeed I want only to illustrate the point that it is such a gigantic and complicated task to host an IMO!

Our team brought home 1 silver and 5 bronze medals. Among 97 teams, we ranked 31. I cannot say that our team did badly. Indeed all our team members managed to get medals, indicating they achieved certain standard. However in these few years, we trailed behind teams like Singapore, Canada, Australia and other teams, not to say the even stronger teams such as China, USA, Korea and Russia, etc. Do we want to do better? Can we recruit better team members? Can we afford time and energy to do that? We have to think about these problems. I can identify some weak points for our team. For example, our team members simply don't like to do geometry and/or combinatorics problems. Our team members usually get stuck in harder problems, presentations and other things. Or perhaps our team members are too much occupied also by other contests? I know for sure IMO team members of teams such as USA, Australia and Canada would not be allowed to compete in other contests such as IOI or IPhO in the same year. Another suggestion is that we do not train our team enough, we have no intensive camp before IMO (compared with China, USA or UK), and perhaps we should start an intensive camp that will also used as a selection criterion of our team. This idea comes from none other than our old team members! We should pause to think about all these for a while, I suppose.

On the other hand, in this IMO, we confirmed that we will host IMO2016, so in 2016, IMO will be held in Hong Kong. Now we just have to do it, and do it right.

I shall discuss the problems of this IMO. First let us see how they were selected. Indeed the host country (Problem Selection Committee) shortlisted about 30 problems from hundred or so problems submitted by various countries. In the last few years, the Jury first chose an easy pair (problem 1 and 4), then a hard pair (problem 3 and 6), then a medium pair (problem 2 and 5). The 6 selected problems will be then juggled to form the papers. However this year, it was proposed (and accepted) 4 easy problems in algebra, combinatorics, geometry and number theory were selected. Likewise 4 medium problems again from the different topics were selected. Then two easy problems were selected from the 4 easy problems, say problems of algebra and conbinatorics were selected. The medium problems of other topics (geometry and number theory) were automatically selected as the medium pair. The idea is to guarantee problems of all topics be selected either as an easy problem or a medium problem. After that it doesn't matter what problems were selected as the hard pair. However, perhaps the end result was not as ideal as we wanted. Eventually in this IMO there are two synthetic geometry problems (Problem 3 and 4). Problem 2, which was supposed to be a combinatorics problem, is actually a problem of combinatorial geometry. Problem 6, which is a combinatorics problem, also has some geometry favor. Problem 1, which was supposed to be a number theory problem, is more like an algebra problem (no prime numbers, no factorization of integers, merely algebraic manipulation and some induction). And finally of course problem 5 is a problem of functional inequalities. So this paper is very much skewed to geometry and with no number theory. Can we say it is balanced? Really at the very beginning, the problems selected were not quite balanced. The problem selection committee suggested there were no easy combinatorics problems and no hard geometry problems! In short, Jury members tended to select problems that demand "ad hoc" considerations, no need to resort to more advanced techniques and/or theorems.

(For the statement of the problems, please see the Olympiad Corner on page 1-*Ed.*)

Problem 1: Problem 1 and 4 (easy pair) turned out to be too easy. Many strong teams get full score in these two problems. For $k=1$, we have

$$1+\frac{2^1-1}{n}=1+\frac{1}{n},$$

and it is already of the required form. Hence it is natural to solve the problem using some kind of induction procedure. Essentially all of us did the problem using iterations. One of our team members did the problem as follows. Denote the statement that $1+(2^k-1)/n$ is of the form

$$\left(1+\frac{1}{m_1}\right)\left(1+\frac{1}{m_2}\right)\cdots\left(1+\frac{1}{m_k}\right)$$

by $S(n,k)$. Note that

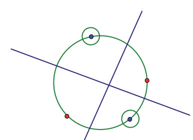
$$1+\frac{2^{k+1}-1}{2n}=\left(1+\frac{1}{2n+2^{k+1}-2}\right)\left(1+\frac{2^k-1}{n}\right),$$

hence if $S(n,k)$ is valid, so is $S(2n,k+1)$. Likewise

$$1+\frac{2^{k+1}-1}{2n-1}=\left(1+\frac{1}{2n-1}\right)\left(1+\frac{2^k-1}{n}\right).$$

Hence if $S(n,k)$ is valid, so is $S(2n-1,k+1)$. Clearly the cases $S(n,1)$ or $S(1,k)$ are valid. Hence by reducing the cases $S(2n,k)$ to $S(n,k-1)$, or $S(2n-1,k)$ to $S(n,k-1)$, (odd or even cases), one can always obtain the cases $S(p,1)$ or $S(1,q)$, and we are done.

