Mathematical Excalibur

Volume 19, Number 2 *September 2014 – October 2014*

Olympiad Corner

Below are the problems of the 2014 Bulgarian National Math Olympiad on May 17-18, 2014.

Problem 1. *(Teodosi Vitanov, Emil Kolev)* Given is a circle k and a point A outside it. The segment BC is a diameter of k. Find the locus of the orthocenter of $\triangle ABC$, when BC is changing.

Problem 2. *(Nikolay Beluhov)* Consider a rectangular $n\times$m table where $n \geq 2$ and $m \geq 2$ are positive integers. Each cell is colored in one of the four colors: white, green, red or blue. Call such a coloring interesting if any 2×2 square contains every color exactly once. Find the number of interesting colorings.

Problem 3. *(Alexander Ivanov)* A real nonzero number is assigned to every point in space. It is known that for any tetrahedron τ the number written in the incenter equals the product of the four numbers written in the vertices of τ. Prove that all numbers equal 1.

Problem 4. *(Peter Boyvalenkov)* Find all prime numbers p and q such that

$$p^2 \mid q^3+1 \quad \text{and} \quad q^2 \mid p^6-1.$$

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
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The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***November 20, 2014***.

For individual subscription for the next five issues for the 14-15 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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IMO2014 and Beyond (II)

Leung Tat-Wing

To discuss the IMO2014 problems, let's proceed from the easier problems to the harder problems.

Problem 1. Let $a_0 < a_1 < a_2 < \cdots$ be an infinite sequence of positive integers. Prove that there exists a unique integer $n \geq 1$ such that

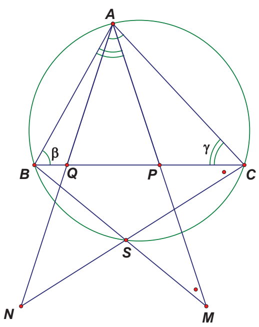
$$a_n < \frac{a_0 + a_1 + \ldots + a_n}{n} \leq a_{n+1}.$$

This problem is nice and easy. It gave us no problem. All of us got full scores in this problem. Nevertheless the problem is not entirely trivial, and indeed about 100 contestants scored nothing in this problem! First notice the middle term is not an arithmetical mean. Really during the question and answer period, some contestants did ask why the sequence doesn't start at index 1. Moreover the problem is not exactly an algebra problem, as it involves a strictly *increasing* sequence of *integers*. Try small cases, say $n = 1$. Then we need $a_1 < a_0+a_1$ sure, but not necessarily $a_0+a_1 \leq a_2$, why is that so? For $n = 2$, then we need $a_2 < (a_0+a_1+a_2)/2$, or $a_2 < a_0+a_1$, not necessarily true, but say when compared with the case of $n = 1$, if it is false, then $a_0+a_1 \leq a_2$ is true and we have an n satisfying the inequality! And the other side $a_0+a_1+a_2 \leq 2a_3$, why true again? If it is false, look at the left hand side for the case of $n = 3$. After several attempts, we really see what is going on. Indeed the inequality is equivalent to $na_n < a_0+a_1+\cdots+a_n \leq na_{n+1}$. The left hand inequality corresponds to $(a_0+a_1+\cdots+a_n) - na_n > 0$, while the right hand inequality corresponds to $(a_0+a_1+\cdots+a_n) - na_{n+1} \leq 0$, same as $(a_0+a_1+\cdots+a_{n+1}) - (n+1)a_{n+1} \leq 0$. Alas, if we define $d_n = (a_0+a_1+\cdots+a_n) - na_n$, then we just have to show there exists a *unique* n such that $d_n > 0 \geq d_{n+1}$! The proof is then complete if we can see (prove) d_n is a strictly *decreasing* sequence of *integers*. Not too bad.

Using induction, or other measures on the expression $(a_0+a_1+\cdots+a_n)/n$, our team members managed to solve the problem.

Problem 4. Points P and Q lie on side BC of acute-angled triangle ABC such that $\angle PAB = \angle BCA$ and $\angle CAQ = \angle ABC$. Points M and N lie on lines AP and AQ, respectively, such that P is the midpoint of AM, and Q is the midpoint of AN. Prove that lines BM and CN intersect on the circumcircle of triangle ABC.

