Mathematical Excalibur

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Olympiad Corner

Below are the problems of the *32nd Balkan Mathematical Olympiad* held in May 5, 2015.

Problem 1. Let a, b and c be positive real numbers. Prove that

$$a^3b^6+b^3c^6+c^3a^6+3a^3b^3c^3 \geq abc(a^3b^3+b^3c^3+c^3a^3)+a^2b^2c^2(a^3+b^3+c^3).$$

Problem 2. Let ABC be a scalene triangle with incenter I and circum-circle (ω). The lines AI, BI, CI intersect (ω) for the second time at the point D, E, F, respectively. The line through I parallel to the sides BC, AC, AB intersect the lines EF, DF, DE at the points K, L, M, respectively. Prove that the points K, L, M are collinear.

Problem 3. A jury of 3366 film critics is judging the Oscars. Each critic makes a single vote for his favorite actor, and a single vote for his favorite actress. It turns out that for every integer $n\in\{1,2,\ldots,100\}$ there is an actor or actress who has been voted for exactly n times. Show that there are two critics who voted for the same actor and the same actress.

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The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***January 7, 2016***.

For individual subscription for the next five issues for the 14-15 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Divisibility Problems

Kin Y. Li

Divisibility problems are common in many math competitions. Below we will look at some of these interesting problems. As usual, for integers a and b with $a\neq 0$, we will write $a \mid b$ to denote <u>b is divisible by a</u> (or in short <u>a divides b</u>).

In dividing b by a, we get a quotient q and a remainder r, we get $b/a=q+r/a$. Notice that b/a is an integer if and only if r/a is an integer. The following examples exploit this observation.

<u>*Example 1.*</u> (1999 *AIME*) Find the greatest positive integer n such that $(n-2)^2(n+1)/(2n-1)$ is an integer.

<u>*Solution.*</u> The numerator is n^3-3n^2+4. So

$$\frac{n^3-3n^2+4}{2n-1}=\frac{1}{2}n^2-\frac{5}{4}n-\frac{5}{8}+\frac{27/8}{2n-1}.$$

Multiplying by 8, we get

$$\frac{8(n^3-3n^2+4)}{2n-1}=4n^2-10n-5+\frac{27}{2n-1}.$$

Then $2n-1|27$. The greatest such n is 14.

<u>*Example 2.*</u> (*1998 IMO*) Determine all pairs (a,b) of positive integers such that ab^2+b+7 divides a^2b+a+b.

<u>*Solution.*</u> We can think of a as a variable and b as a constant, then do division of polynomials to get

$$\frac{a^2b+a+b}{ab^2+b+7}=\frac{1}{b}a-\frac{7a/b-b}{ab^2+b+7}.$$

Multiplying by $b(ab^2+b+7)$, we get

$$b(a^2b+a+b)= (ab^2+b+7)a-(7a-b^2).$$

If $ab^2+b+7 \mid a^2b+a+b$, then

$$ab^2+b+7 \mid \ (ab^2+b+7)a-b(a^2b+a+b) =7a-b^2. \quad (*)$$

<u>*Case 1* $(7a-b^2=0)$.</u> Then $7a=b^2$. So $7\mid b$. Then for some positive integer k, $b=7k$ and $a=7k^2$. We can check $(a,b)=(7k^2,7k)$ are indeed solutions.

<u>*Case 2* $(7a-b^2<0)$.</u> Then $7a < b^2$ and

$$ab^2+b+7 \leq |7a-b^2| = b^2-7a.$$

However, $b^2-7a < b^2 < ab^2+b+7$, which leads to a contradiction.

<u>*Case 3* $(7a-b^2>0)$.</u> Then $ab^2+b+7 \leq 7a-b^2$. If $b \geq 3$, then $ab^2+b+7 \geq 9a > 7a > 7a-b^2$, contradicting (*).

So b = 1 or 2. If b = 1, then (*) yields $a+8 \mid 7a-1 = 7(a+8)-57$. Hence, $a+8 \mid 57$, which leads to a = 11 or 49. Then we can check (a,b) = (11,1) and (49,1) are solutions. If b=2, then (*) yields $4a+9 \mid 7a-4$. Now

$$4a+9 \leq 7a-4 < 8a+18 = 2(4a+9).$$

So $4a+9 = 7a-4$, contradicting a is an integer.

<u>*Example 3.*</u> (*2003* IMO) Determine all pairs of positive integers (a,b) such that $a^2/(2ab^2-b^3+1)$ is a positive integer.

