Olympiad Corner

Below are the problems of the Second Round of the 32nd Iranian Math Olympiad.

Problem 1. A local supermarket is responsible for the distribution of 100 supply boxes. Each box is ought to contain 10 kilograms of rice and 30 eggs. It is known that a total of 1000 kilograms of rice and 3000 eggs are in these boxes, but in some of them the amount of either item is more or less than the amount required. In each step, supermarket workers can choose two arbitrary boxes and transfer any amount of rice or any number of eggs between them. At least how many steps are required so that, starting from any arbitrary initial condition, after these steps the amount of rice and the number of eggs in all these boxes are equal?

Problem 2. Square $ABCD$ is given. Points $N$ and $P$ are selected on sides $AB$ and $AD$, respectively, such that $PN = NC$, and point $Q$ is selected on segment $AN$ such that $\angle NCB = \angle QPN$. Prove that $\angle BCQ = \frac{1}{2} \angle PQA$. Prove that

(continued on page 4)

---

Coloring Problems

Kin Y. Li

In some math competitions, there are certain combinatorial problems that are about partitioning a board (or a set) into pieces like dominos. We will look at some of these interesting problems. Often clever ways of assigning color patterns to the squares of the board allow simple solutions. Below, a $m \times n$ rectangle will mean a $m$-by-$n$ or an $n$-by-$m$ rectangle.

Example 1. A 8×8 chessboard with the the northeast and southwest corner unit squares removed is given. Is it possible to partition such a board into thirty-one dominoes (where a domino is a 1×2 rectangle)?

Solution. For such a board, we can color the unit squares alternatively in black and white, say black is color 1 and white is color 2. Then we have the following pattern.

```
1 2 1 2 1 2 1
2 1 2 1 2 1 2
1 2 1 2 1 2 1
2 1 2 1 2 1 2
1 2 1 2 1 2 1
2 1 2 1 2 1 2
1 2 1 2 1 2 1
2 1 2 1 2 1 2
```

Each domino will cover two adjacent squares, one with color 1 and the other with color 2. If 31 dominos can cover the board, there should be 31 squares with color 1 and 31 squares with color 2. However, in the board above there are 32 squares of color 1 and 30 squares of color 2. So the task is impossible.

Example 2. Eight 1×3 rectangles and one 1×1 square covered a 5×5 board. Prove that the 1×1 square must be over a color B square.

Solution. Consider the following coloring of the 8×8 board.

```
A B C A B  
B C A B C  
C A B C A  
A B C A B  
B C A B C  
C A B C A  
B C A B C  
A B C A B  
```

There are 8 color A squares, 9 color B squares and 8 color C squares. Each 1×3 rectangle covers a color A, a color B and a color C square. So the 1×1 square piece must be over a color B square.

Next, we rotate the coloring of the board (not the board itself) clockwise 90° around the center unit square.

```
B A C B A  
B A C B A  
A B C A B  
C A B C B  
C A B C B  
A B C A B  
B A C B A  
A B C A B  
```

Then observe that the 1×1 square piece must still be over a color B square due to reasoning used in the top paragraph. However, the only color B square that remains color B after the 90° rotation is the center unit square. So the 1×1 square piece must be over the center unit square.

Example 3. Can a 8×8 board be covered by fifteen 1×4 rectangles and one 2×2 square without overlapping?

Solution. Consider the following coloring of the 8×8 board.

(continued on page 2)
In the coloring of the board, there are 32 white and 32 black squares respectively. By simple checking, we can see every 1×4 rectangle will cover 2 white and 2 black squares. The 2×2 square will cover either 1 black and 3 white squares or 3 black and 1 white squares. Assume the task is possible. Then the 16 pieces together should cover either 31 black and 33 white squares on the boundary of the board, odd or even depends only on the 1×1 squares. If two 1×1 squares sharing a common edge have distinct colors, then color every 1×1 square of a board either black or white (but not both). If two 1×1 squares sharing a common edge have distinct colors, then color every 1×1 square of a board either black or white (but not both). If two 1×1 squares sharing a common edge have distinct colors, then color every 1×1 square of a board either black or white (but not both). If two 1×1 squares sharing a common edge have distinct colors, then color every 1×1 square of a board either black or white (but not both). If two 1×1 squares sharing a common edge have distinct colors, then color every 1×1 square of a board either black or white (but not both). If two 1×1 squares sharing a common edge have distinct colors, then color every 1×1 square of a board either black or white (but not both). If two 1×1 squares sharing a common edge have distinct colors, then color every 1×1 square of a board either black or white (but not both).

