Mathematical Excalibur

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Olympiad Corner

Below are the problems of the 28^{nd} *Asian Pacific Math Olympiad*, which was held in March 2016.

Problem 1. We say that a triangle ABC is *great* if the following holds: for any point D on the side BC, if P and Q are the feet of the perpendiculars from D to the lines AB and AC, respectively, then the reflection of D in the line PQ lies on the circumcircle of the triangle ABC. Prove that triangle ABC is great if and only if $\angle A=90^\circ$ and $AB=AC$.

Problem 2. A positive integer is called *fancy* if it can be expressed in the form

$$2^{a_1}+2^{a_2}+\cdots+2^{a_{100}},$$

where $a_1, a_2, \ldots, a_{100}$ are non-negative integers that are not necessarily distinct. Find the smallest positive integer n such that no multiple of n is a fancy number.

Problem 3. Let AB and AC be two distinct rays not lying on the same line, and let ω be a circle with center O that is tangent to ray AC at E and ray AB at F. Let R be a point on segment EF. The line through O parallel to EF intersects line AB at P.

(continued on page 4)

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The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***August 14, 2016***.

For individual subscription for the next five issues for the 15-16 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Inequalities of Sequences

Kin Y. Li

There are many math competition problems on inequalities. While most symmetric inequalities can be solved by powerful facts like the Muirhead and Schur inequalities, there are not many tools for general inequalities involving sequences. Below we will first take a look at some relatively easy examples on inequalities of sequences.

<u>*Example 1*</u>. (1997 *Chinese Math Winter Camp*) Let $a_1, a_2, a_3, \ldots$ be a sequence of nonnegative numbers. If for all positive integers m and n, $a_{n+m} \le a_n+a_m$, then prove that

$$a_n \le ma_1 + \left(\frac{n}{m}-1\right)a_m.$$

<u>*Solution.*</u> Let $n = mq + r$, where q, r are integers and $0 \le r < m$. We have

$$\begin{aligned} a_n &\le a_{mq}+a_r \le qa_m + a_r \\ &= \frac{n-r}{m}a_m + a_r \\ &= \left(\frac{n}{m}-1\right)a_m + \frac{m-r}{m}a_m + a_r \\ &\le \left(\frac{n}{m}-1\right)a_m + \frac{m-r}{m}ma_1 + ra_1 \\ &= \left(\frac{n}{m}-1\right)a_m + ma_1. \end{aligned}$$

<u>*Example 2.*</u> (*IMO* 2014) Let $a_0 < a_1 < a_2 < \cdots$ be an infinite sequence of positive integers. Prove that there exists a unique integer $n \ge 1$ such that

$$a_n < \frac{a_0+a_1+\cdots+a_n}{n} \le a_{n+1}.$$

<u>*Solution.*</u> For $n = 1,2,3,\ldots$, define

$$d_n = (a_0+a_1+\cdots+a_n) - na_n.$$

We have

$$\begin{aligned} &na_{n+1} - (a_0+a_1+\cdots+a_n) \\ &= (n+1)a_{n+1} - (a_0+a_1+\cdots+a_{n+1}) \\ &= -d_{n+1}. \end{aligned}$$

In terms of d_i's, the required conclusion is the same as $d_n > 0 \ge d_{n+1}$ for some unique $n \ge 1$.

Now observe that $d_1 = (a_0+a_1) - a_1 > 0$. Also the d_i's are strictly decreasing as

$$\begin{aligned} &d_{n+1} - d_n \\ &= \sum_{i=1}^{n+1} a_i - (n+1)a_{n+1} - \sum_{i=1}^{n} a_i - na_n \\ &= n(a_n - a_{n+1}) < 0. \end{aligned}$$

Finally, from $d_1 > 0$, the d_i's are integers and strictly decreasing, there must be a first non-positive d_i. So $d_n > 0 \ge d_{n+1}$ for some unique $n \ge 1$.