Problem 2: All our members guessed the correct answer. The trouble is how to present a proof that is complete (no missing cases). Jury members also worried students didn't realize the minimum value of k should work for all possible configurations. Thus they defined the term "Colombian". (Another definition is the "beautiful" labeling in problem 6. In my opinion it was quite unnecessary.) First we show $k \geq 2013$. Indeed we mark 2013 red points and 2013 blue points alternately on a circle, (and another blue point elsewhere), then there are 4026 arcs formed. All these arcs have two endpoints of different colors and there must be a line passing through an arc to separate the two points, also each line passing through an arc will meet another arc only once, so we see at least 4026/2=2013 lines are needed.



A case of 2 red points and 3 blue points

Now we have to show $k=2013$ is indeed enough. The official solution goes as follows. First if there are two points of the same color, say A and B, then one can draw two lines parallel to AB, and are sufficiently close and there are only two points between these lines, namely A and B. This statement is intuitively clear. Draw the convex hull P of the points, and there are two cases.

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Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***November 8, 2013.***

Problem 426. Real numbers a, b, x, y satisfy the property that for all positive integers n, $ax^n+by^n=1+2^{n+1}$. Determine (with proof) the value of x^a+y^b.

Problem 427. Determine all (m,n,k), where m, n, k are integers greater than 1, such that $1!+2!+\cdots+m!=n^k$.

Problem 428. Let $A_1A_2A_3A_4$ be a convex quadrilateral. Prove that the nine point circles of $\Delta A_1A_2A_3$, $\Delta A_2A_3A_4$, $\Delta A_3A_4A_1$ and $\Delta A_4A_1A_2$ pass through a common point.

Problem 429. Inside ΔABC, there is a point P such that $\angle APB=\angle BPC=\angle CPA$. Let $PA=u$, $PB=v$, $PC=w$, $BC=a$, $CA=b$ and $AB=c$. Prove that

$$(u+v+w)^2 \le ab+bc+ca - \left(\sqrt{a(b+c-a)}-\sqrt{b(c+a-b)}\right)^2.$$

Problem 430. Prove that among any $2n+2$ people, there exist two of them, say A and B, such that there exist n of the remaining $2n$ people, each either knows both A and B or does not know A nor B. Here, *x knows y* does not necessarily imply *y knows x*.

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Solutions

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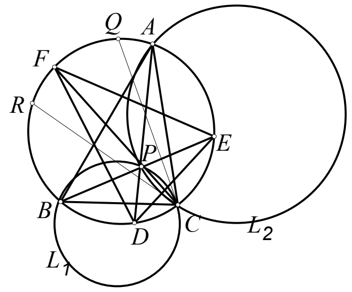
Problem 421. For every acute triangle ABC, prove that there exists a point P inside the circumcircle ω of ΔABC such that if rays AP, BP, CP intersect ω at D, E, F, then $DE: EF: FD = 4:5:6$.

Solution. **Jon GLIMMS** (Vancouver, Canada), **Jeffrey HUI Pak Nam** (La Salle College, Form 6) and **William PENG**.

For such a point P, let us apply the exterior angle theorem to ΔABP and ΔACP. Then we have

$$\angle BPC = \angle BAC + \angle ABE + \angle ACF = \angle BAC + \angle FDE.$$

Similarly, $\angle CPA = \angle CBA + \angle DEF$.



