Mathematical Excalibur

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Olympiad Corner

Below are the problems of the 2014 International Math Olympiad on July 8 and 9, 2014.

Problem 1. Let $a_0 < a_1 < a_2 < \cdots$ be an infinite sequence of positive integers. Prove that there exists a unique integer $n \geq 1$ such that

$$a_n < \frac{a_0 + a_1 + \ldots + a_n}{n} \leq a_{n+1}.$$

Problem 2. Let $n \geq 2$ be an integer. Consider a $n \times n$ chessboard consisting of n^2 unit squares. A configuration of n rooks on this board is *peaceful* if every row and every column contains exactly one rook. Find the greatest positive integer k such that, for each peaceful configuration of n rooks, there is a $k \times k$ square which does not contain a rook on any of its k^2 unit squares.

Problem 3. Convex quadrilateral $ABCD$ has $\angle ABC = \angle CDA = 90°$. Point H is the foot of the perpendicular from A to BD. Points S and T lie on sides AB and AD, respectively, such that H lies inside triangle SCT and $\angle CHS - \angle CSB = 90°$, $\angle THC - \angle DTC = 90°$. Prove that line BD is tangent to the circumcircle of triangle TSH.

(continued on page 4)

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The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***October 12, 2014***.

For individual subscription for the next five issues for the 14-15 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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IMO2014 and Beyond

Leung Tat-Wing

I write this article with three goals in mind: (1) to report on IMO 2014; (2) to give some idea how we can further train our team members and (3) finally and hopefully provide us some help of how to organize IMO 2016.

Itinerary The 55th International Mathematical Olympiad was held in Cape Town, South Africa from 3 July to 13 July, 2014. It took us 13 hours flying from Hong Kong to Johannesburg, waiting for a couple of hours, then another 2 hours' flight to Cape Town. Surely when compared with Argentina and Colombia, it was a much easier trip. Because we have to host IMO 2016, this year several observers (with leaders or deputy leaders) came with us. We have gathered a lot of information in this trip, which will help us tremendously in our preparation. This IMO was held when world cup matches were going on, and we were lucky that we still managed to watch several games, and at better times (6 pm or 10 pm). We missed only the final game, Germany vs Argentina, when we were exactly in our return flight, and I managed to get the result only when we got off the plane.

Weather in South Africa was nice. It was winter, and usually 20˚*C* during the day time and about 10˚*C* during the night. If it was raining, then it got a bit cooler. We first stayed in a hotel, right below a mountain, which I believe belongs to the Table Mountain range. The view, if I may say, is simply majestic. The city structure looks nice. It looks like a decent English town. The hotel is pretty normal and we stayed there for 6 days. Then we moved to the University of Cape Town (UCT) and stayed with the students. Our students arrived Cape Town three days after us, and they were stationed in dormitories of the University all the time. Though accommodation and food were not as good as in the hotel, I believe I can bear it. Only thing is, every entrance of a dormitory in the University is equipped with heavy iron gate and is watched by a security guard, which I found it a bit scary. This reminds me of the security issue in South Africa. Of course, it is a country with high unemployment rate (25%), high Gini index (6.3), and there are racial problems and other things.

Leaders spent three days to select the 6 problems from a shortlist of 30 problems, then refined the wordings and wrote the English version and other official versions. They discussed the marking schemes proposed by the Problem Group and coordinators, and approved the marking schemes. The students then arrived, and the next day leaders and contestants together participated in the Opening Ceremony, with leaders and contestants still separated so that they could not communicate during the Ceremony. Students then wrote the two 4.5 hour contests on the mornings of the next two days, while leaders had the time to do a bit of sight-seeing and the like. After the two contests, leaders were then moved to the University. After the two contests, students were free, then they had the chances to see further things. I knew my students got the chances to see the Cape of Good Hope, and took a cable-car to the top of the Table Mountain. Because I, as a leader, had to participate in the coordination process, had to miss both events. Coordination is a process in which the leader and deputy leader, plus two coordinators of the host county, come together to decide how many points are to be awarded to a particular problem submission of a student. Given that we had nice and detailed marking schemes, and the coordinators are generally very experienced, we encountered little trouble in deciding points. Then we had a final day excursion and the Closing Ceremony, on the same day. The next day we headed home.