<u>*Solution.*</u> Suppose $a^2/(2ab^2-b^3+1)=k$ is a positive integer. Then $a^2-2kb^2a+kb^3-k = 0$. Multiplying by 4 and completing squares, we get

$$(2a-2kb^2)^2 = (2kb^2-b)^2 + (4k-b^2). \quad (**)$$

Let $M = 2a-2kb^2$ and $N = 2kb^2-b$.

<u>*Case 1* $(4k-b^2 = 0)$.</u> Then b is even and $M = \pm N$. If $M=-N$, then $b=2a$. If $M=N$, then $2a = 4kb^2-b = b^4-b$. We get $(a,b) = (b/2,b)$ or $((b^4-b)/2,b)$ with b even. These are easily checked to be solutions.

<u>*Case 2* $(4k-b^2 > 0)$.</u> Then $M^2 > N^2$ and $N = 2kb^2-b = b(2kb-1) \geq 1(2-1) = 1$. So $M^2 \geq (N+1)^2$. Hence, by (**)

$$\begin{aligned}4k-b^2 &= M^2 - N^2\\ &\geq (N+1)^2-N^2=2N+1\\ &= 4kb^2-2b+1,\end{aligned}$$

which implies $4k(b^2-1) + (b-1)^2 \leq 0$.

(continued on page 2)

Then $b = 1$, $k = a/2$ and $(a,b) = (2k,1)$ are easily checked to be solutions for all positive integer k.

Case 3 ($4k-b^2 < 0$). Then $M^2 \le (N-1)^2$. By (**),

$$\begin{aligned}4k-b^2 &= M^2 - N^2\\ &\le (N-1)^2-N^2=-2N+1\\ &= -4kb^2+2b+1.\end{aligned}$$

This implies

$$\begin{aligned}0 &\le (1-4k)b^2+2b+(1-4k)\\ &= (1-4k)\left(b+\frac{1}{1-4k}\right)^2+\frac{8k(2k-1)}{1-4k}<0,\end{aligned}$$

which is a contradiction.

Exercise 1. Find all positive integers *n, a,* and *b* such that

$$n^b-1 \mid n^a+1.$$

For divisibility problems involving exponential terms, like 2^n, often we will need to do modulo arithmetic and apply Fermat's little theorem. A useful fact is *if $m>n\ge 0$, then there exist integers s, t such that $\gcd(m,n) = ms+nt$.* (*Proof.* If $n=0$, then let $s=1$, $t=0$. Suppose it is true for all r with $0 \le r < n$. Then $m=qn+r$, where $q=[m/n]$. We have

$$\begin{aligned}\gcd(m,n) &= \gcd(m,r) = ms+rt\\ &= ms+(m-qn)t = m(s+t)+n(-qt).\ )\end{aligned}$$

So if $d = \gcd(m,n)$ and a^m, $a^n \equiv 1 \pmod k$, then $a^d \equiv 1 \pmod k$ by the fact.

Example 4. (*1972 Putnam Exam*) Show that if n is an integer greater than 1, then 2^n-1 is not divisible by n.

Solution. Assume there exists an integer $n > 1$ such that $n \mid 2^n-1$. Since 2^n-1 is odd, n must be odd. Let p be the least prime divisor of n. Then $p \mid 2^n-1$, which is the same as $2^n\equiv 1 \pmod p$. By Fermat's little theorem, $2^{p-1}\equiv 1 \pmod p$. Let $d = \gcd(n,p-1)$. Then $2^d \equiv 1 \pmod p$. By the definition of p, since $d \mid n$ and $d \le p-1 < p$, we get $d = 1$. Then $2 = 2^d \equiv 1 \pmod p$ lead to a contradiction.

Having seen the last example, here comes a hard problem that one needs to know the last example to get a start.

Example 5. (*1990 IMO*) Determine all integer $n>1$ such that $(2^n+1)/n^2$ is an integer.

Solution. Since 2^n+1 is odd, n must be odd. Let p be the least prime divisor of n. Then $p|2^n+1$, which implies $(2^n)^2\equiv(-1)^2 = 1 \pmod p$. By Fermat's little theorem, $2^{p-1}\equiv 1 \pmod p$. Let $d = \gcd(2n,p-1)\ge 2$. Then $2^d\equiv 1 \pmod p$. By the definition of p, we get $\gcd(n,p-1)=1$. This gives $d=2$ and $4=2^d\equiv 1 \pmod p$ gives $p=3$. Then $n = 3^km$ for some $k \ge 1$ and m satisfying $\gcd(3,m)=1$.