<u>*Example 3.*</u> (1980 *Austrian-Polish Math Competition*) Let a_1, a_2, a_3, … be a sequence of real numbers satisfying the inequality

$$|a_{k+m} - a_k - a_m| \le 1 \quad \text{for all } k, m.$$

Show that the following inequality holds for all positive integers k and m,

$$\left|\frac{a_k}{k} - \frac{a_m}{m}\right| < \frac{1}{k} + \frac{1}{m}.$$

<u>*Solution.*</u> Observe that multiplying by km, the desired inequality is the same as $|ma_k - ka_m| < m+k$. To get this, we will prove for a fixed m, $|a_{km} - ka_m| < k$ holds for all positive integer k by induction. The case $k = 1$ is $|a_m - a_m| = 0 < 1$. Suppose the k-th case is true. Then

$$\begin{aligned} &|a_{(k+1)m} - (k+1)a_m| \\ &= |a_{km+m} - a_{km} - a_m + a_{km} - ka_m| \\ &\le |a_{km+m} - a_{km} - a_m| + |a_{km} - ka_m| \\ &\le 1 + |a_{km} - ka_m| < 1 + k. \end{aligned}$$

This completes the inductive step. Now interchanging k and m, similarly we also have $|a_{km} - ma_k| < m$. Then

$$\begin{aligned} |ma_k - ka_m| &\le |a_{km} - ma_k| + |a_{km} - ka_m| \\ &\le m + k \end{aligned}$$

and we are done.

(continued on page 2)

Example 4. (2006 *IMO Shortlisted Problem*) The sequence of real numbers a_0, a_1, a_2, … is defined recursively by

$$a_0=-1,\ \sum_{k=0}^{n}\frac{a_{n-k}}{k+1}=0\quad for\ \ n\ge 1.$$

Show that $a_n > 0$ for $n \ge 1$.

Solution. Setting n=1, we find a_1=1/2. For $n\ge 1$, reversing the order of the terms in the given sum, we have

$$\sum_{k=0}^{n}\frac{a_k}{n-k+1}=0\ \ and\ \ \sum_{k=0}^{n+1}\frac{a_k}{n-k+2}=0.$$

Suppose a_1 to a_n are positive. Then

$$\begin{aligned}0&=(n+2)\sum_{k=0}^{n+1}\frac{a_k}{n-k+2}-(n+1)\sum_{k=0}^{n}\frac{a_k}{n-k+1}\\&=(n+2)a_{n+1}+\sum_{k=0}^{n}\left(\frac{n+2}{n-k+2}-\frac{n+1}{n-k+1}\right)a_k.\end{aligned}$$

Notice the k=0 term in the last sum is 0. Solving for a_{n+1}, we get

$$\begin{aligned}a_{n+1}&=\frac{1}{n+1}\sum_{k=1}^{n}\left(\frac{n+1}{n-k+1}-\frac{n+2}{n-k+2}\right)a_k\\&=\frac{1}{n+2}\sum_{k=1}^{n}\frac{k}{(n-k+1)(n-k+2)}a_k\end{aligned}$$

is positive as a_1 to a_n are positive.

Next we will study certain examples that require more observation and possibly involve some calculations of limit of sequences.

Example 5. (1988 *IMO Shortlisted Problem*) Let a_1, a_2, a_3, … be a sequence of nonnegative real numbers such that

$$a_k-2a_{k+1}+a_{k+2}\ge 0\ \ \text{and}\ \ \sum_{j=1}^{k}a_j\le 1$$

for all k = 1, 2, …. Prove that

$$0\le a_k-a_{k+1}<\frac{2}{k^2}$$

for all k = 1, 2, ….

Solution. We claim $0\le a_k-a_{k+1}$ for all k. (Otherwise assume for some k, we have $a_k-a_{k+1}<0$. From $a_k-2a_{k+1}+a_{k+2}\ge 0$, we get $a_{k+1}-a_{k+2}\le a_k-a_{k+1}<0$. It follows $a_k < a_{k+1} < a_{k+2} < \cdots$. Then

$$a_k + a_{k+1} + a_{k+2} + \cdots$$

diverges to infinity, which leads to a contradiction.)

Let $b_k=a_k-a_{k+1}$. Then for all positive integer k, we have $b_k\ge b_{k+1}\ge 0$. Now we have

$$\begin{aligned}b_k\sum_{i=1}^{k}i\le\sum_{i=1}^{k}ib_i&=\sum_{i=1}^{k}ia_i-\sum_{i=1}^{k}ia_{i+1}\\&=\sum_{i=1}^{k}ia_i-\sum_{i=2}^{k+1}(i-1)a_i\\&=\sum_{i=1}^{k}a_i-ka_{k+1}\le\sum_{i=1}^{k}a_i\le 1.\end{aligned}$$

Therefore,

$$b_k\le 1\Big/\sum_{i=1}^{k}i=\frac{2}{k(k+1)}<\frac{2}{k^2}.$$

Example 6. (1970 *IMO*) Let $1 = a_0 \le a_1 \le a_2 \le \cdots \le a_n \le \cdots$ be a sequence of real numbers. Consider the sequence defined by

$$b_n=\sum_{k=1}^{n}\left(1-\frac{a_{k-1}}{a_k}\right)\frac{1}{\sqrt{a_k}}.$$

Prove that :

(a) For all positive integers n, $0\le b_n\le 2$.