This is the easiest problem in the competition, yet about 30 contestants did not get anything from it. Altogether more than 10 solutions were received, using synthetic geometry, coordinate geometry, complex numbers and the like. Some of us did it by coordinate geometry, setting the foot of A be (0,0), and coordinates $A(0,a)$, $B(b,0)$ and $C(c,0)$. Then get everything out of it via complicated calculations. But indeed if we can draw the picture properly, and do the angle tracings correctly, the problem is really not hard at all.



Indeed suppose BM and NC meet at S. Let $\angle ABC = \angle CAQ = \beta$ and $\angle ACB = \angle BAP = \gamma$, then $\triangle ABP \sim \triangle CAQ$. Hence

$$\frac{BP}{PM} = \frac{BP}{PA} = \frac{AQ}{QC} = \frac{QN}{QC}.$$

Also, $\angle NQC = \angle BQA = \angle APC = \angle BPM$. The last two statements imply $\triangle BPM \sim \triangle NQC$, hence $\angle BMP = \angle NCQ$. Then we also have $\triangle BPM \sim \triangle BSC$!

Finally, we have $\angle CSB = \angle MPB = \beta+\gamma = 180° - \angle ABC$. So $\angle CSB + \angle BAC = 180°$ and we are done.

Problem 2. Let $n \geq 2$ be an integer. Consider a $n \times n$ chessboard consisting of n^2 unit squares. A configuration of n rooks on this board is *peaceful* if every row and every column contains exactly one rook. Find the greatest positive integer k such that, for each peaceful configuration of n rooks, there is a $k \times k$ square which does not contain a rook on any of its k^2 unit squares.

All of us managed to give (basically) the correct answer ($\lceil \sqrt{n-1} \rceil$) and knew essentially how to tackle the question. There were gaps here and there and few points eventually deducted, but in my opinion, not really serious mistakes. Here n rooks are placed in a $n \times n$ board so that they are not attacking each other, and this time we ask for the largest possible gap (square with no rook). Of course the k^2 squares should be congruent to others and the "gap" square should be in one piece. Indeed several candidates had the same concern. This is really a classical chess board problem and I am not at all sure if the question was asked before somewhere.

First, given a $n \times n$ board with n rooks non-attacking (peaceful configuration). Suppose l is such that $l^2 < n$, then we can find a $l \times l$ square with no rook in it. Indeed there is a rook in the first *column*, consider the l consecutive *rows* starting with the row where the particular rook is placed. Now remove the first $n - l^2$ columns of this piece (hence at least one rook is removed). The remaining $l \times l^2$ piece can be decomposed into l $l \times l$ pieces of squares, but contain at most $l-1$ rooks, hence we have an empty $l \times l$ square.

Now we want to construct a peaceful configuration with largest possible square of size $\lceil \sqrt{n-1} \rceil \times \lceil \sqrt{n-1} \rceil$. Most of us see what the configuration should look like. We first let n be of the form l^2. Label the square with row i and column j as (i,j), with $0 \leq i \leq l-1$ and $0 \leq j \leq l-1$. The rooks are then placed on the positions $(il+j, jl+i)$, $0 \leq i,j \leq l-1$. One can easily check that any $l \times l$ square contains a rook.

Now comes where the most common gap lies. If $n < l^2$, we need to produce a peaceful configuration with no rook in any $l \times l$ square. The idea is of course to remove columns and rows from the previous construction. Only when (say) the top row and the leftmost column removed, *two* rooks may be removed, we have to put a rook back to an appropriate position (naturally where it should be) to return to a peaceful configuration!

(A 9×9 peaceful configuration with 2×2 squares as largest possible empty squares.)

								X
					X			
		X						
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X								

Problem 5. For each positive integer n, the Bank of Cape Town issues coins of denomination $1/n$. Given a finite collection of such coins (of not necessarily different denominations) with total value at most $99+½$, prove that it is possible to split the collection into 100 or fewer groups, such that each group has total value at most 1.

I am happy to see how our students handled this problem. In short, they used various grouping and induction techniques and tricks, and changed the problem to a format they can handle, thus solved the problems. Even though our arguments were sometimes rather unclear and convoluted, thus some points deducted because of gaps and other things, four of us essentially solved the problem. Indeed the main idea of solving the problem is by "merging" or "cleaning" the set of coins. Clearly if the process can still be completed after merging the coins, it can be done before merging!