To get such a point P, we first draw ΔXYZ with $XY=4$, $YZ=5$ and $ZX=6$. Let $\alpha=\angle ZXY$ and $\beta=\angle XYZ$. Next we consider the locus L_1 of point P such that $\angle BPC = \angle BAC+\alpha$, which is a circle through B and C. Also, let L_2 be the locus of point P such that $\angle CPA = \angle CBA+\beta$, which is a circle through C and A.

Let the tangents to L_1 and L_2 at C intersect ω at Q and R. Then

$$\begin{aligned}&\angle QCB+\angle RCA\\ &= 180^\circ-(\angle BAC+\alpha)+180^\circ-(\angle CBA+\beta)\\ &= \angle ACB+\angle YZX > \angle ACB.\end{aligned}$$

This implies L_1 and L_2 intersect at a point P inside ω. Define D, E, F as in the statement of the problem. From the last two paragraphs, we get $\angle ZXY=\alpha=\angle FDE$ and $\angle XYZ=\beta=\angle DEF$. These imply ΔDEF and ΔXYZ are similar. Therefore, $DE: EF: FD = 4:5:6$.

Problem 422. Real numbers $a_1, a_2, a_3, \ldots$ satisfy the relations

$$a_{n+1}a_n+3a_{n+1}+a_n+4=0$$

and $a_{2013}\le a_n$ for all positive integer n. Determine (with proof) all the possible values of a_1.

Solution. **CHEUNG Wai Lam** (Queen Elizabeth School, Form 4), **Jon GLIMMS** (Vancouver, Canada), **William PENG** and **TAM Pok Man** (Sing Yin Secondary School, Form 6).

The recurrence relation can be written as $(a_{n+1}+2)(a_n+2)=(a_n+2)-(a_{n+1}+2)$. If $a_i=-2$ for some i, then all $a_n=-2$ by induction. So $a_1=-2$ is a possible value. Suppose no $a_i=-2$. Then

$$\frac{1}{a_{n+1}+2}=1+\frac{1}{a_n+2}.$$

Letting $b_n=1/(a_n+2)$, we easily get $b_n=n-1+b_1\neq 0$ for all positive integer n. Then $b_1\neq 0,-1,-2,\ldots$ and $a_n=-2+1/(n-1+b_1)$. Now for positive integer n, a_n is least when $n-1+b_1<0$ and nearest 0, i.e.

$$n-1+b_1<0<n+b_1.$$

Setting $n=2013$ and $b_1=1/(a_1+2)$, we can solve the inequality to get

$$-\frac{4025}{2012}<a_1<-\frac{4027}{2013}.$$

Other commended solvers: **Jeffrey HUI Pak Nam** (La Salle College, Form 6) and **LKL Excalibur** (Madam Lau Kam Lung Secondary School of MFBM).

Problem 423. Determine (with proof) the largest positive integer m such that a $m\times m$ square can be divided into seven rectangles with no two having any common interior point and the lengths and widths of these rectangles form the sequence 1,2,3,4,5,6,7,8,9,10, 11,12,13,14.

Solution. **Jon GLIMMS** (Vancouver, Canada), **William PENG** and **ZOLBAYAR Shagdar** (Orchlon International School, Ulaanbaatar, Mongolia).

Let $a_1, a_2, a_3, a_4, \ldots, a_{2n-1}, a_{2n}$ be a permutation of $1, 2, 3, 4, \ldots, 2n-1, 2n$. We claim the maximum of $a_1a_2+a_3a_4+\cdots+a_{2n-1}a_{2n}$ is $S_n=1\times 2+3\times 4+\cdots+(2n-1)\times 2n$. The cases $n=1$ or 2 can be checked. Suppose cases 1 to n are true. For the case $n+1$, if $(2n+1)(2n+2)$ is one of the term, then we can switch it with the last term and apply the case n to get

$$a_1a_2+a_3a_4+\cdots+a_{2n-1}a_{2n}+(2n+1)(2n+2) \le S_n+(2n+1)\times(2n+2)=S_{n+1}.$$