(continued on page 2)

Problem Selection By the end of March 2014, the host country (South Africa) received 141 problem proposals from 43 countries. I don't know when the problem selection group started to work, but surely, it took them more than a month to select 30 shortlisted problems. Furthermore they modified them, supplied alternative solutions and comments, and prepared a booklet for Jury members to consider. Incidentally the problem group was composed of six international members. I was told, they managed to do something before they formally met in South Africa, and also after they left. The selected problems are of course of high quality. However I cannot say I am totally happy with the selection. Indeed I think the problems selected were rather skewed, there were 6 algebra problems, 9 combinatorics problems, 7 geometry problems and 8 number theory problems. Some algebra problems and number theory problems in fact have quite a bit of combinatorics flavor. Moreover, several hard combinatorics problems were simply too hard. The Jury worried very much if one of them was selected, no one would be able to solve it.

When the Jury members met, it was suggested that first we selected 2 out of 4 easy problems, with one problem from each of the topics algebra, combinatorics, geometry and number theory. Again 4 medium problems from the four topics were selected. When the two easy problems were chosen, the two medium problems from the other two categories were automatically selected. Then the hard problems (problem 3 and 6) were chosen arbitrary. The suggestion was adapted. Finally two easy problems of algebra and geometry were selected, and so were two medium problems on combinatorics and number theory. However I am not sure if the easy algebra problem is really an algebra problem, of course it involves some algebraic manipulations, but I think the result very much depends on the discrete structure of integers. It is not an inequality problem nor a functional equation problem anyway. The medium combinatorics problem concerns "holes" within a distribution of rooks in a checker board. The number theory problem again is not really number theory. There is no need for congruence or other number theory things. It basically involves merging or grouping of coins of different values, so it is more like a combinatorics problem. Finally a hard geometry problem and a hard combinatorics problem were selected. It is quite certain in these days two geometry problems are to be selected. Those are the problems contestants cannot easily quote high power theorems or use more specialized techniques. However due to the preference of leaders, in general there is no 3D geometry problems. In this contest, three problems are really of combinatorial flavor. So I think the new method of choosing problems does not guarantee a good distribution of problems. Concerning Problem 6, I have to say I don't like it and I have something more to say, but let's wait.

Coordination The process of coordination was done seriously and rigorously. After the six problems were selected by the Jury (composed of leaders from 101 countries), I believed the chief coordinator then instructed the six problem captains to write up detailed marking schemes, incorporating various solutions supplied by leaders. Each problem captain was responsible for only one specific problem, he knew essentially everything concerning that problem, originality, various solutions, etc. The marking schemes were then formally approved by the Jury. After the two contests, they scanned all the answers scripts of the students. We leaders then got back answer scripts of our students and tried to allocate suitable points for our contestants. A minor mishap was, the scanner could not scan marks of correcting fluid, and thus I was asked several times why were there correcting fluids found on my students' scripts. Luckily of course was, we did not add anything new.

Detailed schedules were given to us, so leaders knew when and where to go. The process of coordination was done formally within two days. I believe because of language issue and other reasons, coordinators were recruited internationally. They were composed of old time leaders, experienced problem solvers etc. Some we met more than 10 years' ago. They were very experienced and were able to spot errors made by students, whether an error is trivial (no point deducted), minor (1 or 2 points deducted) or major (at least 4 to 5 points deducted). I thank my deputy leader, Ching Tak Wing, our old-time trainee and IMO gold medalist, who helped us to go through the many convoluted arguments of our members. We were able to discuss (or argue) with our coordinators, to convince them that our members did do somethings of certain parts of a problem or so, and thus got few extra points. On the whole, I think our papers were fairly marked and the process of coordination was done well.

Results of our Students We got 4 silvers and 2 bronzes, ranked (unofficially) 18 out of 101 countries. Indeed 3 of our 4 silver medalists solved essentially 4 problems and the other silver medalist got 3 problems correct. Also our 2 bronze medalists essentially got 3 problems correct and were real close to silver. I don't think I can blame our students for not trying hard. Indeed they picked up a lot of techniques in these few years, learned (and are still learning) to face a problem fairly and squarely. I observed when they were doing problem 2 and 5 (medium problems), they had generated the habit of gathering data and information, using various grouping and simplification methods, induction and other techniques to solve them, even though their approaches were later found to be a bit clumsy and there were a few gaps (thus few points deducted). Because a lot of time were spent on problems 2 and 5, no one could do problems 3 and 6, thus no one could tackle the hard problems. Four of our six members were old-timers, and they are leaving us for universities. I think we need 2 to 3 years to have another group of members of this caliber.