Using $x^3+1=(x+1)(x^2-x+1)$ for $x=2^m$, 2^{3m}, 2^{9m}, …, we have

$$2^n+1=(2^m+1)\prod_{j=0}^{k-1}(2^{2\cdot 3^j m}-2^{3^j m}+1).\quad (*)$$

For odd c, $2^c\equiv 2,-1,-4 \pmod 9$ implies $2^{2c}-2^c+1\equiv 3 \pmod 9$. From the binomial expansion, we see $2^m+1 = (3-1)^m+1 \equiv 3m \equiv 3$ or $6 \pmod 9$. So each of the factor on the right side of (*) is divisible by 3, but not by 9. So $2^n+1=3^{k+1}s$ for some integer s satisfying $\gcd(3,s)=1$. Now $n^2 = 3^{2k}m^2 \mid 2^n+1 = 3^{k+1}s$, which implies $k=1$ and $n=3m$.

Assume $m>1$. Let q be the least prime divisor of m. Now q is odd and $q>3$. Then $\gcd(m,q-1)=1$. Since $q \mid m \mid n$, we have $q^2 \mid n^2 \mid 2^n+1$. Then 2^{q-1} and $2^{2n}\equiv 1 \pmod q$ lead to $2^w\equiv 1 \pmod q$, where $w = \gcd(2n, q-1)$. Then $w|2n=6m$. Also, from $w|q-1$ and $\gcd(m,q-1) = 1$, we get $w|6$. Now $q>3$, $w=1,2,3,6$ and $2^w\equiv 1 \pmod q$ imply $q=7$. Then $7=q \mid 2^n+1$, but $2^n\equiv 1, 2, 4\not\equiv -1 \pmod 7$, contradiction. Therefore, $m=1$ and $n=3$. Indeed, $3^2=9 \mid 2^3+1$.

Exercise 2. (1999 *IMO*) Find all pairs of positive integers (x,p) such that p is prime, $x\le 2p$, and x^{p-1} divides $(p-1)^x+1$.

In the following examples, we will see there is a very clever trick in solving certain divisibility problems.

Example 6. (*1988 IMO*) Let a and b positive integers such that $ab+1$ divides a^2+b^2. Show that $(a^2+b^2)/(ab+1)$ is square of an integer.

Solution. Let $k = (a^2+b^2)/(ab+1)$. Assume there is a case k is an integer, but not a square. Among all such cases, consider the case when $\max\{a,b\}$ is least possible. Note $a=b$ implies $0 < k = 2a^2/(a^2+1) < 2$, which implies $k=1=1^2$. So in the least case, $a\ne b$, say $a>b$. Now $k = (a^2+b^2)/(ab+1)>0$ and it can be rewritten as $a^2-kba+b^2-k=0$. Note $k\ne b^2$ implies $a\ne 0$.

Other than a, let c be the second root of $x^2-kbx+b^2-k=0$. Then $k = (c^2+b^2)/(cb+1)$, $a+c=kb$ and $ac=b^2-k$. So $c=kb-a=(b^2-k)/a$ is an integer. Now $cb+1=(c^2+b^2)/k > 0$ and $c=(b^2-k)/a\ne 0$ imply c is a positive integer. Finally, $c = (b^2-k)/a < (a^2-k)/a < a$. Now $k = (c^2+b^2)/(cb+1)$ is an integer, not a square and $\max\{b,c\} < a = \max\{a,b\}$. This contradicts $\max\{a,b\}$ is the least.

Example 7. (*2007 IMO*) Let a and b be positive integers. Show that if $4ab-1$ divides $(4a^2-1)^2$, then $a=b$.

Solution. We can consider a as variable and b as constant to do a division as in example 2, but a nicer way is as follows: from $(4a^2-1)b=a(4ab-1)+(a-b)$, we get

$$(4a^2-1)^2b^2 = J(4ab-1)+(a-b)^2,$$

where $J=a^2(4ab-1)+2a(a-b)$. Observe that $\gcd(b^2,4ab-1) =1$ (otherwise prime $p \mid \gcd(b^2,4ab-1)$ would imply $p|b$ and $p|4ab-(4ab-1)=1$). Hence,

$$4ab-1|(4a^2-1)^2 \Leftrightarrow 4ab-1|\ (a-b)^2.$$

Now $k= (a-b)^2/(4ab-1) > 0$ and it can be rewritten as $a^2-(4k+2)ba+b^2+k=0$.