(b) Given an arbitrary $0 \le b < 2$, there is a sequence a_0, a_1, …, a_n, … of the above type such that $b_n > b$ is true for infinitely many natural numbers n.

Solution. (a) For all k, we have

$$\begin{aligned}0&\le\left(1-\frac{a_{k-1}}{a_k}\right)\frac{1}{\sqrt{a_k}}\\&=\frac{(\sqrt{a_k}+\sqrt{a_{k-1}})(\sqrt{a_k}-\sqrt{a_{k-1}})}{a_k\sqrt{a_k}}\\&\le 2\frac{\sqrt{a_k}-\sqrt{a_{k-1}}}{\sqrt{a_ka_{k-1}}}\\&=2\left(\frac{1}{\sqrt{a_{k-1}}}-\frac{1}{\sqrt{a_k}}\right).\end{aligned}$$

Then

$$\begin{aligned}0\le b_n&\le 2\sum_{k=1}^{n}\left(\frac{1}{\sqrt{a_{k-1}}}-\frac{1}{\sqrt{a_k}}\right)\\&=2\left(1-\frac{1}{\sqrt{a_n}}\right)<2.\end{aligned}$$

(b) Let $0<q<1$. Then $a_n=q^{-2n}$ for n = 0, 1, 2, … satisfy $1 = a_0 < a_1 < a_2 < \cdots$ and the sequence $b_n=q(1+q)(1-q^n)$ has

$$\lim_{n\to\infty}b_n=q(1+q).$$

For an arbitrary $0 \le b < 2$, take q satisfy

$$\frac{\sqrt{1+4b}-1}{2}<q<1.$$

Then $0<q<1$ and $q(1+q)>b$. So eventually the sequence b_n (on its way to $q(1+q)$) will be greater than b.

Example 7. (2006 *IMO Shortlisted Problem*) Prove the inequality

$$\sum_{1\le i<j\le n}\frac{a_ia_j}{a_i+a_j}\le\frac{n}{2(a_1+a_2+\cdots+a_n)}\sum_{1\le i<j\le n}a_ia_j$$

for positive real numbers a_1, a_2, …, a_n.

Solution. Let S be the sum of the n numbers. Let L and R be the left and the right expressions in the inequality. Observe that

$$\sum_{1\le i<j\le n}(a_i+a_j)=(n-1)\sum_{k=1}^{n}a_k=(n-1)S$$

and

$$\begin{aligned}L&=\sum_{1\le i<j\le n}\frac{a_ia_j}{a_i+a_j}\\&=\sum_{1\le i<j\le n}\frac{1}{4}\left(a_i+a_j-\frac{(a_i-a_j)^2}{a_i+a_j}\right)\\&=\frac{n-1}{4}S-\frac{1}{4}\sum_{1\le i<j\le n}\frac{(a_i-a_j)^2}{a_i+a_j}.\end{aligned}$$

Next we will write the expression R in two ways. On one hand, we have

$$R=\frac{n}{2S}\sum_{1\le i<j\le n}a_ia_j=\frac{n}{4S}\left(S^2-\sum_{i=1}^{n}a_i^2\right).$$

On the other hand,

$$\begin{aligned}R&=\frac{n}{4S}\sum_{1\le i<j\le n}\left(a_i^2+a_j^2-(a_i-a_j)^2\right)\\&=\frac{n(n-1)}{4S}\sum_{i=1}^{n}a_i^2-\frac{n}{4S}\sum_{1\le i<j\le n}(a_i-a_j)^2.\end{aligned}$$

Multiplying the first of these equations by $n-1$ and adding it to the second equation, then dividing the sum by n, we get

$$R=\frac{n-1}{4}S-\frac{1}{4}\sum_{1\le i<j\le n}\frac{(a_i-a_j)^2}{S}.$$

Comparing L and R and using $S\ge a_i+a_j$, we get $L\le R$.

