

Mathematical Excalibur

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Olympiad Corner

Below are the problems of the 2015 Canadian Mathematical Olympiad held in January 28, 2015.

Notation: If V and W are two points, then VW denotes the line segment with endpoints V and W as well as the length of this segment.

Problem 1. Let $\mathbb{N} = \{1, 2, 3, \dots\}$ denote the set of positive integers. Find all functions f , defined on \mathbb{N} and taking values in \mathbb{N} , such that $(n-1)^2 < f(n)f(f(n)) < n^2+n$ for every positive integer n .

Problem 2. Let ABC be an acute-angled triangle with altitudes AD , BE and CF . Let H be the orthocenter, that is, the point where the altitudes meet. Prove that

$$\frac{AB \cdot AC + BC \cdot BA + CA \cdot CB}{AH \cdot AD + BH \cdot BE + CH \cdot CF} \leq 2.$$

Problem 3. On a $(4n+2) \times (4n+2)$ square grid, a turtle can move between squares sharing a side. The turtle begins in a corner square of the grid and enters each square exactly once, ending in the square where she started.

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On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is **August 27, 2015**.

For individual subscription for the next five issues for the 14-15 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Tournament of the Towns

Kin Yin Li

In 1980, Kiev, Moscow and Riga participated in a mathematical problem solving contest for high school students, later called the *Tournament of the Towns*. At present thousands of high school students from dozens of cities all over the world participate in this contest. In this article, we present some very interesting math problems from this contest. At the end of the article, there are some information on where interested readers can find past problems and solutions of this contest.

Here are some examples we enjoy.

Example 1. (*Junior Questions, Spring 1981, proposed by A. Andjans*) Each of 64 friends simultaneously learns one different item of news. They begin to phone one another to tell them their news. Each conversation last exactly one hour, during which time it is possible for two friends to tell each other all of their news. What is the minimum number of hours needed in order for all of the friends to know all the news?

Solution. More generally, suppose there are 2^n friends. After n rounds, the most anyone can learn are 2^n pieces of gossip. Hence n rounds are necessary. We now prove by induction on n that n rounds are also sufficient. For $n=1$, the result is trivial. Suppose the result holds up to $n-1$ for some $n \geq 2$. Consider the next case with 2^n friends. Have them call each other impairs in the first round. After this, divide them into two groups, each containing one member from each pair who had exchanged gossip. Each group has 2^{n-1} friends who know all the gossip among them. By the induction hypothesis, $n-1$ rounds are sufficient for everyone within each group to learn everything. This completes the induction argument. In particular, with 64 friends, 6 rounds are both necessary and sufficient.

Example 2. (*Senior Questions, Spring 1983, proposed by A. Andjans*) There are K boys placed around a circle. Each of them has an even number of sweets. At a command each boy gives half of his sweets to the boy on his right. If, after that, any boy has an odd number of sweets, someone outside the circle gives him one more sweet to make the number even. This procedure can be repeated indefinitely. Prove that there will be a time at which all boys have the same number of sweets.

Solution. Suppose initially the maximum number of sweets a boy has is $2m$, and the minimum is $2n$. We may as well assume that $m > n$. After a round of exchange and possible augmentation, we claim that the most any boy can have is $2m$ sweets. This is because he could have kept at most m sweets, and received m more in the exchange, but will not be augmented if he already has $2m$ sweets.

On the other hand, at least one boy who had $2n$ sweets will have more than that, because as long as $m > n$, one of these boys will get more than he gives away. It follows that while the maximum cannot increase, the minimum must increase until all have the same number of sweets.

Example 3. (*Junior Questions, Autumn 1984*) Six musicians gathered at a chamber music festival. At each scheduled concert some of these musicians played while the others listened as members of the audience. What is the least number of such concerts which would need to be scheduled in order to enable each musician to listen, as a member of the audience, to all the other musicians?

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Solution. Let the musicians be A, B, C, D, E and F . Suppose there are only three concerts. Since each of the six must perform at least once, at least one concert must feature two or more musicians. Say both A and B perform in the first concert. They must still perform for each other. Say A performs in the second concert for B and B in the third for A . Now C, D, E and F must all perform in the second concert, since it is the only time B is in the audience. Similarly, they must all perform in the third. The first concert alone is not enough to allow C, D, E and F to perform for one another. Hence we need at least four concerts. This is sufficient, as we may have A, B and C in the first, A, D and E in the second, B, D and F in the third and C, E and F in the fourth.

