Mathematical Excalibur

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Olympiad Corner

Below are the problems of the IMO2015 Hong Kong Team Selection Test 2 held on 25th October, 2014.

Problem 1. Assume the dimensions of an answer sheet to be 297 mm by 210 mm. Suppose that your pen leaks and makes some non-intersecting ink stains on the answer sheet. It turns out that the area of each ink stain does not exceed 1 mm^2. Moreover, any line parallel to an edge of the answer sheet intersects at most one ink stain. Prove that the total area of the ink stains is at most 253.5 mm^2. (You may assume a stain is a connected piece.)

Problem 2. Let $\{a_n\}$ be a sequence of positive integers. It is given that a_1=1, and for every n≥1, a_{n+1} is the smallest positive integer greater than a_n which satisfies the following condition: for any integers i, j, k, with $1 \le i, j, k \le n+1$, $a_i+a_j \ne 3a_k$. Find a_{2015}.

Problem 3. Let ABC be an equilateral triangle, and let D be a point on AB between A and B. Next, let E be a point on AC with DE parallel to BC. Further, let F be the midpoint of CD and G the circumcentre of $\triangle ADE$. Determine the interior angles of $\triangle BFG$.

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Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***January 31, 2015***.

For individual subscription for the next five issues for the 14-15 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Variations and Generalisations to the Rearrangement Inequality

Law Ka Ho

A. The rearrangement inequality

In *Math Excalibur*, vol. 4, no. 3, we can find the following

Theorem 1 (Rearrangement inequality) Let $a_1 \le a_2 \le \cdots \le a_n$ and $b_1 \le b_2 \le \cdots \le b_n$ be two increasing sequences of real numbers. Then amongst all **random sums** of the form

$$a_1b_{\sigma_1} + a_2b_{\sigma_2} + \cdots + a_nb_{\sigma_n},$$

where $(\sigma_1,\sigma_2,\ldots,\sigma_n)$ is a permutation of $(1,2,\ldots,n)$,

- the greatest is the **direct sum** $a_1b_1+a_2b_2+\cdots+a_nb_n$;
- the smallest is the **reverse sum** $a_1b_n+a_2b_{n-1}+\cdots+a_nb_1$.

A well-known corollary of the rearrangement inequality is the following

Theorem 2 (Chebyshev's inequality) With the same setting in Theorem 1, the quantity

$$\frac{(a_1+a_2+\cdots+a_n)(b_1+b_2+\cdots+b_n)}{n}$$

lies between the direct sum and the reverse sum, again with equality if and only if at least one of the two sequences is constant.

B. A variation --- from 'sum' to 'product'

The different 'sums' in the rearrangement inequality are in fact 'sums of products'. For this reason we shall from now on call them **P-sums**, to remind ourselves that we take products and then sum them up. Naturally, we ask what happens if we look at 'product of sums' (**S-products**) instead.

A little trial *suggests* that, opposite to the case of P-sums, the direct S-product is minimum while the reverse S-product is maximum. For example we may take the sequences 1≤2≤3≤4 and 5≤6≤7≤8. The direct S-product of these sequences is (1+5)(2+6)(3+7)(4+8) = 5760 and the reverse S-product of the sequences is (1+8)(2+7)(3+6)(4+5) = 6561. We can also check some random S-products, e.g we have (1+6)(2+5)(3+8)(3+7) = 5929 and (1+6)(2+7)(3+8)(4+5) = 6237.

But then a little further thought shows that this is not quite right. For instance we may take 1≤2≤3≤4 and −5≤−2 ≤1≤ 2 and end up with a reverse S-product (1+2)(2+1)[3+(−2)][4+(−5)], which is negative. Yet, some random S-products, such as [1+(−2)](2+2)(3+1)[4+(−5)], can be positive.

It turns out that we have to require the variables to be non-negative for the result to hold.

Theorem 3 (Rearrangement inequality for S-products) Let $a_1 \le a_2 \le \cdots \le a_n$ and $b_1 \le b_2 \le \cdots \le b_n$ be two increasing sequences of non-negative real numbers. Then amongst all random S-products of the form

$$(a_1+b_{\sigma_1})(a_2+b_{\sigma_2})\cdots(a_n+b_{\sigma_n})$$

where $(\sigma_1,\sigma_2,\ldots,\sigma_n)$ is a permutation of $(1,2,\ldots,n)$,

- the smallest is the direct S-product $(a_1+b_1)(a_2+b_2)\cdots(a_n+b_n)$;
- the greatest is the reverse S-product $(a_1+b_n)(a_2+b_{n-1})\cdots(a_n+b_1)$.