Example 8. (1998 *IMO Longlisted Problem*) Let

$$a_n=[\sqrt{(1+n)^2+n^2}],\quad n=1,2,\ldots,$$

where $[x]$ denotes the integer part of x. Prove that

(a) there are infinitely many positive integers m such that $a_{m+1}-a_m>1$;

(b) there are infinitely many positive integers m such that $a_{m+1}-a_{m=1}=1$.

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Problem Corner

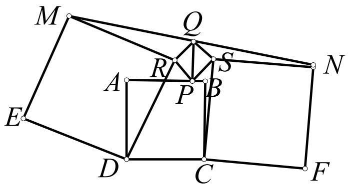
We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***August 14, 2016.***

Problem 486. Let a_0=1 and

$$a_n = \frac{\sqrt{1+a_{n-1}^2}-1}{a_{n-1}}.$$

for n=1,2,3,…. Prove that $2^{n+2}a_n > \pi$ for all positive integers n.

Problem 487. Let $ABCD$ and $PSQR$ be squares with point P on side AB and $AP>PB$. Let point Q be outside square $ABCD$ such that $AB \perp PQ$ and $AB=2PQ$. Let $DRME$ and $CSNF$ be squares as shown below. Prove Q is the midpoint of line segment MN.



Problem 488. Let $\mathbb{Q}$ denote the set of all rational numbers. Let f: $\mathbb{Q} \to \{0,1\}$ satisfy $f(0)=0, f(1)=1$ and the condition $f(x) = f(y)$ implies $f(x) = f((x+y)/2)$. Prove that if $x \geq 1$, then $f(x) = 1$.

Problem 489. Determine all prime numbers p such that there exist positive integers m and n satisfying $p=m^2+n^2$ and m^3+n^3-4 is divisible by p.

Problem 490. For a parallelogram $ABCD$, it is known that $\triangle ABD$ is acute and AD=1. Prove that the unit circles with centers A, B, C, D cover $ABCD$ if and only if

$$AB \leq \cos\angle BAD + \sqrt{3}\sin\angle BAD.$$

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Solutions

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Problem 481. Let S={1,2,…,2016}. Determine the least positive integer n such that whenever there are n numbers in S satisfying every pair is relatively prime, then at least one of the n numbers is prime.

Solution. **BOBOJONOVA Latofat** (academic lycuem S.H.Sirojiddinov, Tashkent, Uzbekistan), **KWOK Man Yi** (Baptist Lui Ming Choi Secondary School, S5), **Toshihiro SHIMIZU** (Kawasaki, Japan),**WONG Yat.**

Let k_0=1 and k_i be the square of the i-th prime number. Then $k_{14}=43^2<2016$. Since the numbers $k_0,k_1,\ldots,k_{14}$ are in S and are pairwise coprime, so $n \geq 16$.

Next suppose $A=\{a_1,a_2,\ldots,a_{16}\} \subset S$ with no a_i prime and a_r, a_s are coprime for $r \neq s$.

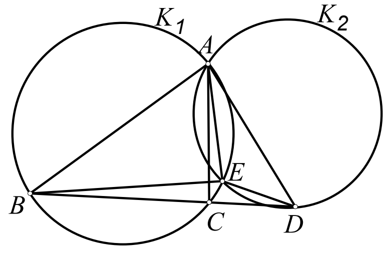
Then in case $1 \notin A$, let p_i be the least prime divisor of a_i. We have $a_i \geq p_i^2$. As the a_i's are pairwise coprime, no two p_i's are the same. Now the 15th prime is 47. So the largest p_i is at least 47, which leads to some $a_i \geq p_i^2 \geq 47^2 > 2016$, a contradiction.

Otherwise, $1 \in A$. For the 15 numbers in A that is not 1, let a_i be their maximum, then $a_i \geq p_i^2 \geq 47^2 > 2016$, again contradiction. So the least n is 16.

Other commended solvers: **Joe SPENCER.**

Problem 482. On $\triangle ABD$, C is a point on side BD with $C \neq B,D$. Let K_1 be the circumcircle of $\triangle ABC$. Line AD is tangent to K_1 at A. A circle K_2 passes through A and D and line BD is tangent to K_2 at D. Suppose K_1 and K_2 intersect at A and E with E inside $\triangle ACD$. Prove that $EB/EC=(AB/AC)^3$.

Solution. **Jafet Alejandro BACA OBANDO** (IDEAS High School, Nicaragua), **KWOK Man Yi** (Baptist Lui Ming Choi Secondary School, S5), **MANOLOUDIS Apostolos** (4 High School of Korydallos, Piraeus, Greece), **Vijaya Prasad NALLURI** and **Toshihiro SHIMIZU** (Kawasaki, Japan).