Indeed the problem can be generalized as follows. Given coins of total value at most $N-½$, they can be split into N groups each of value at most 1. The problem then can be completed by the following steps.

(i) Two coins of values $1/(2k)$ may be merged into a coin of value $2 \times 1/(2k) = 1/2$, thus for every even number m, we may assume there is at most one coin of value $1/m$.

(ii) For every odd number m, there are at most $m-1$ coins of such value, otherwise they can be merged to form a coin of value 1 first.

(iii) Coins of value 1 must form a group of itself. Thus if there are d coins of value 1 in a group of N coins, we might as well consider a group of $N-d$ coins of values less than 1.

(iv) Now consider coins of values $1/(2k-1)$ and $1/(2k)$, with $k=1,2,\ldots,N$. We first place them into N groups according different values of k. In each group, the total value is at most

$$(2k-2)\times\frac{1}{2k-1}+\frac{1}{2k}=1-\frac{1}{2k-1}+\frac{1}{2k}<1.$$

The total value of all N groups is at most $N-½$. By taking average, there exists a group of total value at most

$$\frac{1}{N}(N-\frac{1}{2})=1-\frac{1}{2N}.$$

(v) All the remaining coins are of values less than $1/(2N)$. We may put them one by one into each group, as long as the value of each group does not exceed $1-1/(2N)$ and we are done!

The problem is meant to be a number theory problem, but is really more like a combinatorial problem. Our members managed to give different proofs to this problem and it is very nice. But indeed it is natural to consider coins of larger values (greedy method) first then consider coins of small values (a lot of them).

Problem 3. Convex quadrilateral $ABCD$ has $\angle ABC = \angle CDA = 90°$. Point H is the foot of the perpendicular from A to BD. Points S and T lie on sides AB and AD, respectively, such that H lies inside triangle SCT and $\angle CHS - \angle CSB = 90°$, $\angle THC - \angle DTC = 90°$. Prove that line BD is tangent to the circumcircle of triangle TSH.

In these few years, problems of this kind appear rather frequently. Proving a certain line is tangent to a certain (hidden) circle, or two (hidden) circles will touch each other, or the like, are generally not too easy. Still one should be able to handle them by first finding out some related geometric properties, and then obtain final results still by using only basic geometric properties and techniques.

Let us look at this problem. It is not easy to draw an accurate and nice picture, let alone proving it.

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***November 20, 2014.***

Problem 451. Let P be an n-sided convex polygon on a plane and $n>3$. Prove that there exists a circle passing through three consecutive vertices of P such that every point of P is inside or on the circle.

Problem 452. Find the least positive real number r such that for all triangles with sides a,b,c, if $a \ge (b+c)/3$, then

$$c(a+b-c) \le r\,((a+b+c)^2+2c(a+c-b)).$$

Problem 453. Prove that there exist infinitely many pairs of relatively prime positive integers a,b with $a>b$ such that b^2-5 is divisible by a and a^2-5 is divisible by b.

Problem 454. Let Γ_1, Γ_2 be two circles with centers O_1, O_2 respectively. Let P be a point of intersection of Γ_1 and Γ_2. Let line AB be an external common tangent to Γ_1, Γ_2 with A on Γ_1, B on Γ_2 and A, B, P on the same side of line O_1O_2. There is a point C on segment O_1O_2 such that lines AC and BP are perpendicular. Prove that $\angle APC=90°$.

Problem 455. Let a_1, a_2, a_3, … be a permutation of the positive integers. Prove that there exist infinitely many positive integer n such that the greatest common divisor of a_n and a_{n+1} is at most $3n/4$.

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Solutions

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Problem 446. If real numbers a and b satisfy $3^a+13^b=17^a$ and $5^a+7^b=11^b$, then prove that $a < b$.