Example 4. (Junior Questions, Autumn 1984, proposed by V. G. Ilichev) On the Island of Camelot live 13 grey, 15 brown and 17 crimson chameleons. If two different chameleons of different colours meet, they both simultaneously change colour to the third colour (eg. If a grey and a brown chameleon meet each other they both change to crimson). Is it possible that they will eventually all be the same colour?

Solution. In this case the numbers of chameleons of each colour at the start have remainders of 0, 1 and 2 when divided by three. Each “meeting” maintains such a situation (not necessarily in any order) as two of these remainders must either be reduced by 1 (or increased by 2) while the other must be increased by 2 (or reduced by 1). Thus at least two colours are present at any stage, guaranteeing the possibility of obtaining all of the three colours in fact by future meetings.

Note. The only way of getting chameleons to be of the same colour would be getting an equal number of two colours first. This would mean getting two with the same remainder on division by three. This would have been possible if we had started with, say 15 of each colour. From this position we can obtain sets with remainders equal to $\{0,0,0\}$, $\{1,1,1\}$ and $\{2,2,2\}$.

Example 5. (Junior Questions, Spring 1985, proposed by S. Fomin) There are 68 coins, each coin having a different

weight that that of each other. Show how to find the heaviest and lightest coin in 100 weighings on a balance beam.

Solution 1. First divide into 34 pairs and perform 34 weighings, each time identifying the heavier and lighter coins. Put all the heavier coins into one group and the lighter coins into another. Divide the group with heavier coins into 17 pairs, and perform 17 weighings on these to identify the 17 heavier coins. Continue this process with the group of heavier coins each time. If there is an odd number of coins at any stage, the odd coin out must be carried over to the following stage. There will be a total of $17+8+4+2+1+1=33$ such weighings required for identifying the heaviest coin.

A similar 33 weighings of the lighter group will identify the lightest coin. The total number of weighing is thus $34+33+33=100$, as required.

Solution 2. More generally, we show that $3n-2$ weighings are sufficient for $2n$ coins. We first divide the coins into n pairs, and use n weighings to sort them out into a “heavy” pile and a “light” pile. The heaviest coin is among the n coins in the “heavy” pile. Each weighing eliminates 1 coin. Since there are n coins, $n-1$ weighings are necessary and sufficient. Similarly, $n-1$ weighings will locate the lightest coin in the “light” pile. Thus the task can be accomplished in $3n-2$ weighings.

Example 6. (Junior Questions, Spring 1987, proposed by D. Fomin) A certain number of cubes are painted in six colours, each cube having six faces of different colours (the colours in different cubes may be arranged differently). The cubes are placed on a table so as to form a rectangle. We are allowed to take out any column of cubes, rotate it (as a whole) along its long axis and place it in a rectangle. A similar operation with rows is also allowed. Can we always make the rectangle monochromatic (i.e. such that the top faces of all the cubes are the same colour) by means of such operations?

Solution. The task can always be accomplished, and we can select the top colour in advance, say red. By fixing a cube, we mean bringing its red face to the top. Given a rectangular block, we fix one cube at a time, from left to right, and from front to back.

Suppose that the cube in the i -th row and the j -th column is the next to be fixed. Suppose that we need to rotate the i -th row. In order not to unfix the first $j-1$ cubes of this row, we rotate each of the first $j-1$ columns so that all red faces are to the left. They remain to the left when the i -th row is rotated. We can now refix the first $j-1$ columns.

Similarly, if we need to rotate the j -th column, we can go through an analogous three-step process.

Example 7. (Senior Questions, Autumn 1987, proposed by A. Andjans) A certain town is represented as an infinite plane, which is divided by straight lines into squares. The lines are streets, while the squares are blocks. Along a certain street there stands a policeman on each 100th intersection. Somewhere in the town there is a bandit, whose position and speed are unknown, but he can move only along the streets. The aim of the police is to see the bandit. Does there exist an algorithm available to the police to enable them to achieve their aim?

Solution. We assume that (a) there is no limit to how far a policeman can see along the street he is on; (b) there is no overall time limit, and (c) if the bandit is ever on the same street as a policeman he will be seen.

Let i, j and k denote integers, let the North-South streets be $x=i$ for all i , the East-West streets $y=j$ for all j and suppose the k -th policeman is at $(100k,0)$.

For all even k the k -th policeman remains stationary throughout. This traps the bandit in the infinite strip between $x=200k$ and $x=200(k+1)$ for some k , say k^* .