Solution. From 1004 distinct points, we can draw $K_{1004}$ line segments connecting pairs of them. Among these, there exists a longest segment $AB$. Now the midpoints of the line segments joining $A$ to the other 1003 points lie inside or on the circle center at $A$ and radius $\frac{1}{2}AB$. Similarly, the midpoints of the line segments joining $B$ to the other 1003 points lie inside or on another circle center at $B$ and radius $\frac{1}{2}AB$. These two circles intersect only at the midpoint of $AB$. Then there are at least $2 \times 1003 - 1 = 2005$ black midpoints generated by the line segments.

To construct an example of a set of 1004 points generating exactly 2005 black midpoints, consider the line segments connecting pairs of them.

Example 5. There are 1004 distinct points on a plane. Connect each pair of these points. Among these, there exists a finite number of distinct pairs in the board. So whether $S$ is even or odd is totally determined by the set of type 2 squares.

Next we will look at problems about coloring elements of some sets.

Example 4. Let $m,n$ be integers greater than 2. Color every 1×1 square of a $m \times n$ board either black or white (but not both). If two 1×1 squares sharing a common edge have distinct colors, then color every 1×1 square of a board either black or white (but not both).

In coloring problems, other than assigning different colors to all the squares, sometimes assigning different numerical values for different types of squares can be useful in solving the problem. Below is one such example.

Example 6. Find all ways of coloring all positive integers such that

1. every positive integer is colored either black or white (but not both) and

2. the sum of two numbers with distinct colors is always colored black and their product is always colored white.

Next for every pair of 1×1 squares sharing a common edge, write the product of the values in the two squares on their common edge. Let $H$ be the product of these values on all the common edges. For every type 1 square, it has four neighbor squares sharing a common edge with it. So the number in a type 1 square appears four times as factors in $H$. For every type 2 square, it has three neighbor squares sharing a common edge with it. So the number in a type 2 square appears three times as factors in $H$. For every type 3 square, it has four neighbor squares sharing a common edge with it. So the number in a type 3 square appears four times as factors in $H$. Hence,

$$H = (abcd)^4 (x_{1,1} \cdots x_{2n-2,2n-4}) (y_{1,1} \cdots y_{2n-2,2n-4})^4 = (x_1 \cdots x_{2n-4})^4 (y_1 \cdots y_{2n-4})^4.$$

If $x_{1,1} \cdots x_{2n-2,2n-4} = 1$, then $H = 1$ and there are an even number of distinct pairs in the board. If $y_{1,1} \cdots y_{2n-2,2n-4} = -1$, then $H = -1$, and there are an odd number of distinct pairs in the board. So whether $S$ is even or odd is totally determined by the set of type 2 squares.

Next for every pair of 1×1 squares sharing a common edge, write the product of the values in the two squares on their common edge. Let $H$ be the product of these values on all the common edges. For every type 1 square, it has four neighbor squares sharing a common edge with it. So the number in a type 1 square appears four times as factors in $H$. For every type 2 square, it has three neighbor squares sharing a common edge with it. So the number in a type 2 square appears three times as factors in $H$. For every type 3 square, it has four neighbor squares sharing a common edge with it. So the number in a type 3 square appears four times as factors in $H$. Hence,

$$H = (abcd)^4 (x_{1,1} \cdots x_{2n-2,2n-4}) (y_{1,1} \cdots y_{2n-2,2n-4})^4 = (x_1 \cdots x_{2n-4})^4 (y_1 \cdots y_{2n-4})^4.$$

If $x_{1,1} \cdots x_{2n-2,2n-4} = 1$, then $H = 1$ and there are an even number of distinct pairs in the board. If $y_{1,1} \cdots y_{2n-2,2n-4} = -1$, then $H = -1$, and there are an odd number of distinct pairs in the board. So whether $S$ is even or odd is totally determined by the set of type 2 squares.
Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver’s name, home (or email) address and school affiliation. Please send submissions to Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong. The deadline for sending solutions is February 29, 2016.

Problem 481. Let \( S = \{1, 2, \ldots, 2016\} \). Determine the least positive integer \( n \) such that whenever there are \( n \) numbers in \( S \) satisfying every pair is relatively prime, then at least one of the \( n \) numbers is prime.