Otherwise, $2n+1$ and $2n+2$ are in different terms. We can switch terms so that $a_{2n-1}=2n+1$ and $a_{2n+1}=2n+2$. If we try switching $(2n+1)a_{2n}+(2n+2)a_{2n+2}$ to $a_{2n}a_{2n+2}+(2n+1)(2n+2)$, then since a_{2n} and a_{2n+2} are at most $2n$, we have

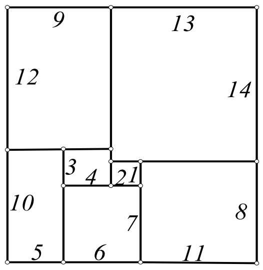
$$[(2n+2)-a_{2n}][(2n+1)-a_{2n+2}]>0.$$

Expanding, we see

$$\begin{aligned}&a_{2n}a_{2n+2}+(2n+1)(2n+2)\\ &>(2n+1)a_{2n}+(2n+2)a_{2n+2}\\ &= a_{2n-1}a_{2n}+a_{2n+1}a_{2n+2}.\end{aligned}$$

Adding $a_1a_2+a_3a_4+\cdots+a_{2n-3}a_{2n-2}$ and using case $n-1$, we see S_{n+1} again is the maximum.

For the problem, the claim implies $m^2\le S_7=1\times 2+3\times 4+\cdots+13\times 14=504$. Then $m\le 22$. To finish, we show a 22×22 square which can be so divided.



Other commended solvers: **LKL Excalibur** (Madam Lau Kam Lung Secondary School of MFBM).

Problem 424. (*Due to Prof. Marcel Chirita, Bucuresti, Romania*) In ΔABC, let a=BC, b=CA, c=AB and R be the circumradius of ΔABC. Prove that

$$\max(a^2+bc,b^2+ca,c^2+ab)\geq\frac{2\sqrt{3}abc}{3R}.$$

Solution. **Jeffrey HUI Pak Nam** (La Salle College, Form 6), **TAM Pok Man** (Sing Yin Secondary School, Form 6), **Alex TUNG Kam Chuen** (La Salle College), **ZOLBAYAR Shagdar** (Orchlon International School, Ulaanbaatar, Mongolia) and **Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania).

By the extended sine law, $c/\sin C = 2R$. Let $[ABC]$ denote the area of ΔABC. Then $[ABC]$ = ½$ab\sin C = abc/(4R)$. So $ab = 2[ABC]/\sin C$. Using these below, we have

$$\begin{aligned}&3\max(a^2+bc,b^2+ca,c^2+ab)\\&\geq a^2+bc+b^2+ca+c^2+ab\\&\geq 2(ab+bc+ca)\\&=4[ABC]\left(\frac{1}{\sin C}+\frac{1}{\sin A}+\frac{1}{\sin B}\right)\\&\geq 4[ABC]\frac{3}{\sin((A+B+C)/3)}\\&=\frac{2\sqrt{3}abc}{R},\end{aligned}$$

where the second inequality is by expanding $(a-b)^2+(b-c)^2+(c-a)^2\geq 0$ and the third inequality is by applying Jensen's inequality to $f(x)=1/\sin x$.

Other commended solvers: **Ioan Viorel CODREANU** (Secondary School Satulung, Maramures, Romania) and **KWOK Man Yi** (Baptist Lui Ming Choi Secondary School, Form 2).

Problem 425. Let p be a prime number greater than 10. Prove that there exist distinct positive integers a_1, a_2, …, a_n such that $n \leq (p+1)/4$ and

$$\frac{(p-a_1)(p-a_2)\cdots(p-a_n)}{a_1a_2\cdots a_n}$$

is a positive integral power of 2.

Solution. **Jeffrey HUI Pak Nam** (La Salle College, Form 6), **LKL Excalibur** (Madam Lau Kam Lung Secondary School of MFBM), **Alex TUNG Kam Chuen** (La Salle College) and **ZOLBAYAR Shagdar** (Orchlon International School, Ulaanbaatar, Mongolia).