Proof Take any random S-product

$$(a_1+b_{\sigma_1})(a_2+b_{\sigma_2})\cdots(a_n+b_{\sigma_n})$$

which is not the direct S-product. Then there exists $i<j$ such that $b_{\sigma_i} > b_{\sigma_j}$.

Let's see what happens if we swap σ_i and σ_j. In that case only two terms are changed. Consider the two products

$$P_1 = (a_i+b_{\sigma_i})(a_j+b_{\sigma_j}) \text{ and}$$

$$P_2 = (a_i+b_{\sigma_j})(a_j+b_{\sigma_i}).$$

(continued on page 2)

After expanding, cancelling and factoring, we have

$$P_2 - P_1 = (a_i - a_j)(b_{\sigma_j} - b_{\sigma_i}),$$

which is non-positive since $a_i - a_j \le 0$ and $b_{\sigma_i} > b_{\sigma_j}$. So $P_2 \ge P_1$. This means swapping σ_i and σ_j leads to a larger (or equal) S-product. It follows that the direct S-product is the minimum amongst all random S-products. In a similar manner we can prove that the reverse S-product is the maximum.

Example 4 (IMO 1966) In the interior of sides BC, CA, AB of ΔABC, points K, L, M respectively, are selected. Prove that the area of at least one of the triangles AML, BKM, CLK is less than or equal to one quarter of the area of ΔABC.

Solution Let a, b, c denote the lengths of the sides opposite A, B, C respectively. Let also a_1 and a_2 denote the lengths of the two segments after the side with length a is cut into two parts by the point K (i.e. $BK = a_1$ and $KC = a_2$), and similarly for b_1, b_2, c_1, c_2. The six variables a_1, a_2, b_1, b_2, c_1, c_2 can be ordered to form an increasing sequence. By the rearrangement inequality for S-products, the direct S-product

$$(a_1+a_1)(a_2+a_2)(b_1+b_1)(b_2+b_2)(c_1+c_1)(c_2+c_2)$$
$$= 64a_1a_2b_1b_2c_1c_2$$

is less than or equal to the random S-product

$$(a_1+a_2)(a_2+a_1)(b_1+b_2)(b_2+b_1)(c_1+c_2)(c_2+c_1)$$
$$= a^2b^2c^2.$$

Let S denote the area of ΔABC. If triangles AML, BKM, CLK all have areas greater than $S/4$, then using the above result we have

$$\left(\frac{S}{4}\right)^3 < \left(\frac{1}{2}c_1b_2\sin A\right)\left(\frac{1}{2}c_2a_1\sin B\right)\left(\frac{1}{2}a_2b_1\sin C\right)$$
$$\le \frac{a^2b^2c^2}{8\cdot 64}\cdot\sin A\sin B\sin C$$
$$= \frac{1}{64}\left(\frac{1}{2}ab\sin C\right)\left(\frac{1}{2}bc\sin A\right)\left(\frac{1}{2}ca\sin B\right)$$
$$= \left(\frac{S}{4}\right)^3$$

which is a contradiction.

Example 5 (IMO 1984) Prove that

$$0 \le xy + yz + zx - 2xyz \le 7/27,$$

where x, y and z are non-negative real numbers for which $x+y+z=1$.

Solution The left-hand inequality is pretty easy. We have

$$xy+yz+zx-2xyz$$
$$=(xy-xyz)+(yz-xyz)+(zx-xyz)+xyz$$
$$=xy(1-z)+yz(1-x)+zx(1-y)+xyz$$
$$=xy(x+y)+yz(y+z)+zx(z+x)+xyz \ge 0.$$

For the right-hand inequality, it is well-known that

$$1 = (x+y+z)^2$$
$$= x^2+y^2+z^2+xy+yz+zx$$
$$\ge 3(xy+yz+zx)$$

and so $xy + yz + zx \le 1/3$. By the rearrangement inequality for S-products, we have

$$(1-2x)(1-2y)(1-2z)$$
$$\le\left(\frac{1-2x}{2}+\frac{1-2y}{2}\right)\left(\frac{1-2y}{2}+\frac{1-2z}{2}\right)\left(\frac{1-2z}{2}+\frac{1-2x}{2}\right)$$
$$=zxy\ .$$