Problem 482. On \( \triangle ABD \), \( C \) is a point on side \( BD \) with \( C \neq B, D \). Let \( K_1 \) be the circumcircle of \( \triangle ABC \). Line \( AD \) is tangent to \( K_1 \) at \( A \). A circle \( K_2 \) passes through \( A \) and \( D \) and line \( BD \) is tangent to \( K_2 \) at \( D \). Suppose \( K_1 \) and \( K_2 \) intersect at \( A \) and \( E \) with inside \( \triangle ACD \). Prove that \( EB/EC = (AB/AC)^2 \).

Problem 483. In the open interval (0,1), \( n \) distinct rational numbers \( a/b, (i=1,2,\ldots,n) \) are chosen, where \( n>1 \) and \( a, b \) are positive integers. Prove that the sum of the \( b/i \)’s is at least \( (n/2)^{3/2} \).

Problem 484. In a multiple choice test, there are four problems. For each problem, there are choices \( A, B, C \) and \( D \). For any three students who took the test, there exist a problem the three students selected distinct choices. Determine the maximum number of students who took the test.

Problem 485. Let \( m \) and \( n \) be integers such that \( m \geq 1 \), \( S = \{1, 2, \ldots, m\} \) and \( T = \{a_1, a_2, \ldots, a_n\} \) is a subset of \( S \). It is known that every two numbers in \( T \) do not both divide any number in \( S \). Prove that

\[
\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} < \frac{m+n}{n}.
\]

***************

Solutions

***************

Problem 476. Let \( p \) be a prime number. Define sequence \( a_k \) by \( a_0 = 0 \), \( a_1 = 1 \) and \( a_{k+2} = a_{k+1} + pa_k \). If one of the terms of the sequence is \(-1\), then determine all possible value of \( p \).

**Solution.** Jon GLIMMS and KWOK Man Yi (Baptist Lui Ming Choi Secondary School, S5).

Observe that \( p \neq 2 \) (otherwise beginning with \( a_2 \), the rest of the terms will be even, then \(-1 \) cannot appear). On one hand, using the recurrence relation, we get

\[ a_{k+2} = 2a_{k+1} \equiv \cdots \equiv 2^{k+1}a_1 \equiv 2^{k+1} \pmod{p}. \]

If \( a_0 = -1 \) for some \( m \geq 2 \), then letting \( k = m - 2 \), we get

\[ -1 = a_m \equiv 2^{-m} \pmod{p}. \]

On the other hand, using the recurrence relation again, we also have

\[ a_{k+2} = 2a_{k+1} - a_k \pmod{p-1}, \]

which implies \( a_{k+2} - a_{k+1} = a_{k+1} - a_k = \cdots = a_1 - a_0 = 1 \pmod{p-1} \). Then

\[ -1 = a_m = m+2a_0 = m \pmod{p-1}, \]

which implies \( p \) divides \( m+1 \). By Fermat’s little theorem and (*), we get

\[ 1 \equiv 2^{m+1} \equiv 4 \cdot 2^{m-1} \equiv -4 \pmod{p}. \]

Then \( p = 5 \). Finally, if \( p = 5 \), then \( a_3 = -1 \).

**Problem 477.** In \( \triangle ABC \), points \( D, E \) are on sides \( AC, AB \) respectively. Lines \( BD \) and \( CE \) intersect at a point \( P \) on the bisector of \( \angle BAC \).

Prove that quadrilateral \( ADPE \) has an inscribed circle if and only if \( AB = AC \).

**Solution.** Adnan ALI (Atomic Energy Central School 4, Mumbai, India), KWOK Man Yi (Baptist Lui Ming Choi Secondary School, S5), MANOLOUDIS Apostolos (4 High School of Korydallas, Piraeus, Greece), Jaťet Alejandro Baca OBANDO (IDEAS High School, Nicaragua) and Toshihiro SHIMIZU (Kawasaki, Japan).

Suppose \( ADPE \) has an inscribed circle \( \Gamma \). Since the center of \( \Gamma \) is on the bisector of \( \angle BAC \), the center is on line \( AP \). Similarly, \( AP \) also bisects \( \angle BAC \), so \( ADPE \equiv \angle BAPD \).

It also follows that \( \angle APB = \angle APC \), since \( \angle EPB = \angle DPC \). By ASA, we get \( \Delta APB \equiv \Delta APC \) with \( AP \) common. Then \( AB = AC \).