Think of this issue the other way. If we want to keep our ranking, surely several silver and bronze medals are required. If we want to be ranked within the top 10 countries, for instance, we need two or three gold medals, and some silvers and bronzes. It depends on really what we want. For me, I think it is fine if we can produce a bunch of well-trained students, good and brave to face problems and are ready to pick up necessary skills and other things in the process. Getting a gold medal in an IMO is a process, is part of a training process, but not necessarily is an end, (not like getting a world cup).

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong*. The deadline for sending solutions is ***October 12, 2014.***

Problem 446. If real numbers a and b satisfy $3^a+13^b=17^a$ and $5^a+7^b=11^b$, then prove that $a < b$.

Problem 447. For real numbers x, y, z, find all possible values of $\sin(x+y) + \sin(y+z) + \sin(z+x)$ if

$$\frac{\cos x+\cos y+\cos z}{\cos(x+y+z)}=\frac{\sin x+\sin y+\sin z}{\sin(x+y+z)}.$$

Problem 448. Prove that if s,t,u,v are integers such that $s^2-2t^2+5u^2-3v^2=2tv$, then $s = t = u = v = 0$.

Problem 449. Determine the smallest positive integer k such that no matter how $\{1,2,3,\ldots,k\}$ are partitioned into two sets, one of the two sets must contain two distinct elements m, n such that mn is divisible by $m+n$.

Problem 450. (*Proposed by Michel BATAILLE*) Let $A_1A_2A_3$ be a triangle with no right angle and O be its circumcenter. For i = 1,2,3, let the reflection of A_i with respect to O be A_i' and the reflection of O with respect to line $A_{i+1}A_{i+2}$ be O_i (subscripts are to be taken modulo 3). Prove that the circumcenters of the triangles OO_iA_i' (i = 1,2,3) are collinear.

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Solutions

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Problem 441. There are six circles on a plane such that the center of each circle lies outside of the five other circles. Prove there is no point on the plane lying inside all six circles.

Solution. **Kaustav CHATTERJEE** (MCKV Institute of Engineering College, India), **William FUNG**, **KWOK Man Yi** (Baptist Lui Ming Choi Secondary School, S4), **Corneliu Mănescu-Avram** (Transportation High school, Ploieşti, Romania), **Math Activity Center** (Carmel Alison Lam Foundation Secondary School),

Assume there is a point P inside all six circles $C_1, C_2, \ldots, C_6$ with centers $O_1, O_2, \ldots, O_6$ and radii $r_1, r_2, \ldots, r_6$ respectively. Then $O_iP < r_i$ for i = 1,2,…,6. Connecting the six O_i to P, since the six angles about P sum to 360˚, there exists $\angle O_mPO_n \le 60°$. Then in ΔO_mPO_n, either $O_mO_n \le O_mP < r_m$ or $O_mO_n \le O_nP < r_n$. This leads to either O_n is inside C_m or O_m is inside C_n, which is a contradiction.

Problem 442. Prove that if $n > 1$ is an integer, then n^5+n+1 has at least two distinct prime divisors.

Solution. **Adnan ALI** (Atomic Energy Central School 4, Mumbai, India), **Ioan Viorel CODREANU** (Secondary School Satulung, Maramures, Romania), **Luke Minsuk KIM** (Stanford University) and **KWOK Man Yi** (Baptist Lui Ming Choi Secondary School, S4).

The case n = 2 is true as $n^5+n+1=5\times 7$. For $n\ge 3$, we have $n^5+n+1=(n^3-n^2+1)(n^2+n+1)$ and $n^3-n^2+1=(n^2+n+1)(n-2)+(n+3)$. Then $n^3-n^2+1 > n^2+n+1 > 1$. Assume n^5+n+1 is a power of some prime p. Then $n^3-n^2+1= p^s$ and $n^2+n+1=p^t$ with $s > t \ge 1$. Now

$$n+3=n^3-n^2+1-(n^2+n+1)(n-2)=p^s-p^t(n-2)$$

is a multiple of $p^t = n^2+n+1$. This leads to $n+3 \ge p^t=n^2+n+1$, i.e. $2 \ge n^2$, contradiction.