More generally, we prove this is true for all odd integers $p\geq 3$. Let

$$X=\frac{(p-a_1)(p-a_2)\cdots(p-a_n)}{a_1a_2\cdots a_n}$$

If $p\equiv 1$ (mod 4), then let $n=(p-1)/4$ and for $i=1,2,\ldots,n$, let $a_i=2i-1$. We have

$$\begin{aligned}X&=\frac{4n(4n-2)\cdots(2n+2)}{1\cdot 3\cdots(2n-1)}\\&=\frac{4n(4n-2)\cdots(2n+2)}{1\cdot 3\cdots(2n-1)}\times\frac{2\cdot 4\cdots(2n)}{2\cdot 4\cdots(2n)}\\&=2^{2n}.\end{aligned}$$

If $p\equiv 3$ (mod 4), then let $n=(p+1)/4$ and for $i=1,2,\ldots,n$, let $a_i=2i-1$. We have

$$\begin{aligned}X&=\frac{(4n-2)(4n-4)\cdots(2n)}{1\cdot 3\cdots(2n-1)}\\&=\frac{(4n-2)(4n-4)\cdots(2n)}{1\cdot 3\cdots(2n-1)}\times\frac{2\cdot 4\cdots(2n-2)}{2\cdot 4\cdots(2n-2)}\\&=2^{2n-1}.\end{aligned}$$

Olympiad Corner

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Problem 3. Let the excircle of triangle ABC opposite the vertex A be tangent to the side BC at the point A_1. Define the points B_1 on CA and C_1 on AB analogously, using the excircles opposite B and C, respectively. Suppose that the circum-centre of triangle $A_1B_1C_1$lies on the circumcircle of triangle ABC. Prove that triangle ABC is right-angled.

Problem 4. Let ABC be an acute-angled triangle with orthocenter H, and let W be a point on the side BC, lying strictly between B and C. The points M and N are the feet of the altitudes from B and C, respectively. Denote by ω_1 the circumcircle of BWN, and let X be the point on ω_1 such that WX is a diameter of ω_1. Analogously, denote by ω_2 the circumcircle of CWM, and let Y be the point on ω_2 such that WY is a diameter of ω_2. Prove that X, Y, H are collinear.

Problem 5. Let $\mathbb{Q}_{>0}$ be the set of positive rational numbers. Let f: $\mathbb{Q}_{>0}\rightarrow\mathbb{R}$ be a function satisfying the following three conditions:

(i) for all $x,y\in\mathbb{Q}_{>0}$, we have $f(x)f(y)\geq f(xy)$;
(ii) for all $x,y\in\mathbb{Q}_{>0}$, we have $f(x+y)\geq f(x)+f(y)$;
(iii) there exists a rational number $a>1$ such that $f(a)=a$.

Prove that $f(x)=x$ for all $x\in\mathbb{Q}_{>0}$.

Problem 6. Let $n\geq 3$ be an integer, and consider a circle with n+1 equally spaced points marked on it. Consider all labellings of these points with the numbers 0,1,…, n such that each label is used exactly once; two such labellings are considered to be the same if one can be obtained from the other by a rotation of the circle. A labelling is called *beautiful* if, for any four labels $a<b<c<d$ with $a+d=b+c$, the chord joining the points labelled a and d does not intersect the chord joining the points labelled b and c.

Let M be the number of beautiful labellings, and let N be the number of ordered pairs (x,y) of positive integers such that $x+y\leq n$ and $gcd(x,y)$=1. Prove that $M=N+1$.

IMO 2013–Leader Report (I)

(continued from page 2)

Case 1. If there is a red point A on the convex hull P, we can draw a line separating A draw all other points. Then we pair up the remaining 2012 red points into 1006 pairs, and as remarked, draw 1006 pairs of parallel lines (2012 lines), separating each pair of red points from all other points. Thus 2012+1=2013 lines are needed.

Case 2. All vertices of the convex hull P are blue. Take any pair of consecutive blue points A and B, separating them from all other points by a line (one line) parallel to AB. Then pair up the remaining 2012 blue points into 1006 pairs as before, separating each pair from all other points by 1006 pairs of parallel lines (2012 lines). Thus again 2013 lines are used.

(To be continued)