Assume there exists (a,b) such that k is an integer and $a\ne b$, say $a>b$. Among all such cases, consider the case when $a+b$ is least possible.

Other than a, let c be the second root of $x^2-(4k+2)bx+b^2+k=0$. Then $k = (c-b)^2/(4cb-1)$, $a+c = (4k+2)b$ and $ac = b^2+k$. So $c = (4k+2)b-a = (b^2+k)/a$ is a positive integer. So (c,b) is another case k is an integer. Since $a+b$ is least possible, we would have $c \ge a > b$. Now $c = (b^2+k)/a \ge a$ leads to $k \ge a^2-b^2$. Then

$$(a-b)^2 = k(4ab-1) \ge (a^2-b^2)(4ab-1).$$

Canceling $a-b$ on both sides, we get

$$a-b \ge (a+b)(4ab-1) > a,$$

a contradiction.

The next example is short and cute.

Example 8. (*2005 IMO Shortlisted Problem*) Let a and b be positive integers such that a^n+n divides b^n+n for every positive integer n. Show that $a=b$.

Solution. Assume $a\ne b$. For $n = 1$, we have $a+1|b+1$ and so $a < b$. Let p be a prime greater than b. Then let $n = (a+1)(p-1)+1$. By Fermat's little theorem, $a^n=(a^{p-1})^{a+1}a\equiv a \pmod p$.

So $a^n+n \equiv a+n = (a+1)p \equiv 0 \pmod p$. Then $p \mid a^n+n \mid b^n+n$. By Fermat's little theorem,

$$0\equiv b^n+n=(b^{p-1})^{a+1}b+n\equiv b-a \pmod p,$$

which contradicts $0 < a < b < p$.

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Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***January 7, 2016.***

Problem 476. Let p be a prime number. Define sequence a_n by $a_0=0$, $a_1=1$ and $a_{k+2}=2a_{k+1}-pa_k$. If one of the terms of the sequence is -1, then determine all possible value of p.

Problem 477. In $\triangle ABC$, points D, E are on sides AC, AB respectively. Lines BD, CE intersect at a point P on the bisector of $\angle BAC$.

Prove that quadrilateral $ADPE$ has an inscribed circle if and only if $AB=AC$.

Problem 478. Let a and b be a pair of coprime positive integers of opposite parity. If a set S satisfies the following conditions:

(1) $a, b \in S$;
(2) if $x,y,z\in S$, then $x+y+z\in S$,

then prove that every positive integer greater than $2ab$ belongs to S.

Problem 479. Prove that there exists infinitely many positive integers k such that for every positive integer n, the number $k2^n+1$ is composite.

Problem 480. Let m, n be integers with $n > m > 0$. Prove that if $0 < x < \pi/2$, then

$$2|\sin^n x-\cos^n x| \le 3|\sin^m x-\cos^m x|.$$

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Solutions

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Problem 471. For $n \ge 2$, let $A_1, A_2, \ldots, A_n$ be positive integers such that $A_k \le k$ for $1 \le k \le n$. Prove that $A_1+A_2+\cdots+A_n$ is even if and only if there exists a way of selecting + or – signs such that

$$A_1 \pm A_2 \pm \cdots \pm A_n = 0.$$

Solution. **Adithya BHASKAR** (Atomic Energy School 2, Mumbai, India), **Jon GLIMMS** and **Toshihiro SHIMIZU** (Kawasaki, Japan).

If $A_1 \pm A_2 \pm \cdots \pm A_n = 0$, then using $A_i \equiv \pm A_i \pmod 2$, we get $A_1+A_2+\cdots+A_n \equiv 0 \pmod 2$. Hence $A_1+A_2+\cdots+A_n$ is even.

Conversely, we will prove by induction that for t from n to 1 that there exists a way of selecting signs so that

$$0 \le S_t = \pm A_t \pm A_{t+1} \pm \cdots \pm A_n \le t.$$

The case $t=n$ is $0 < A_n \le n$. Suppose the case $t=k$ is true, that is

$$0 \le S_k = \pm A_k \pm A_{k+1} \pm \cdots \pm A_n \le k.$$

If $A_{k-1} \le S_k$, then let $S_{k-1} = -A_{k-1}+S_k$ and we have $0 \le S_{k-1} = S_k - A_{k-1} \le k-1$. If $A_{k-1} > S_k$, then let $S_{k-1} = A_{k-1} - S_k$ (here $-S_k$ means reversing all the signs of S_k) and we have $0 \le S_{k-1} \le A_{k-1} \le k-1$. This completes the induction.