Other commended solvers: **Christian Pratama BUNAIDI** (SMA YPK Ketapang I, Indonesia), **CHAN Long Tin** (Cambridge University, Year 2), **Kaustav CHATTERJEE** (MCKV Institute of Engineering College, India), **Victorio Takahashi CHU** (Pontifícia Universidade Católica - São Paulo SP, Brazil), **Gabriel Cheuk Hung LOU**, **Corneliu Mănescu-Avram** (Transportation High school, Ploieşti, Romania), **Math Activity Center** (Carmel Alison Lam Foundation Secondary School), **NGUYEN Van Thien** (Luong The Vinh High School, Dong Nai, Viet Nam), **Milan PAVIC** (Serbia), **Mamedov SHATLYK** (School of Young Physics and Mathematics No. 21, Dashoguz, Turkmenistan), **Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania).

Problem 443. Each pair of n ($n\ge 6$) people play a game resulting in either a win or a loss, but no draw. If among every five people, there is one person beating the other four and one losing to the other four, then prove that there exists one of the n people beating all the other n–1 people.

Solution. **Jon GLIMMS** (Vancouver, Canada).

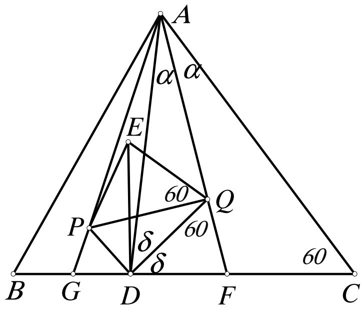
Assume no one beat all other n–1 people. Then the number of wins for each of the n people is 0,1,…, n–2. By the pigeonhole principle, there exist two people, say A and B with the same number of wins. Now, say A beat B. Due to same wins, there exists C such that A beat B, B beat C and C beat A.

Next add two other people to A, B, C. By given condition, one of these five lost to the other four. Observe that this one cannot be A, B, C, say it is D. Since $n\ge 6$, ignoring D, we can add two other people to A, B, C. Again, by given condition, one of these five lost to the other four. Observe that this one cannot be A, B, C, D, say it is E. Then none of A,B,C,D,E beat the other four, contradicting the given condition.

Other commended solvers: **Kaustav CHATTERJEE** (MCKV Institute of Engineering College, India), **KWOK Man Yi** (Baptist Lui Ming Choi Secondary School, S4) and **Math Activity Center** (Carmel Alison Lam Foundation Secondary School).

Problem 444. Let D be on side BC of equilateral triangle ABC. Let P and Q be the incenters of ΔABD and ΔACD respectively. Let E be the point so that ΔEPQ is equilateral and D, E are on opposite sides of line PQ. Prove that lines BC and DE are perpendicular.

Solution. **Jon GLIMMS** (Vancouver, Canada) and **T. W. LEE** (Alumni of New Method College).



We have $\angle QDP = \angle QDA+\angle PDA = ½(\angle CDA+\angle BDA)= 90°$. Also, $\angle QDA = \angle QDC=90°-\angle PDB$. To show $BC\perp DE$, i.e. $\angle PDE+\angle PDB=90°$, it suffices to show $\angle QDA = \angle PDE$. This is the same as showing lines AD, ED

are symmetric with respect to the angle bisector of $\angle QDP$. For convenience, we refer to this condition by saying lines AD, ED are isogonal with respect to $\angle QDP$.

This will follow from the isogonal conjugacy theorem (see comments below) if we can show that (1) lines AQ, EQ are isogonal with respect to $\angle PQD$ and (2) lines AP, EP are isogonal with respect to $\angle DPQ$. For (1), we have $\angle AQD = 180° - ½(\angle CAD + \angle CDA) = 120°$. Let lines AQ, BC meet at F. Then $\angle FQD = 180° - \angle AQD = 60° = \angle EQP$ implies (1). For (2), similarly $\angle APD = 120°$. Let lines AP, BC meet at G. Then $\angle GPD = 180° - \angle APD = 60° = \angle EPQ$ implies (2).

Comments: If we have (1) and (2), we can write down the two trigonometric forms of Ceva's theorem for points A and E with respect to $\triangle QDP$. Cancelling common factors in the two equations leads to

$$\frac{\sin\angle QDA}{\sin\angle ADP} = \frac{\sin\angle PDE}{\sin\angle EDQ}.$$

Then $\angle QDA = \angle PDE$ follows from $f(x) = \sin x / \sin(\angle QDP - x)$ is strictly increasing for $0 < x < \angle QDP$.

Other commended solvers: **CHAN Long Tin** (Cambridge University, Year 2) and **Math Activity Center** (Carmel Alison Lam Foundation Secondary School).