Solution. **Kaustav CHATTERJEE** (MCKV Institute of Engineering College, India), **Ioan Viorel CODREANU** (Secondary School Satulung, Maramures, Romania), **KWOK Man Yi** (Baptist Lui Ming Choi Secondary School, S4), **Elaine LAM** (Tsuen Wan Secondary School), **Corneliu MĂNESCU-AVRAM** (Transportation High school, Ploieşti, Romania), **NGUYÊN Viêt Hoàng** (Hà Nôi, Viêt Nam), **PANG Lok Wing, YAN Yin Wang** (United Christian College (Kowloon East), Teaching Staff) and **Simon YAU.**

If $a \ge b$, then $3^a+13^a \ge 3^a+13^b=17^a$. (*) Since $3/17<13/17<1$, the function $f(x) =(3/17)^x+(13/17)^x$ is strictly decreasing. By (*), $f(a) \ge 1> f(1)$. So $a < 1$.

Next, $5^b+7^b \le 5^a+7^b = 11^b$. (**) Since $5/11 < 7/11 < 1$, the function $g(x) = (5/11)^x + (7/11)^x$ is strictly decreasing. By (**), $g(b) \le 1 < g(1)$. So $b>1>a$, contradiction.

Other commended solvers: **Math Activity Center** (Carmel Alison Lam Foundation Secondary School), **Nicuşor ZLOTA** ("Traian Vuia" Technical College, Focşani, Romania), **Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania).

Problem 447. For real numbers x, y, z, find all possible values of $\sin(x+y) + \sin(y+z) + \sin(z+x)$ if

$$\frac{\cos x+\cos y+\cos z}{\cos(x+y+z)}=\frac{\sin x+\sin y+\sin z}{\sin(x+y+z)}.$$

Solution. **KWOK Man Yi** (Baptist Lui Ming Choi Secondary School, S4), **Corneliu MĂNESCU-AVRAM** (Transportation High school, Ploieşti, Romania), **YAN Yin Wang** (United Christian College (Kowloon East), Teaching Staff), **Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania).

Let $S=x+y+z$. Cross multiply and transfer all terms to one side. We get

$$\begin{aligned}0 &= \sin S\cos x - \cos S\sin x + \sin S\cos y\\ &\quad - \cos S\sin y + \sin S\cos z - \cos S\sin z\\ &= \sin(S-x) + \sin(S-y) + \sin(S-z)\\ &= \sin(y+z) + \sin(z+x) + \sin(x+y).\end{aligned}$$

Other commended solvers: **Kaustav CHATTERJEE** (MCKV Institute of Engineering College, India), **Ioan Viorel CODREANU** (Secondary School Satulung, Maramures, Romania) and **Math Activity Center** (Carmel Alison Lam Foundation Secondary School).

Problem 448. Prove that if s,t,u,v are integers such that $s^2-2t^2+5u^2-3v^2=2tv$, then $s = t = u = v = 0$.

Solution. **Ioan Viorel CODREANU** (Secondary School Satulung, Maramures, Romania), **KWOK Man Yi** (Baptist Lui Ming Choi Secondary School, S4), **Corneliu MĂNESCU-AVRAM** (Transportation High school, Ploieşti, Romania), **Math Activity Center** (Carmel Alison Lam Foundation Secondary School), **NGUYÊN Viêt Hoàng** (Hà Nôi, Viêt Nam), **YAN Yin Wang** (United Christian College (Kowloon East), Teaching Staff), **Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania).

Assume s,t,u,v are not all zeros. By cancelling all common factors of s,t,u,v, we may assume they are relatively prime. We can rewrite the equation as

$$2(s^2+5u^2) = (2t+v)^2+5v^2. \qquad (\dagger)$$

For $0 \le x, y \le 4$, we have $2x^2 \equiv y^2 \pmod 5$ if and only if $x \equiv y \equiv 0 \pmod 5$. (‡) So $s^2+5u^2 \equiv 2t+v \equiv 0 \pmod 5$, which implies $s = 5m$ and $2t+v = 5n$ for some integers m,n. Substituting these into (†), we get $2(5m^2+u^2)=5n^2+v^2$. By (‡), u, v are divisible by 5. Then s,t,u,v are divisible by 5, contradicting they are relatively prime. So s,t,u,v are all zeros.

Other commended solvers: **Kaustav CHATTERJEE** (MCKV Institute of Engineering College, India),

Problem 449. Determine the smallest positive integer k such that no matter how $\{1,2,3,\ldots,k\}$ are partitioned into two sets, one of the two sets must contain two distinct elements m, n such that mn is divisible by $m+n$.