All other policemen first travel along $y=0$ towards $(0,0)$ until they reach the first cross street $x=s$ for which there is a policeman on every street $x=i$ for i between 0 and s . Police are to travel at regulation speed, say one block per minute, but nevertheless there will come a time, dependent only on k^* , when every street $x=i$ on the k^* strip will be policed.

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Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is **August 27, 2015.**

Problem 466. Let k be an integer greater than 1. If $k+2$ integers are chosen among $1, 2, 3, \dots, 3k$, then there exist two of these integers m, n such that $k < |m - n| < 2k$.

Problem 467. Let p be a prime number and q be a positive integer. Take any pq consecutive integers. Among these integers, remove all multiples of p . Let M be the product of the remaining integers. Determine the remainder when M is divided by p in terms of q .

Problem 468. Let $ABCD$ be a cyclic quadrilateral satisfying $BC > AD$ and $CD > AB$. E, F are points on chords BC, CD respectively and M is the midpoint of EF . If $BE = AD$ and $DF = AB$, then prove that $BM \perp DM$.

Problem 469. Let m be an integer greater than 4. On the plane, if m points satisfy no three of them are collinear and every four of them are the vertices of a convex quadrilateral, then prove that all m of the points are the vertices of a m -sided convex polygon.

Problem 470. If $a, b, c > 0$, then prove that

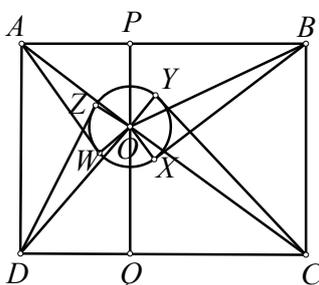
$$\frac{a}{b(a^2 + 2b^2)} + \frac{b}{c(b^2 + 2c^2)} + \frac{c}{a(c^2 + 2a^2)} \geq \frac{3}{ab + bc + ca}.$$

Solutions

Problem 461. Inside rectangle $ABCD$, there is a circle. Points W, X, Y, Z are on the circle such that lines AW, BX, CY, DZ are tangent to the circle. If $AW=3, BX=4, CY=5$, then find DZ with proof.

Solution. Adnan ALI (Atomic Energy Central School 4, Mumbai, India), Adithya BHASKAR (Atomic Energy School 2, Mumbai, India), Ioan Viorel CODREANU (Secondary School Satulung, Maramures, Romania), DHRUV Nevatia (10th Standard, Ramanujan Academy, Nashik, India), KWOK Man Yi (Baptist Lui Ming Choi Secondary School, S4), LKL Excalibur (Madam Lau Kam Lung Secondary School

FANCHINI (Cantú, Italy), William FUNG, KWOK Man Yi (Baptist Lui Ming Choi Secondary School, S4), Jon GLIMMS, LKL Excalibur (Madam Lau Kam Lung Secondary School of MFBM), Corneliu MĂNESCU-AVRAM ("Henri Mathias Berthelot" Secondary School, Ploiești, Romania), MANOLOUDIS Apostolos (4 High School of Korydallos, Piraeus, Greece), Vijaya Prasad NALLURI (Retired Principal, AP Educational Service, India), Alex Kin-Chit O (G.T. (Ellen Yeung) College), Toshihiro SHIMIZU (Kawasaki, Japan), Titu ZVONARU (Comănești, Romania) and Neculai STANCIU ("George Emil Palade" Secondary School, Buzău, Romania).



Let r be the radius of the circle. By Pythagoras' theorem, we have

$$r^2 = AW^2 - AO^2 = BX^2 - BO^2 = CY^2 - CO^2 = DZ^2 - DO^2. \quad (*)$$

Let P, Q be the feet of perpendiculars from O to AB, CD respectively. Then

$$AO^2 - BO^2 = (AP^2 + PO^2) - (BP^2 + PO^2) = (DQ^2 + QO^2) - (CQ^2 + QO^2) = DO^2 - CO^2.$$

Using (*), we get $AW^2 - BX^2 = AO^2 - BO^2 = DO^2 - CO^2 = DZ^2 - CY^2$. Then

$$DZ = \sqrt{AW^2 - BX^2 + CY^2} = 3\sqrt{2}.$$

Problem 462. For all $x_1, x_2, \dots, x_n \geq 0$, let $x_{n+1} = x_1$, then prove that

$$\sum_{k=1}^n \sqrt{\frac{1}{(x_k + 1)^2} + \frac{x_{k+1}^2}{(x_{k+1} + 1)^2}} \geq \frac{n}{\sqrt{2}}.$$