(The rearrangement inequality for S-products applies only if the three terms on the left hand side are non-negative. However, if this is not true then exactly one of them is negative and the result therefore still holds.) Expanding gives

$$1-2(x+y+z)+4(xy+yz+zx)-8xyz \le xyz$$

or $9xyz \ge 4(xy+yz+zx)-1$. From this, we have

$$xy+yz+zx-2xyz$$
$$\le xy+yz+zx-2\left(\frac{4(xy+yz+zx)-1}{9}\right)$$
$$=\frac{xy+yz+zx+2}{9} \le \frac{2\frac{1}{3}}{9} = \frac{7}{27}.$$

C. A generalisation — from two sequences to more

Another natural direction of generalising the rearrangement inequality (for P-sums) is to consider the case in which there are more than two sequences. This time we need two subscripts to index the terms, one for the index of the sequence and one for the index of a particular term of a sequence. Again, we need to restrict ourselves to sequences of non-negative numbers (for both P-sums and S-products), otherwise one can easily construct counter- examples. Also, note that there is no such thing as 'reverse P-sum/S-product' when there are more than two sequences.

Theorem 6 (Rearrangement inequality for multiple sequences) Suppose there are m increasing sequences (each with n terms) of non-negative numbers, say, $a_{i1} \le a_{i2} \le \cdots \le a_{in}$, where $i = 1, 2, \ldots, m$. Then

- the direct P-sum $\sum_{j=1}^{n} a_{1j}a_{2j}\cdots a_{mj}$ is greater than or equal to any other random P-sum of the form $\sum_{j=1}^{n} a_{1\sigma_{1j}}a_{2\sigma_{2j}}\cdots a_{m\sigma_{mj}}$;
- the direct S-product $\prod_{j=1}^{n}(a_{1j}+a_{2j}+\cdots+a_{mj})$ is smaller than or equal to any other random S-product of the form $\prod_{j=1}^{n}(a_{1\sigma_{1j}}+a_{2\sigma_{2j}}+\cdots+a_{m\sigma_{mj}})$.

Here $(\sigma_{i1},\sigma_{i2},\ldots,\sigma_{in})$ is a permutation of $(1,2,\ldots,n)$ for $i = 1,2, \ldots, m$.

Remarks.

(1) Theorem 6 is sometimes known as '微微對偶不等式' in Chinese.

(2) A less clumsy way to express Theorem 6 is to use matrices. With the above m sequences we may form the matrices

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \text{ and}$$

$$B = \begin{pmatrix} a_{1\sigma_{11}} & a_{1\sigma_{12}} & \cdots & a_{1\sigma_{1n}} \\ a_{2\sigma_{21}} & a_{2\sigma_{22}} & \cdots & a_{2\sigma_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m\sigma_{m1}} & a_{m\sigma_{m2}} & \cdots & a_{m\sigma_{mn}} \end{pmatrix}.$$

Here each row of A is in ascending order (corresponding to one of the m increasing sequences) while each row of B is a permutation of the terms in the corresponding row of A (corresponding to a permutation of the corresponding sequence). Then Theorem 6 says

- the sum of column products (P-sum) in A is greater than or equal to that in B;
- the product of column sums (S-product) in A is less than or equal to that in B.

(3) The proof of Theorem 6 is essentially the same as that of Theorem 3, and is therefore omitted.

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Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***January 31, 2015.***

Problem 456. Suppose $x_1, x_2, \ldots, x_n$ are non-negative and their sum is 1. Prove that there exists a permutation σ of $\{1,2,\cdots,n\}$ such that

$$x_{\sigma(1)}x_{\sigma(2)}+x_{\sigma(2)}x_{\sigma(3)}+\cdots+x_{\sigma(n)}x_{\sigma(1)}\le 1/n.$$

Problem 457. Prove that for each n = 1,2,3,…, there exist integers a, b such that if integers x, y are relatively prime, then $\sqrt{(a-x)^2+(b-y)^2}>n.$

Problem 458. Nonempty sets A_1, A_2, A_3 form a partition of $\{1,2,\ldots,n\}$. If $x+y=z$ have no solution with x in A_i, y in A_j, z in A_k and $\{i,j,k\}=\{1,2,3\}$, then prove that A_1, A_2, A_3 cannot have the same number of elements.

Problem 459. H is the orthocenter of acute $\triangle ABC$. D,E,F are midpoints of sides BC, CA, AB respectively. Inside $\triangle ABC$, a circle with center H meets DE at P,Q, EF at R,S, FD at T,U. Prove that $CP=CQ=AR=AS=BT=BU$.