Line AD tangent to K_1 at A implies $\angle DAC = \angle DBA$. With $\angle ADC = \angle BDA$, we see $\triangle DAC$ is similar to $\triangle DBA$. Now $BD/CD = [DBA]/[DAC] = AB^2/CA^2$. Then we have

$$\left(\frac{AB}{AC}\right)^3 = \frac{BD}{CD}\cdot\frac{AB}{AC} = \frac{BD/AC}{CD/AB}. \quad (*)$$

Next, $\angle DBE = \angle CBE = \angle CAE$ and $\angle BDE = \angle DAE = \angle ACE$ implies $\triangle DBE$ is similar to $\triangle CAE$. Similarly, $\angle ECD = \angle EAB$ and $\angle EDC = \angle EAD = \angle EBA$ implies $\triangle ECD$ is similar to $\triangle EAB$. Then

$$\frac{BD/CA}{CD/AB} = \frac{EB/AE}{EC/EA} = \frac{EB}{EC}. \quad (**)$$

Therefore, combining (*) and (**), we have $EB/EC=(AB/AC)^3$.

Other commended solvers: **BOBOJONOVA Latofat** (academic lycuem S. H. Sirojiddinov, Tashkent, Uzbekistan) and **WONG Yat**.

Problem 483. In the open interval (0,1), n distinct rational numbers a_i/b_i (i=1,2,…,n) are chosen, where n>1 and a_i, b_i are positive integers. Prove that the sum of the b_i's are at least $(n/2)^{3/2}$.

Solution. **Toshihiro SHIMIZU** (Kawasaki, Japan).

Without loss of generality, we may suppose the numbers a_i/b_i are sorted so that the denominators are in ascending order. We have the following lemma.

Lemma. Let k be an integer in [1,n] and b be the denominator of the k-th number. Then we have

$$b \geq \left(\frac{k}{2}\right)^{3/2} - \left(\frac{k-1}{2}\right)^{3/2}.$$

Proof. We first consider the number of denominators that are at most b. For every i = 1, 2, …, b, the number of denominators equal to i is at most $i-1$. Thus,

$$k \leq \sum_{i=1}^{b}(i-1) = \frac{b(b-1)}{2} \leq \frac{b^2}{2}.$$

This implies $b \geq \sqrt{2k}$. We will show

$$\sqrt{2k} \geq \left(\frac{k}{2}\right)^{3/2} - \left(\frac{k-1}{2}\right)^{3/2}.$$

It is equivalent to

$$4\sqrt{k} \geq k\sqrt{k} - (k-1)\sqrt{k-1}$$

or $(k-1)\sqrt{k-1} \geq (k-4)\sqrt{k}$.

For k=1,2,3,4, the left hand side is greater than the right hand side is non-positive. For $k\geq 5$, squaring the inequality, it is equivalent to $(k-1)^3 \geq (k-4)^2 k$ or $5k^2-13k+1\geq 0$. The larger roots of the left hand side is $(13+\sqrt{149})/10$, which is less than 2.6. Then the left hand side is always positive for $k\geq 5$. QED

Using the lemma and summing the cases k=1, 2, …,n, we get the result.

Other commended solvers: **Jim GLIMMS, Joe SPENCER** and **WONG Yat**.

Problem 484. In a multiple choice test, there are four problems. For each problem, there are choices *A, B* and *C.* For any three students who took the test, there exist a problem the three students selected distinct choices. Determine the maximum number of students who took the test.

Solution. **Jon GLIMMS** and **Toshihiro SHIMIZU** (Kawasaki, Japan).

More generally, suppose there are n problems with $n\geq 4$. Let S_n be the maximum number of students who took the test with n problems. If $S_1>3$, then there would exist 2 students with the same choice and 1 problem cannot distinguish these 2 students. Now $S_1=3$ is certainly possible by given condition. In general if there is a problem which 3 students have different choices, then we say the problem *distinguish* them.

By pigeonhole principle, for problem 1, there is a choice among *A, B, C*, which at most $[S_n/3]$ selected. For the remaining at least $S_n-[S_n/3]$ students, problem 1 does not distinguish any 3 of them. So problem 2 to n will be used to distinguish these remaining students. Then $S_{n-1}\geq S_n-[S_n/3] \geq 2S_n/3$. Hence, $S_n\leq 3S_{n-1}/2$. So $S_2\leq 4$, $S_3\leq 6$ and $S_4\leq 9$.