Solution. Adnan ALI (Atomic Energy Central School 4, Mumbai, India), Adithya BHASKAR (Atomic Energy School 2, Mumbai, India), Ioan Viorel CODREANU (Secondary School Satulung, Maramures, Romania), DHRUV Nevatia (10th Standard, Ramanujan Academy, Nashik, India), KWOK Man Yi (Baptist Lui Ming Choi Secondary School, S4), LKL Excalibur (Madam Lau Kam Lung Secondary School

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By squaring both sides or RMS-AM inequality, we have for all $a, b \geq 0$,

$$\sqrt{a^2 + b^2} \geq \frac{a + b}{\sqrt{2}}.$$

Applying this, we get

$$\begin{aligned} & \sum_{k=1}^n \sqrt{\frac{1}{(x_k + 1)^2} + \frac{x_{k+1}^2}{(x_{k+1} + 1)^2}} \\ & \geq \sum_{k=1}^n \frac{1}{\sqrt{2}} \left(\frac{1}{x_k + 1} + \left(1 - \frac{1}{x_{k+1} + 1}\right) \right) = \frac{n}{\sqrt{2}}. \end{aligned}$$

Problem 463. Let S be a set with 20 elements. N 2-element subsets of S are chosen with no two of these subsets equal. Find the least number N such that among any 3 elements in S , there exist 2 of them belong to one of the N chosen subsets.

Solution. Jon GLIMMS, KWOK Man Yi (Baptist Lui Ming Choi Secondary School, S4), LKL Excalibur (Madam Lau Kam Lung Secondary School of MFBM) and Toshihiro SHIMIZU (Kawasaki, Japan).

Let $x \in S$ be contained in k of the N 2-elements subsets of S , where k is least among the elements of S .

Let x_1, x_2, \dots, x_k be the other elements in k of the N 2-element subsets with x . As k is least, so each of the x_i 's is also contained in at least k of the N 2-element subsets of S .

Also, there are $m = 19 - k$ elements $w_1, w_2, \dots, w_m \in S$ not in any of the N 2-element subsets of S with x . For

every pair w_r, w_s of these, $\{w_r, w_s\}$ is one of these N 2-element subsets of S (otherwise, no two of x, w_r, w_s form one of the N 2-element subsets). Then

$$N \geq \binom{k+1}{2} + \binom{19-k}{2} = (k-9)^2 + 90 \geq 90.$$

To get the least case of $N=90$, we divide the 20 elements into two groups of 10 elements. Then take all 2-element subsets of each of the two groups to get $45+45=90$ 2-element subsets of S .

Problem 464. Determine all positive integers n such that for n , there exists an integer m with $2^n - 1$ divides $m^2 + 289$.

Solution. Adnan ALI (Atomic Energy Central School 4, Mumbai, India), Adithya BHASKAR (Atomic Energy School 2, Mumbai, India), KWOK Man Yi (Baptist Lui Ming Choi Secondary School, S4), LKL Excalibur (Madam Lau Kam Lung Secondary School of MFBM), Corneliu MĂNESCU-AVRAM ("Henri Mathias Berthelot" Secondary School, Ploiești, Romania), PANG Lok Wing and Toshihiro SHIMIZU (Kawasaki, Japan).

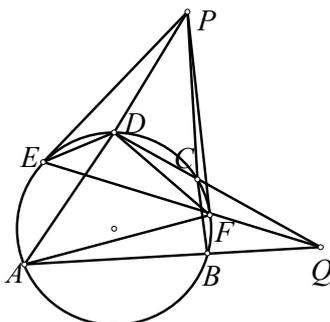
The case $n=1$ is a solution. For $n>1$, we first show if a prime q of the form $4k+3$ divides a^2+b^2 , then q divides a and b . Assume $\gcd(q,a)=1$. Let $c=a^{q-2}$. Then by Fermat's little theorem, $ac=a^{q-1} \equiv 1 \pmod{q}$. As $q|a^2+b^2$, so $b^2 \equiv -a^2 \pmod{q}$. Then $(bc)^2 \equiv -(ac)^2 \equiv -1 \pmod{q}$ and $(bc)^{q-1} = (bc)^{2(2k+1)} \equiv -1 \pmod{q}$, contradicting Fermat's little theorem. So q divides a (and b similarly).