Problem 460. If $x,y,z>0$ and $x+y+z+2=xyz$, then prove that

$$x+y+z+6\ge 2\left(\sqrt{yz}+\sqrt{zx}+\sqrt{xy}\right).$$

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Solutions

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Problem 451. Let P be an n-sided convex polygon on a plane and $n>3$. Prove that there exists a circle passing through three consecutive vertices of P such that every point of P is inside or on the circle.

Solution. **Adnan ALI** (Atomic Energy Central School 4, Mumbai, India) and **T.W. LEE** (Alumni of New Method College).

Let R_{XYZ} denote the radius of the circle through vertices X,Y,Z of P. Let circle Γ through vertices A,B,C of P be one with maximal radius. Without loss of generality, we may assume $\angle ABC$ and $\angle ACB<90°$. If there is a vertex D of P outside Γ, let AD meet Γ at E. Then $\angle ADC<\angle AEC=\angle ABC$. By the extended sine law

$$R_{ADC}=\frac{AC}{2\sin\angle ADC}>\frac{AC}{2\sin\angle ABC}=R_{ABC},$$

contradicting maximality of Γ. So all vertices of P is on or inside Γ.

Let F be the vertex of P next to A (toward C). If F is inside Γ, then $AFCB$ is convex and $\angle AFC+\angle ABC>180°$. Hence $0°<180°-\angle AFC<\angle ABC<90°$. Then

$$R_{AFC}=\frac{AC}{2\sin\angle AFC}>\frac{AC}{2\sin\angle ABC}=R_{ABC},$$

contradiction. So F is on Γ. Similarly, the vertex of P next to A (toward B) is on Γ.

Problem 452. Find the least positive real number r such that for all triangles with sides a,b,c, if $a\ge(b+c)/3$, then

$$c(a+b-c)\le r\,((a+b+c)^2+2c(a+c-b)).$$

Solution. **Jon GLIMMS** and **Samiron SADHUKHAN** (Kendriya Vidyalaya, India).

Let $I=a+b-c$. Then $a\ge(b+c)/3$ implies $a-b\ge-(a+b-c)/2=-I/2$ (*)

Using $a+b+c=I+2c$, (*) and the *AM-GM* inequality, we have

$$J=\frac{(a+b+c)^2+2c(a+c-b)}{2c(a+b-c)}$$
$$=\frac{I^2+4cI+4c^2}{2cI}+\frac{a+c-b}{I}$$
$$=\frac{I}{2c}+2+\frac{3c}{I}+\frac{a-b}{I}$$
$$\ge\frac{3}{2}+\frac{I}{2c}+\frac{3c}{I}\ge\frac{3}{2}+2\sqrt{\frac{3}{2}}.$$

Equality hold if $a=(b+c)/3$ and $I^2=6c^2$, i.e. $a:b:c=2+\sqrt{6}:2+3\sqrt{6}:4$. The least r such that $1/(2J)\le r$ is $(\sqrt{24}-3)/15$.

Problem 453. Prove that there exist infinitely many pairs of relatively prime positive integers a,b with $a>b$ such that b^2-5 is divisible by a and a^2-5 is divisible by b.

Solution. **Adnan ALI** (Atomic Energy Central School 4, Mumbai, India), **LKL Excalibur** (Madam Lau Kam Lung Secondary School of MFBM) and **Samiron SADHUKHAN** (Kendriya Vidyalaya, India).

Note $(a,b)=(11,4)$ is a solution. From any solution (a,b) with $a>b\ge4$, we get $a^2-5=bc$ and $b^2-5=ad$ for some positive integers c and d. Now we show (c,a) is another such solution. First $bc=a^2-5>a^2-a=a(a-1)\ge ab$ implies $c>a$. If a prime p divides $\gcd(a,c)$, then $a^2-5=bc$ and $b^2-5=ad$ imply $b^2=ad+5=ad+a^2-bc$ is divisible by p. Since $\gcd(a,b)=1$, we get $\gcd(c,a)=1$. Using $\gcd(a,b)=1$ and $a(a+d)=a^2+b^2-5=b(b+c)$, we see a divides $b+c$. Then a divides $(b+c)(c-b)+(b^2-5)=c^2-5$. So there are infinitely many solutions.

Other commended solvers: **Corneliu MĂNESCU-AVRAM** (Transportation High school, Ploieşti, Romania), **O Kin Chit** (G. T. (Ellen Yeung College), **WONG Yat** (G. T. (Ellen Yeung) College), **Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania).