The following table will show $S_4=9$:

Student\problem	I	II	III	IV
1	A	A	A	A
2	A	B	B	B
3	A	C	C	C
4	B	A	C	B
5	B	B	A	C
6	B	C	B	A
7	C	A	B	C
8	C	B	C	A
9	C	C	A	B

Other commended solvers: **Joe SPENCER**.

Problem 485. Let m and n be integers such that $m>n>1$, $S=\{1,2,\ldots,m\}$ and $T=\{a_1, a_2, \ldots, a_n\}$ is a subset of S. It is known that every two numbers in T do not both divide any number in S. Prove that

$$\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}<\frac{m+n}{n}.$$

Solution. **Jon GLIMMS** and **Toshihiro SHIMIZU** (Kawasaki, Japan).

For i=1,2,…,n, let

$$T_i=\{\, k\in S : k \text{ is divisible by } a_i \,\}.$$

Then T_i has $[m/a_i]$ elements. Since every pair of numbers in T do not both divide any number in S, so if $i\neq k$, then T_i and T_k are disjoint. Now the number of elements in the union of the sets $T_1, T_2, \ldots, T_n$ is

$$\left[\frac{m}{a_1}\right]+\left[\frac{m}{a_2}\right]+\cdots+\left[\frac{m}{a_n}\right]\leq m.$$

Using $m/a_i < [m/a_i]+1$, we have

$$\sum_{i=1}^{n}\frac{m}{a_i}\leq\sum_{i=1}^{n}\left[\frac{m}{a_i}\right]+\sum_{i=1}^{n}1\leq m+n.$$

Then $m\sum_{i=1}^{n}\frac{1}{a_i}<m+n.$ Therefore,

$$\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}<\frac{m+n}{m}<\frac{m+n}{n}.$$

Other commended solvers: **Joe SPENCER**.

Olympiad Corner

(Continued from page 1)

Problem 3. (Continued) Let N be the intersection of lines PR and AC, and let M be the intersection of line AB and the line through R parallel to AC. Prove that line MN is tangent to ω.

Problem 4. The country Dreamland consists of 2016 cities. The airline Starways wants to establish some one-way flights between pairs of cities in such a way that each city has exactly one flight out of it. Find the smallest integer k such that no matter how Starways establishes its flights, the city can always be partitioned into k groups so that from any city it is not possible to reach another city in the same group by using at most 28 flights.

Problem 5. Find all functions f : $\mathbb{R}^+\to\mathbb{R}^+$ such that

$$(z+1)f(x+y)=f(xf(z)+y)+f(yf(z)+x),$$

for all positive real numbers x, y, z.

Inequalities of Sequences

(Continued from page 2)

<u>Solution.</u> For every integer n, we have

$$\begin{aligned}\sqrt{2}n-3&<[\sqrt{2}(n-1)]\\ &<a_n=[\sqrt{2(n^2-n+1/2)}] \qquad (*)\\ &\leq[\sqrt{2}n]<\sqrt{2}n.\end{aligned}$$

From this, we get

$$n^2+(n+1)^2-(n-1)^2-n^2=4n>2a_n+1.$$

Hence,

$$\begin{aligned}a_{n+1}&=[\sqrt{n^2+(n+1)^2}]\geq[\sqrt{a_n^2+4n}]\\ &\geq[\sqrt{a_n^2+2a_n+1}]=a_n+1\end{aligned}$$

for n=1,2,3,…. If (a) is false, then there exists N such that

$$a_{k+1}-a_k=1 \quad \text{for all } k\geq N. \qquad (**)$$

So $a_{N+k}=a_N+k$ for k=0,1,2,3,…. By (*), for k=0,1,2,3,…, we have

$$\sqrt{2}(N+k)-3<a_{N+k}=a_N+k,$$

i.e. $(\sqrt{2}-1)k<a_N+3-\sqrt{2}N$. Since N is constant, when k is large, this leads to a contradiction. So (a) must be true.

Next assume (b) is false. By (**), we can see there exists N such that

$$a_{k+1}-a_k\geq 2 \quad \text{for all } k\geq N.$$

Then $a_{N+k}\geq a_N+2k$ for k = 0,1,2,3,…. By (*), we have

$$a_N+2k<\sqrt{2}(N+k),$$

which is the same as

$$(2-\sqrt{2})k<\sqrt{2}N-a_N.$$

This leads to a contradiction when k is large. So (b) must be true.