If $n>1$, then $2^n - 1 \equiv 3 \pmod{4}$. Hence $2^n - 1$ has a prime divisor $q \equiv 3 \pmod{4}$. By the fact above, q divides $m^2 + 289$ implies q divides m and 17. Then $q=17 \not\equiv 3 \pmod{4}$, contradiction.

Problem 465. Points A, E, D, C, F, B lie on a circle Γ in clockwise order. Rays AD, BC , the tangents to Γ at E and at F pass through P . Chord EF meets chords AD and BC at M and N respectively. Prove that lines AB, CD, EF are concurrent.

Comments. A number of solvers pointed out if lines AB, CD are parallel, then by symmetry lines AB, CD, EF are all parallel. So below, we present solutions for the case when lines AB and CD intersect at a point.

Solution 1. Jon GLIMMS.



Let lines AB, CD meet at Q . We have

- (1) $\angle AFE = \angle ADE = 180^\circ - \angle PDE$,
- (2) $\angle EFD = \angle PED$,
- (3) $\angle FDQ = \angle PFC$,
- (4) $\angle QAF = \angle FCB = 180^\circ - \angle PCF$.
- (5) $\angle DAQ = \angle DCP$,
- (6) $\angle QDA = 180^\circ - \angle PDC$.

Then

$$\frac{\sin \angle AFE}{\sin \angle EFD} = \frac{\sin \angle PDE}{\sin \angle PED} = \frac{PE}{PD},$$

$$\frac{\sin \angle FDQ}{\sin \angle QAF} = \frac{\sin \angle PFC}{\sin \angle PCF} = \frac{PC}{PF},$$

$$\frac{\sin \angle DAQ}{\sin \angle QDA} = \frac{\sin \angle DCP}{\sin \angle PDC} = \frac{PD}{PC}.$$

Multiplying these and using $PE=PF$, we have

$$\frac{\sin \angle AFE \cdot \sin \angle FDQ \cdot \sin \angle DAQ}{\sin \angle EFD \cdot \sin \angle QAF \cdot \sin \angle QDA} = \frac{PE}{PD} \cdot \frac{PC}{PF} \cdot \frac{PD}{PC} = 1.$$

Applying the converse of the trigonometric form of Ceva's theorem to $\triangle ADF$ and point Q , we get lines AB, CD, EF are concurrent at Q .

Solution 2. Adnan ALI (Atomic Energy Central School 4, Mumbai, India), Adithya BHASKAR (Atomic Energy School 2, Mumbai, India) and William FUNG.

Since the tangents to Γ at E and at F intersect at P , line EF is the polar of P . Since lines AD, BC intersect at P , the polar of P (that is, line EF) passes through the intersection of lines AB and CD .

Other commended solvers: KWOK Man Yi (Baptist Lui Ming Choi Secondary School, S4), MANOLOUDIS Apostolos (4 High School of Korydallos, Piraeus, Greece) and Toshihiro SHIMIZU (Kawasaki, Japan).

Olympiad Corner

(Continued from page 1)

Problem 3 (Cont'd). In terms of n , what is the largest positive integer k such that there must be a row or a column that the turtle has entered at least k distinct times?

Problem 4. Let ABC be an acute-angled triangle with circumcenter O . Let Γ be a circle with centre on the altitude from A in ABC , passing through vertex A and points P and Q on sides AB and AC . Assume that $BP \cdot CQ = AP \cdot AQ$. Prove that Γ is tangent to the circumcircle of triangle BOC .

Problem 5. Let p be a prime number for which $(p-1)/2$ is also prime, and let a, b, c be integers not divisible by p . Prove that there are at most $1 + \sqrt{2p}$ positive integers n such that $n < p$ and p divides $a^n + b^n + c^n$.

Tournament of the Towns

(Continued from page 2)

When this happens the bandit will be trapped on some street $y=j^*$, on a single block between $x=i^*$ and $x=i^*+1$ for some i^* .

For each k , as soon as all streets on the k -th strip are policed, one of the policemen travels north and another travels south. For $k=k^*$ this will inevitably reveal the bandit.

After reading these examples, should anyone want to read more, below are websites, which books on this contest can be ordered or problems and solutions of the recent Tournament of the Towns can be found.

www.amtt.com.au/ProductList.php?page=1&startpage=1

www.artofproblemsolving.com/community/c3239_tournament_of_towns

www.math.toronto.edu/oz/turgor/archives.php