Problem 445. For each positive integer n, prove there exists a polynomial $p(x)$ of degree n with integer coefficients such that $p(0)$, $p(1)$, …, $p(n)$ are distinct and each is of the form $2\times2014^k+3$ for some positive integer k.

Solution. **Math Activity Center** (Carmel Alison Lam Foundation Secondary School).

Let $a = 2014$. Write $n! = n_1n_2$, where n_2 is the greatest divisor of $n!$ that is relatively prime to a. Then n_1 and a have the same prime divisors. By Euler's theorem, for $t=\varphi(n_2)$, we have $a^t \equiv 1 \pmod{n_2}$. For the polynomial

$$f(x) = \sum_{i=0}^{n} \frac{x(x-1)\cdots(x-i+1)}{i!}(a^t-1)^i$$

and $j=0,1,\ldots,n$, we have

$$f(j) = \sum_{i=0}^{j} \binom{j}{i}(a^t-1)^i = a^{tj}.$$

Let s be the maximum of the exponents appeared in the prime factorization of n_1. Then $a^s(a^t-1)/n!$ is a positive integer and $p(x)=2a^sf(x)+3$ is a polynomial of degree n with integer coefficients such that $p(j)=2a^{s+tj}+3$ for $j = 0,1,\ldots,n$.

Olympiad Corner

(*Continued from page 1*)

Problem 4. Points P and Q lie on side BC of acute-angled triangle ABC such that $\angle PAB = \angle BCA$ and $\angle CAQ = \angle ABC$. Points M and N lie on lines AP and AQ, respectively, such that P is the midpoint of AM, and Q is the midpoint of AN. Prove that lines BM and CN intersect on the circumcircle of triangle ABC.

Problem 5. For each positive integer n, the Bank of Cape Town issues coins of denomination $1/n$. Given a finite collection of such coins (of not necessarily different denominations) with total value at most 99+½, prove that it is possible to split the collection into 100 or fewer groups, such that each group has total value at most 1.

Problem 6. A set of lines in the plane is in *general position* if no two are parallel and no three pass through the same point. A set of lines in general position cuts the plane into regions, some of which have finite areas; we call these its *finite regions*. Prove that for all sufficiently large n, in any set of n lines in general position it is possible to color at least $\sqrt{n}$ of the lines blue in such a way that none of its finite regions has a completely blue boundary.

Notes: Results with $\sqrt{n}$ replaced by $c\sqrt{n}$ will be awarded points depending on the value of the constant c.

IMO2014 and Beyond

(*Continued from page 2*)

So far, about 10 Fields' medalists participated in the IMOs, but not everyone was a gold medalist (about half of them were). Even Terry Tao got bronze in his first year, then silver, then gold. Yes, of course I realize some administrators may think otherwise and have different ideas of what it means by sending a team to an IMO.

I heard many theories why we cannot produce even stronger team. Our students have to devote too much time on DSE, in particular SBA. We have no specialized schools, unlike Vietnam and Singapore. Our pool is too small, trainers are no good, training time are not enough, etc. All these are hard to rebuke (no counter-examples?), and not sure how to verify. They may well be so and so what can we do? Indeed in these few years we have strengthened our training process, more tests, asking our members to present and substantiating their views, etc. Indeed we received many suggestions from our former trainees.

We observed a few things by simply looking at the overall results. For instance, despite political trouble in the east, the Ukrainian team still did very good. They ranked 6 out of 101. The Israelites did as well as us (ranked 18). The Koreans, as usual, did very good, but not as formidable as last year. Indeed, Republic of Korea was ranked 7 and the Democratic Republic of Korea was ranked 14. During these 20 or so years, the North Koreans missed the contest altogether for 10 years, but during the times they were around, they did reasonably well. Although we were not as good as the populous countries like China (ranked 1), and USA (ranked 2). We did better than India (ranked 40) and Indonesia (ranked 30). This year we did slightly better than Thailand (ranked 22), the country to host IMO 2015. They have been good, and I was told they put a lot of money into the event and in training their team. We also did better than several traditionally strong countries such as Poland (ranked 28), Iran (ranked 21) and Bulgaria (ranked 37). Indeed Bulgaria has a long tradition of mathematical competitions, and their competition materials are often very well sought. As in the last few years, we still did not do as well as Singapore (ranked 8). However, when I looked closely at their results, I found their gold medalists were not really much better than our silver medalists and I think we can do as well? In short, it is very interesting by simply looking at the results of countries during the years, we may gather some ideas on how we should train our members in the future.

(to be continued)