Problem 454. Let Γ_1, Γ_2 be two circles with centers O_1, O_2 respectively. Let P be a point of intersection of Γ_1 and Γ_2. Let line AB be an external common tangent to Γ_1, Γ_2 with A on Γ_1, B on Γ_2 and A, B, P on the same side of line O_1O_2. There is a point C on segment O_1O_2 such that lines AC and BP are perpendicular. Prove that $\angle APC=90°$.

Solution. **Serik JUMAGULOV** (Karaganda State University, Qaragandy City, Kazakhstan).

Other than P, let the circles also meet at Q. If $PQ\cap AB=M$, then M is the midpoint of AB as $MA^2=MP\times MQ=MB^2$. Let $PQ\cap O_1O_2=K$, $BP\cap AC=N$ and AL be a diameter of the circle with center O_1. Since $PQ\perp O_1O_2$ and $BN\perp AC$, $PNCK$ is cyclic. Now $\angle PBM=90°-\angle NAB=\angle CAO_1$ and $\angle BPM=\angle KPN=\angle ACO_1$. So $\Delta ACO_1\sim\Delta BPM$. Then $AC/BP=AO_1/BM=AL/BA$. So $\Delta ACL\sim\Delta BPA$. Then $\angle ALP=\angle BAP=\angle ALC$. So L,C,P are collinear. As AL is a diameter, $\angle APC=90°$.

Other commended solvers: **Andrea FANCHINI** (Cantú, Italy), **Titu ZVONARU** (Comăneşti, Romania) and **Neculai STANCIU** ("George Emil Palade" Secondary School, Buzău, Romania).

Problem 455. Let $a_1, a_2, a_3, \ldots$ be a permutation of the positive integers. Prove that there exist infinitely many positive integer n such that the greatest common divisor of a_n and a_{n+1} is at most $3n/4$.

Solution. **Jon GLIMMS** and **Samiron SADHUKHAN** (Kendriya Vidyalaya, India).

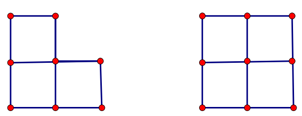
Assume that there exists N such that for all $n\ge N$, $\gcd(a_n,a_{n+1})>3n/4$. Then for all $n\ge 4N$, $a_n\ge\gcd(a_n,a_{n+1})>3n/4\ge 3N$. Since $a_1, a_2, a_3, \ldots$ is a permutation of the positive integers, we see $\{1,2,\cdots,3N\}$ is a subset of $\{a_1, a_2,\cdots, a_{4N-1}\}$. Now the intersection of $\{1,2,\cdots,3N\}$ and $\{a_{2N}, a_{2N+1},\cdots,a_{4N-1}\}$ has at least $3N-(2N-1)$

$= N+1$ elements. By the pigeonhole principle, there exists k such that $2N \le k < 4N-1$ and a_k, $a_{k+1} \le 3N$. Then $\gcd(a_k, a_{k+1}) \le \frac{1}{2}\max\{a_k, a_{k+1}\} \le 3N/2 \le 3k/4$, contradiction.

Olympiad Corner

(*Continued from page 1*)

Problem 4. A 11×11 grid is to be covered completely without overlapping by some 2×2 squares and L-shapes each composed of three unit cells. Determine the smallest number of L-shapes used. (Each shape must cover some grids entirely and cannot be placed outside the 11×11 grid. The L-shapes may be reflected or rotated when placed on the grid.)



Variations and Generalisations

(*Continued from page 2*)

Example 7 Let x_1, x_2, …, x_n be non-negative real numbers whose sum is at most 1/2. Show that $(1-x_1)(1-x_2)\cdots(1-x_n) \ge 1/2$.

Solution Form the $n\times n$ matrix

$$A = \begin{pmatrix} 1-x_1 & 1 & \cdots & 1 \\ 1-x_2 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1-x_n & 1 & \cdots & 1 \end{pmatrix}$$

whose rows are in ascending order. Consider the matrix

$$B = \begin{pmatrix} 1-x_1 & 1 & \cdots & 1 \\ 1 & 1-x_2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1-x_n \end{pmatrix}$$

in which each row is a permutation of the terms in the corresponding row of A. By the rearrangement inequality for multiple sequences, the P-sum in A is greater than the P-sum in B, i.e.