Conversely, if \( AB = AC \), then \( \angle ABC \) is symmetric with respect to \( AP \). Thus, lines \( BP \) and \( CP \) (hence also \( D \) and \( E \)) are symmetric with respect to \( AP \). By symmetry, the bisectors of \( \angle ADP \) and \( \angle AEP \) meet at a point \( I \) on \( AP \). Then the distances from \( I \) to lines \( EA, EP, DP \) are the same. So \( ADPE \) has an inscribed circle with center \( I \).

**Other commended solvers:** Mark LAU Tin Wai (Pui Ching Middle School), Titu ZVONARU (Comănești, Romania) and Neculai STANCIU (“George Emil Palade” Secondary School, Buzău, Romania).

**Problem 478.** Let \( a \) and \( b \) be a pair of coprime positive integers of opposite parity. If a set \( S \) satisfies the following conditions:

1. \( a, b \in S \);
2. if \( x, y, z \in S \), then \( x+y+z \in S \),

then prove that every positive integer greater than \( 2ab \) belongs to \( S \).

**Solution.** Toshihiro SHIMIZU (Kawasaki, Japan).

Without loss of generality, we assume that \( a \) is odd and \( b \) is even. Let \( n > 2ab \). Since \( a \) and \( b \) are coprime, the equation \( ax = n \pmod{b} \) has a solution satisfying \( 0 \leq x < b \). Then \( y = (n-ax) \pmod{b} \) is a positive integer. Now

\[
a = 2ab - ab < \frac{n - ax}{b} = y \leq 2ab - 2a.
\]

Let \( x' = x + b \), \( y' = y - a \). Then \( x' \) and \( y' \) are positive and \( ax' + by' = n \). Observe \( x' + y' = x + y - b + a \) are of opposite parity. So we may assume \( x + y \) is odd (otherwise take \( x' + y' \) ). Then \( x + y \geq 3 \) and by (1) and (2),

\[
n = a + \cdots + a + b + \cdots + b \in S,
\]

where \( a \) appeared \( x \) times and \( b \) appeared \( y \) times.

**Other commended solvers:** KWOK Man Yi (Baptist Lui Ming Choi Secondary School, S5) and Mark LAU Tin Wai (Pui Ching Middle School).

**Problem 479.** Prove that there exists infinitely many positive integers \( k \) such that for every positive integer \( n \), the number \( k^2 + 1 \) is composite.

**Solution.** KWOK Man Yi (Baptist Lui Ming Choi Secondary School, S5).
By the Chinese remainder theorem, there exist infinitely many positive integers $k$ such that

$$k \equiv 1 \pmod{3}$$
$$k \equiv 1 \pmod{5}$$
$$k \equiv 3 \pmod{7}$$
$$k \equiv 10 \pmod{13}$$
$$k \equiv 1 \pmod{17}$$
$$k \equiv -1 \pmod{241}$$

If $n = 1 \pmod{2}$, then $k^{2+n+1} \equiv 2+1 \equiv 0 \pmod{3}$. Otherwise $2n$. If $n = 2 \pmod{4}$, then $k^{2+n+1} \equiv 2+1 \equiv 0 \pmod{5}$. Otherwise $4n$. If $n = 4 \pmod{8}$, then $k^{2+n+1} \equiv 2+1 \equiv 0 \pmod{7}$. Otherwise $8n$. Then we have three cases:

**Case 1:** $n \equiv 8 \pmod{24}$. By Fermat’s little theorem, $2^{24} \equiv (2^{21})^{2} \equiv 1 \pmod{13}$. So $2^{n} = 2^{\frac{n}{2} \cdot 24} \equiv 256 \equiv -4 \pmod{13}$ and $k^{2+n+1} \equiv 10^{4}(-4) \equiv 0 \pmod{13}$.

**Case 2:** $n \equiv 16 \pmod{24}$. Since $2^{24} = (2^{2})^{3} \equiv 1 \pmod{7}$, we have $2^{n} = 2^{16+24} \equiv 2^{1+5(6n)} \equiv 2 \pmod{7}$ and $k^{2+n+1} \equiv 3 \cdot 2 \cdot 1 \equiv 0 \pmod{7}$.

**Case 3:** $n \equiv 0 \pmod{24}$. Since $2^{24} = (2^{3})^{8} \equiv 15^{8} \equiv 256 \equiv -16-15 \equiv 1 \pmod{241}$. So $2^{n} = 2^{24n} \equiv 1 \pmod{241}$ and Then $k^{2+n+1} \equiv -1+1 \equiv 0 \pmod{241}$.