The case $t=1$ gives us $0 \le \pm A_1 \pm A_2 \pm \cdots \pm A_n \le 1$. As $\pm A_1 \pm A_2 \pm \cdots \pm A_n$ is an even integer, $\pm A_1 \pm A_2 \pm \cdots \pm A_n = 0$.

Problem 472. There are $2n$ distinct points marked on a line, n of them are colored red and the other n points are colored blue. Prove that the sum of the distances of all pairs of points with same color is less than or equal to the sum of the distances of all pairs of points with different color.

Solution. **Jon GLIMMS**, **Toshihiro SHIMIZU** (Kawasaki, Japan) and **Raul A. SIMON** (Chile).

Let the points be on the real axis with red points having coordinates $x_1<x_2<\cdots<x_n$ and the blue points having coordinates $y_1<y_2<\cdots<y_n$. Let S_n denote the sum of distances of all pairs of points with same color and D_n denote the sum of distances of all pairs of points with different color. We will prove $S_i \le D_i$ for all i by induction. Now $S_1=0\le|x_1-y_1|=D_1$. Suppose $S_n \le D_n$. For case $n+1$,

$$S_{n+1}-S_n = \sum_{i=1}^{n}(x_{n+1}-x_i)+(y_{n+1}-y_i)$$

$$\le |x_{n+1}-y_{n+1}| + \sum_{i=1}^{n}|x_{n+1}-y_i| + |y_{n+1}-x_i|$$

$$= D_{n+1}-D_n.$$

Then $S_{n+1}-D_{n+1} \le S_n - D_n \le 0$. So $S_{n+1} \le D_{n+1}$.

Problem 473. Determine all functions f: $\mathbb{R} \to \mathbb{R}$ such that for all $x,y\in\mathbb{R}$,

$$f(x)\,f(y\,f(x)-1) = x^2 f(y) - f(x).$$

Solution. **Coco YAU** (Pui Ching Middle School).

The zero function is a solution. Suppose f is a solution that is not the zero function. Then there exists $a\in\mathbb{R}$ such that $f(a)\neq 0$. Denote the functional equation by (*). Setting $x=0$ in (*), we get

$$f(0)(\,f(yf(0)-1)+1\,)=0.$$

If $f(0)\neq 0$, then $f(yf(0)-1) = -1$. Since $\{yf(0)-1: y\in\mathbb{R}\}=\mathbb{R}$, we can see f is the constant function -1. Then (*) with $x=1$ yields $(-1)^2 = -1^2+1$, which is a contradiction. So $f(0)=0$.

Now setting $x=a$, $y=0$ in (*), we can get

$$f(-1) = -1.$$

Also, if $f(b)=0$, then setting $x=b$ and $y=a$, we get $b=0$. Hence,

$$f(x) = 0 \Leftrightarrow x = 0.$$

Next by setting $x=y=1$ in (*), we get $f(1)f(f(1)-1)=0 \Leftrightarrow f(1)-1=0 \Leftrightarrow f(1)=1$.

Setting $x=1$ in (*), we get

$$f(y-1)=f(y)-1. \qquad (1)$$

Applying (1) to $f(yf(x)-1)$ in (*), we can simplify (*) to

$$f(x)\,f(yf(x)) = x^2 f(y). \qquad (2)$$

Setting $x=-1$ in (2), we get $-f(-y)=f(y)$. So f is an odd function.

Applying induction to (1), we get for $n = 1,2,3,\ldots$,

$$f(y-n) = f(y)-n. \qquad (3)$$

Setting $y=0$, this gives $f(-n)=-n$. As f is odd, we get $f(n)=n$ for all integers n. Setting $x=n$ in (2), we get

$$f(ny) = nf(y). \qquad (4)$$

Setting $y=1/n$ and $y=1/m$ we get $1=nf(1/n)$ and $f(n/m)=nf(1/m)=n/m$. So $f(x) = x$ for all rational x.