$$(1-x_1)(1-x_2)\cdots(1-x_n) + n - 1 \ge (1-x_1) + (1-x_2) + \cdots + (1-x_n).$$

It follows that

$$(1-x_1)(1-x_2)\cdots(1-x_n) \ge 1 - (x_1 + x_2 + \cdots + x_n) \ge 1-1/2 = 1/2.$$

Example 8 Let x_1, x_2, …, x_n be positive real numbers with sum 1. Show that

$$\frac{x_1x_2\cdots x_n}{(1-x_1)(1-x_2)\cdots(1-x_n)} \le \frac{1}{(n-1)^n}.$$

Solution Without loss of generality assume $x_1 \le x_2 \le \cdots \le x_n$. Form the $(n-1)\times n$ matrix

$$A = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ x_1 & x_2 & \cdots & x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_1 & x_2 & \cdots & x_n \end{pmatrix}$$

whose rows are in ascending order. The S-product of A is thus $(n-1)^n x_1x_2\cdots x_n$. Now the matrix B given by

$$B = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ x_2 & x_3 & \cdots & x_1 \\ \vdots & \vdots & \ddots & \vdots \\ x_{n-1} & x_n & \cdots & x_{n-2} \end{pmatrix}$$

has the property that each of its rows is a permutation of the terms in the corresponding row of A. Furthermore, since x_1, x_2, …, x_n have sum 1, the S-product of B is equal to $(1-x_1)(1-x_2)\cdots(1-x_n)$. By the rearrangement inequality for multiple sequences, we have $(n-1)^n x_1x_2\cdots x_n \le (1-x_1)(1-x_2)\cdots(1-x_n)$.

D. Proofs of some classic inequalities

The rearrangement inequality for multiple sequences can be used to prove a number of classic inequalities. We look at some such examples in this final section.

Theorem 9 (Bernoulli inequality) For real numbers x_1, x_2, …, x_n , where either all are non-negative or all are negative but not less than –1, we have

$$\prod_{i=1}^{n}(1+x_i) \ge 1 + \sum_{i=1}^{n} x_i.$$

Proof Without loss of generality assume $x_1 \le x_2 \le \cdots \le x_n$. Suppose x_1, x_2, …, x_n are all non-negative. Consider the $n\times n$ matrices

$$A = \begin{pmatrix} 1 & 1 & \cdots & 1+x_1 \\ 1 & 1 & \cdots & 1+x_2 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1+x_n \end{pmatrix} \quad \text{and}$$

$$B = \begin{pmatrix} 1+x_1 & 1 & \cdots & 1 \\ 1 & 1+x_2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1+x_n \end{pmatrix}.$$

Then A and B satisfy the properties stated in Theorem 6. Thus the P-sum in A is greater than or equal to that in B,

i.e. $n-1+\prod_{i=1}^{n}(1+x_i) \ge \sum_{i=1}^{n}(1+x_i)$.

This gives $\prod_{i=1}^{n}(1+x_i) \ge 1+\sum_{i=1}^{n}x_i$.

The proof in the latter case (in which x_1, x_2, …, x_n are negative but not less than –1) is essentially the same; just move the rightmost column of A to the leftmost.

Theorem 10 (Generalised Chebyshev's inequality) For m increasing sequences (each with n terms) of non-negative real numbers, say, $a_{i1} \le a_{i2} \le \cdots \le a_{in}$, where i=1,2,…, m,

the direct P-sum $\sum_{j=1}^{n} a_{1j}a_{2j}\cdots a_{mj}$ is

greater than or equal to

$$\frac{1}{n^{m-1}}\prod_{i=1}^{m}\left(a_{i1}+a_{i2}+\cdots+a_{in}\right).$$

Proof Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

Now we can randomly form a matrix B as follows. The first row of B is the same as that of A. Each other row of B is obtained by shifting the corresponding row of A to the right by k places, where k is randomly chosen from 0, 1, 2, …,$n-1$. (For instance, if k=1, then the second row of B will be $(a_{2\,n}, a_{2\,1}, \cdots, a_{2\,n-1}$.) Thus a total of n^{m-1} different possible B's can be formed. Each of them has a P-sum less than or equal to that of A, according to Theorem 6. The sum of all the P-sums for these n^{m-1} is precisely

$$\prod_{i=1}^{m}\left(a_{i1}+a_{i2}+\cdots+a_{in}\right),$$

which should therefore be less than or equal to n^{m-1} times the P-sum of A, i.e. n^{m-1} times the direct P-sum. This gives us the desired result.