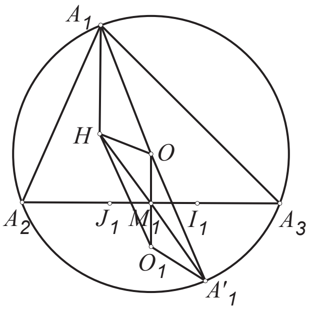
Solution. **Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania).

Call distinct positive integers m,n a <u>*good*</u> pair if mn is divisible by $m+n$. Collect all good pairs with $m,n \le 40$. We will try to separate m,n first. Let $A =\{1,2,3,5,8, 10, 12, 13, 14, 18, 19, 21, 22, 23, 30, 31,32,33,34\}$ and $B = \{4, 6, 7, 9, 11, 15, 16, 17, 20, 24, 25, 26, 27, 28, 29, 35, 36, 37, 38, 39\}$. Each of A and B do not contain any good pair. For $1 \le k \le 39$, we can remove integers greater than k from A and B to get 2 disjoint subsets of $\{1, 2, \ldots, k\}$ with no good pair in each subset.

For k=40, put 6, 12, 24, 40, 10, 15 and 30 around a circle. Notice any two consecutive terms in this circle is a good pair. No matter how we divide $\{1,2,\ldots,40\}$ into 2 disjoint subsets, one of the subsets will contain at least 4 of 7 numbers in the circle. So there will be a good pair in that subset. Therefore, 40 is the desired least integer.

Other commended solvers: **NGUYÊN Viêt Hoàng** (Hà Nôi, Viêt Nam).

Problem 450. (*Proposed by Michel BATAILLE*) Let $A_1A_2A_3$ be a triangle with no right angle and O be its circumcenter. For i = 1,2,3, let the reflection of A_i with respect to O be A_i' and the reflection of O with respect to line $A_{i+1}A_{i+2}$ be O_i (subscripts are to be taken modulo 3). Prove that the circumcenters of the triangles OO_iA_i' (i = 1,2,3) are collinear.



Solution. **KWOK Man Yi** (Baptist Lui Ming Choi Secondary School, S4).

Notice that O_1 is the reflection of O with respect to the midpoint M_1 of A_2A_3. By the nine point circle theorem (see <u>*Math Excalibur*</u>, *vol*.3, *no* 1, *p*,1), AH, OM_1 are parallel and their lengths are 2:1. Now A_1O=OA_1'. So, in $\triangle A_1A_1'H$, M_1 is the midpoint of $A_1'H$, i.e. H is the reflection of A_1' with respect to M_1.

Let I_1 be the circumcenter of $\triangle OO_1A_1'$. Then I_1 lies on the perpendicular bisector A_2A_3 of OO_1. Reflect I_1 with respect to M_1 to J_1. Then J_1 also lies on A_2A_3. With respect to M_1, J_1 is the circumcenter of the reflection of $\triangle OO_1A_1'$, i.e. $\triangle OO_1H$. So, J_1 also lies on the perpendicular bisector of OH.

Define I_2, I_3, J_2, J_3 similarly. As J_2, J_3 also lie on the perpendicular bisector of OH by a similar proof, J_1, J_2, J_3 are collinear. Then by Menelaus' theorem,

$$\frac{A_2J_1}{J_1A_3}\cdot\frac{A_3J_2}{J_2A_1}\cdot\frac{A_1J_3}{J_3A_2}=-1.$$

As A_3I_1/I_1A_2=A_2J_1/J_1A_3 (due to I_1, J_1 are reflection of the midpoint of A_2A_3) and similarly for I_2, J_2, I_3, J_3, we have

$$\frac{A_3I_1}{I_1A_2}\cdot\frac{A_1I_2}{I_2A_3}\cdot\frac{A_2I_3}{I_3A_1}=-1.$$

By the converse of Menelaus' Theorem, I_1, I_2, I_3 are collinear as desired.