Setting $y=1$ in (2), we get

$$f(x)\,f(\,f(x)) = x^2. \qquad (5)$$

Setting x,y to be $f(x)$ in (2), we also get

$$f(f(x))\,f(\,f(x)\,f(f(x)\,) = f(x)^2 f(f(x)).$$

Cancelling $f(f(x))$ on both sides, we get

$$f(x)^2 = f(\,f(x)\,f(f(x))) = f\,(x^2),$$

where the second equality follows from applying f to both sides of (5). Then we see $w>0$ implies $f(w)>0$.

For irrational $w > 0$, assume $f(w) > w$. Take rational $q=n/m$ such that $m>0$ and $f(w) > q > w$. We have $m(q-w) > 0$. So $f(n-mw) = f(m(q-w)) > 0$. As f is odd, using (4) and (3), we get

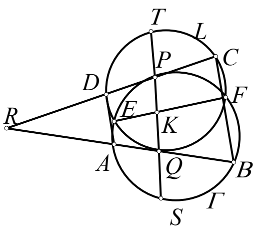
$$mf(w)-n = f(mw)-n = f(mw-n) < 0,$$

which contradicts $f(w) > q$. Similarly, $f(w) < w$ will lead to a contradiction. Therefore, $f(w)=w$ for all w and we can check (*) holds in this case.

Other commended solvers: **Toshihiro SHIMIZU** (Kawasaki, Japan).

Problem 474. Quadrilateral *ABCD* is convex and lines *AB, CD* are not parallel. Circle Γ passes through *A, B* and side *CD* is tangent to Γ at *P*. Circle *L* passes through *C, D* and side *AB* is tangent to *L* at *Q*. Circles Γ and *L* intersect at *E* and *F*. Prove that line *EF* bisects line segment *PQ* if and only if lines *AD, BC* are parallel.

Solution. **Jon GLIMMS** and **Toshihiro SHIMIZU** (Kawasaki, Japan).



Let *EF* meet *PQ* at *K*. Extend *PQ* to meet Γ and *L* at *S* and *T* respectively. Let lines *AB, CD* meet at *R*. We have

$$RP^2=RA\cdot RB \text{ and } RQ^2=RC\cdot RD. \quad (*)$$

By the intersecting chord theorem, we have $KP\cdot KS=KE\cdot KF=KQ\cdot KT$. Then $KP(KQ+QS) = KQ(KP+PT)$. Cancel $KP\cdot KQ$. We have

$$KP\cdot QS= KQ\cdot PT.$$

Then

$$\begin{aligned}&KP=KQ\\ \Leftrightarrow\ &QS=PT\\ \Leftrightarrow\ &PQ\cdot QS=QP\cdot PT\\ \Leftrightarrow\ &AQ\cdot QB=DP\cdot PC.\end{aligned}$$

Using $AQ=RQ-RA$, $QB=RB-RQ$, $DP=RP-RD$, $PC=RC-RP$ and (*), we get

$$\begin{aligned}&AQ\cdot QB=DP\cdot PC\\ \Leftrightarrow\ &RQ(RA+RB)=RP(RC+RD)\\ \Leftrightarrow\ &RC\cdot RD(RA+RB)^2=RA\cdot RB(RC+RD)^2\\ \Leftrightarrow\ &(RA\cdot RC-RB\cdot RD)^2=0\\ \Leftrightarrow\ &\frac{RA}{RB}=\frac{RD}{RC}\\ \Leftrightarrow\ &AD \parallel BC.\end{aligned}$$

Problem 475. Let *a, b, n* be integers greater than 1. If b^n-1 is a divisor of *a*, then prove that in base *b*, *a* has at least *n* digits not equal to zero.

Solution. **Jon GLIMMS** and **Toshihiro SHIMIZU** (Kawasaki, Japan).

Among all numbers that are multiples of b^n-1, suppose the least number of nonzero digits in base *b* of these numbers is *s*. Let *A* be one of these numbers with least digit sum, say

$$A = a_1b^{n_1} + a_2b^{n_2} + \cdots + a_sb^{n_s},$$

where $n_1 > n_2 > \cdots > n_s \ge 0$ and $1 \le a_i < b$ for $i=1,2,\cdots,s$.