**Comment:** We may wonder why modulo $3, 5, 7, 13, 17, 241$ work. It may be that in dealing with $n = 8, 16, 0 \pmod{24}$, we want $2^{24} \equiv 1 \pmod{p}$ for some useful primes $p$. Then we notice

$$2^{24} - 1 = (2^{3} - 1)(2^{3} + 1)(2^{12} + 1) = 7 \cdot 33 \cdot 5 \cdot 13 \cdot 17 \cdot 241.$$  

**Other commended solvers:** Ioan Viorel CODREANU (Secondary School Satulung, Maramures, Romania), Prishtina Math Gymnasium Problem Solving Group (Republic of Kosova), Toshihiro SHIMIZU (Kawasaki, Japan), Titu ZVONARU (Comănești, Romania) and Neculai STANCIU (“George Emil Palade” Secondary School, Buzău, Romania).

**Olympiad Corner**

(Continued from page 1)

**Problem 3.** Let $x, y$ and $z$ be nonnegative real numbers. Knowing that $2(x^{2}+y^{2}+z^{2}) = x^{4}+y^{2}+z^{2}$, prove

$$\frac{x+y+z}{3} \geq \sqrt[3]{2xyz}.$$  

**Problem 4.** Find all of the solutions of the following equation in natural numbers:

$$n^{x} = m^{w}.$$  

**Problem 5.** A non-empty set $S$ of positive real numbers is called powerful if for any two distinct elements of it like $a$ and $b$, at least one of the numbers $a^{b}$ or $b^{a}$ is an element of $S$.

a) Present an example of a powerful set having four elements.

b) Prove that a finite powerful set cannot have more than four elements.

**Problem 6.** In the Majestic Mystery Club (MMC), members are divided into several groups, and groupings change by the end of each week in the following manner: in each group, a member is selected as king; all of the kings leave their respective groups and form a new group. If a group has only one member, that member goes to the new group and his former group is deleted. Suppose that MMC has $n$ members and at the beginning of all of them form a single group. Prove that there comes a week for which thereafter each group will have at most $1 + \sqrt{2n}$ members.

**Coloring Problems**

(Continued from page 2)

**Example 8.** Numbers $1, 2, 3, 4, 5, 6, 7, 8, 9$ are divided into two groups, each having at least one number. Prove that there always exists a three term arithmetic progression (AP in short) in one of the two groups.

**Solution.** Assume no three term AP is in any of the two groups. Color numbers in one group red and the other group blue. Since $5 > 2$, among $1, 3, 5, 7, 9$, there exist three of them assigned the same color, say they are red. By assumption, they are not the terms of an AP. Below are the possibilities of these red numbers: $\{1, 3, 7\}$, $\{1, 3, 9\}$, $\{1, 5, 7\}$, $\{1, 7, 9\}$, $\{3, 5, 9\}$ or $\{3, 7, 9\}$.

If $1, 3, 7$ are red, then as $1, 2, 3$ and $3, 5, 7$ are AP, so $2, 4, 5$ are blue. As $4, 5, 6$ and $2, 5, 8$ are AP, so $6, 8$ are red. So $6, 7, 8$ is a red AP, contradiction.

If $1, 3, 9$ are red, then as $1, 2, 3$ and $3, 5, 7$ are AP, so $2, 4, 5$ are blue. As $4, 5, 6$ and $2, 5, 8$ are AP, so $6, 8$ are red. So $6, 7, 8$ is a red AP, contradiction.

If $1, 5, 7$ are red, then as $1, 2, 3$ and $3, 5, 7$ are AP, so $2, 4, 5$ are blue. As $4, 5, 6$ and $2, 5, 8$ are AP, so $6, 8$ are red. So $6, 7, 8$ is a red AP, contradiction.

If $1, 7, 9$ are red, then as $1, 2, 3$ and $3, 5, 7$ are AP, so $2, 4, 5$ are blue. As $4, 5, 6$ and $2, 5, 8$ are AP, so $6, 8$ are red. So $6, 7, 8$ is a red AP, contradiction.

If $1, 5, 9$ are red, then as $1, 2, 3$ and $3, 5, 7$ are AP, so $2, 4, 5$ are blue. As $4, 5, 6$ and $2, 5, 8$ are AP, so $6, 8$ are red. So $6, 7, 8$ is a red AP, contradiction.

If $3, 7, 9$ are red, then as $3, 5, 7$ and $5, 7, 9$ are AP, so $3, 6, 9$ are blue. Then $3, 6, 9$ is a red AP, contradiction.