Other commended solvers: **Andrea FANCHINI** (Cantú, Italy), **Corneliu Mănescu-Avram** (Transportation High school, Ploieşti, Romania), **NGUYÊN Viêt Hoàng** (Hà Nôi, Viêt Nam), **Samiron SADHUKHAN** (Kendriya Vidyalaya, Barrackpore, Kolkata, India), **Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania).

Olympiad Corner

(*Continued from page 1*)

Problem 5. *(Nikolay Nikolov)* Find all functions $f:\mathbb{Q}^+\to\mathbb{R}^+$ such that

$f(xy) = f(x+y)(f(x)+f(y))$ for any $x,y\in\mathbb{Q}^+$.

Problem 6. *(Nikolay Beluhov)* The quadrilateral $ABCD$ is inscribed in the circle k. The lines AC and BD meet in E and the lines AD and BC meet in F. Show that the line through the incenters of ΔABE and ΔABF and the line through the incenters of ΔCDE and ΔCDF meet on k.

IMO2014 and Beyond (II)

(*Continued from page 2*)

First, let the line passing through C and is perpendicular to SC meets AB at Q. Then $\angle SQC$=90°−$\angle BSC$=180°−$\angle SHC$. So C, H, S, Q are concyclic. Moreover SQ is a diameter of this circle, thus the circumcenter K of SHC lies on AB. Likewise, circumcenter L of the circle CHT lies on AD. To show the circumcircle of the triangle SHT is tangent to BD, it suffices to show the perpendicular bisectors of HS and HT meet at AH. But the two perpendicular bisectors coincide with the angle bisectors of AKH and ALH, thus by the bisector theorem, it suffices to show AK/KH=AL/LH. Let M be the midpoint of CH, then B,C,M,K are concyclic, L,C,M,D are concyclic. By the sine law, AK/AL= $\sin\angle ALK$ / $\sin\angle AKL$ = $(DM/CL)/(BM/CK)$ = CK/CL = KH/LH.

Problem 6. A set of lines in the plane is in *general position* if no two are parallel and no three pass through the same point. A set of lines in general position cuts the plane into regions, some of which have finite areas; we call these its *finite regions*. Prove that for all sufficiently large n, in any set of n lines in general position it is possible to color at least $\sqrt{n}$ of the lines blue in such a way that none of its finite regions has a completely blue boundary.

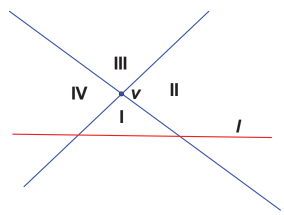
Notes: Results with $\sqrt{n}$ replaced by $c\sqrt{n}$ will be awarded points depending on the value of the constant c.

I have to admit that I don't like this problem at all. Indeed it was meant to be an "open end" problem, that students may produce different results with different degrees of difficulty. But when I first saw the problem, I thought we should give an algorithm, say a greedy algorithm, or other heuristic that gives good pattern (with as many blue colored lines as possible), and then analyze the pattern and give an estimate. Not so. (I guess I have become kind of intuitionist.) I doubt if there was any algorithmic solution anyway. Indeed in the official solution, a best possible solution is assumed, surely it exists, but we were not told how to get there.

Let me reproduce a part of the proof as follows. Given a set of n lines colored blue and red, and the lines colored blue is as large as possible (maximality argument), so that every finite region still has at least one boundary line colored red. Assume k lines are colored blue. Call a vertex which is the intersection of two blue lines *blue* as well, so there are ${}_kC_2$ blue vertices.

Now take any red line l, using the maximality argument, there exists at least one region with this red line l as the *only* red side, (for if all regions have two or more red lines, surely we can change one more red line to blue). In this region there is at least one blue vertex v since any finite region has at least three lines. We then associate the blue vertex with the red line. Now finally every blue vertex v belongs to four regions, (some may be unbounded), hence it may be associated with at most four red lines. Therefore the total number of red lines is at most $4{}_kC_2$=$2k(k-1)$.

On the other hand, there are $n-k$ red lines, thus, $n-k\le 2k(k-1)$. Solving for n, we get $n\le 2k^2-k\le 2k^2$. Hence, $k\ge\left\lceil\sqrt{n/2}\right\rceil$ and we get an estimate on the number of blue lines!



By putting some weights on the blue vertices, or by refining local analysis, one may get the stronger result $k\geq\sqrt{n}$.