Assume there are i,j such that $1\le i<j\le s$ and $n_i\equiv n_j\equiv r \pmod n$ with $0\le r\le n-1$. Then consider

$$B = A - a_ib^{n_i} - a_jb^{n_j} + (a_i + a_j)b^{nn_1+r}.$$

From $b^n \equiv 1 \pmod{b^n-1}$, we get $B \equiv 0 \pmod{b^n-1}$. If $a_i+a_j<b$, then the number of nonzero digits of *B* in base *b* is $s-1$, contradicting the choice of *A*. So we must have $b \le a_i+a_j < 2b$. Let $a_i+a_j = b+q$, where $0\le q<b$. Then

$$\begin{aligned}B = b^{nn_1+r+1} &+ qb^{nn_1+r} + a_1b^{n_1} + \cdots\\ &+ a_{i-1}b^{n_i-1} + a_{i+1}b^{n_{i+1}} + \cdots\\ &+ a_{j-1}b^{n_{j-1}} + a_{j+1}b^{n_{j+1}} + \cdots + a_sb^{n_s}.\end{aligned}$$

Then the digit sum of *B* is

$$\begin{aligned}&\sum_{k=1}^{s} a_k - (a_i + a_j) + 1 + q\\ &= \sum_{k=1}^{s} a_k + 1 - b\\ &< \sum_{k=1}^{s} a_k,\end{aligned}$$

which is the digit sum of *A*. This contradicts the choice of *A*. So $n_1, n_2, \ldots, n_s \pmod n$ are pairwise distinct. Then $s\le n$.

Assume $s<n$. Then let $n_i\equiv r_i \pmod n$ with $0\le r_i<n$ and consider

$$C = a_1b^{r_1} + a_2b^{r_2} + \cdots + a_sb^{r_s}.$$

Since $b^{n_i} \equiv b^{r_i} \pmod{b^n-1}$, so *C* is a multiple of b^n-1. Now $s<n$ implies

$$\begin{aligned}0 < C &\le (b-1)b + (b-1)b^2\\ &\quad + \cdots + (b-1)b^{n-1}\\ &< b^n - 1,\end{aligned}$$

contradiction. Therefore, $s = n$.

Other commended solvers: **Mark LAU Tin Wai** (Pui Ching Middle School) and **LEUNG Kit Yat** (St. Paul's College, Hong Kong).

Olympiad Corner

(*Continued from page 1*)

Problem 4. Prove that among any 20 consecutive positive integers there exists an integer d such that for each positive integer *n* we have the inequality

$$n\sqrt{d}\,\{n\sqrt{d}\} > \frac{5}{2}$$

where $\{x\}$ denotes the fractional part of the real number *x*. The fractional part of a real number *x* is *x* minus the greatest integer less than or equal to *x*.

Divisibility Problems

(*Continued from page 2*)

Solution of Exercise 1. Let $a=qb+r$ with $0 \le r \le b-1$. Then

$$\frac{n^a+1}{n^b-1} = n^r\sum_{j=0}^{q-1} n^{bj} + \frac{n^r+1}{n^b-1}.$$

So we need to find when $n^b-1 \mid n^r+1$. If $b=1$, then $r=0$ and we get $n=2,3$. If $b>1$, then $n>1$ and $n^b \ge 4$. For $n^b>4$, we have $0<n^r+1\le n^{b-1}+1\le n^b/2+1<n^b-1$, hence no solution. For $n^b\le 4$, we have three cases, namely (n,b,a) = $(2,2,2k-1)$, $(3,1,k)$ and $(2,1,k)$, where $k=1,2,3,\ldots$.

Solution of Exercise 2. For $x<3$ or $p<3$, the solutions are $(x,p)=(2,2)$ and (1,prime). For x and $p\ge 3$, since p is odd, $(p-1)^x+1$ is odd, so x is odd. Let q be the least prime divisor of x, which must be odd. We have $q \mid x \mid x^{p-1} \mid (p-1)^x+1$. So $(p-1)^x\equiv -1 \pmod q$. By Fermat's little theorem, $(p-1)^{q-1}\equiv 1 \pmod q$. By the definition of q, we have $\gcd(x,q-1) =1$. Then there are integers a,b such that $ax=b(q-1)+1$ is odd. Then a is odd. Now

$$p-1\equiv(p-1)^{b(q-1)+1}=(p-1)^{ax}\equiv -1 \pmod q$$

implies $q\mid p$. So $q=p$. Since x is odd, $p = q \mid x$ and the problem require the condition $x\le 2p$, we must have $x=p$ for the cases $x,p \ge 3$. Observe that

$$p^{p-1} \mid (p-1)^p + 1 = p^2(mp+1)$$

for some m. Then $p-1\le 2$. So $x=p=3$ is the only solution.