Mathematical Excalibur

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Olympiad Corner

Below are the problems of the 20th Hong Kong (China) Mathematical Olympiad held on December 2, 2017. Time allowed is 3 hours.

Problem 1. The sequence $\{x_n\}$ is defined by $x_1=5$ and $x_{k+1}=x_k^2-3x_k+3$ for $k=1,2,3,\ldots$. Prove that $x_k > 3^{2^{k-1}}$ for all positive integer k.

Problem 2. Suppose $ABCD$ is a cyclic quadrilateral. Produce DA and DC to P and Q respectively such that $AP=BC$ and $CQ=AB$. Let M be the midpoint of PQ. Show that $MA\perp MC$.

Problem 3. Let k be a positive integer. Prove that there exists a positive integer ℓ with the following property: if m and n are positive integers relatively prime to ℓ such that $m^m\equiv n^n$ (mod ℓ), then $m\equiv n$ (mod k).

Problem 4. Suppose 2017 points in a plane are given such that no three points are collinear. Among the triangles formed by any three of these 2017 points, those triangles having the largest area are said to be *good*. Prove that there cannot be more than 2017 good triangles.

Acknowledgment: Thanks to Sindy Ting, Math. Dept., HKUST for general assistance.

On-line: http://www.math.ust.hk/excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***February 10, 2018***.

For individual subscription for the next five issues for the 17-18 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Functional Inequalities

Kin Y. Li

In the volume 8, number 1 issue of Math Excalibur, we provided a number of examples of functional equation problems. In the volume 10, number 5 issue of Math Excalibur, problem 243 in the problem corner section was the first functional inequality problem we posed. That one was from the 1998 Bulgarian Math Olympiad. In this article, we would like to look at some functional inequality problems that appeared in various math Olympiads.

<u>*Example 1 (2016 Chinese Taipei Math Olympiad Training Camp.*</u> Let function $f:[0,+\infty)\to[0,+\infty)$ satisfy

(1) for arbitrary $x,y\ge 0$, we have

$$f(x)f(y)\le y^2 f\left(\frac{x}{2}\right)+x^2 f\left(\frac{y}{2}\right);$$

(2) for arbitrary $0\le x\le 1$, we have $f(x)\le 2016$.

Prove that for arbitrary $x\ge 0$ we have $f(x)\le x^2$.

<u>*Solution.*</u> In (1), let $x=y=0$, then $f(0)=0$. Assume there is $x_0>0$ such that $f(x_0)>x_0^2$. By (1), we see $f(x_0/2)>x_0^2/2$. By math induction, for all positive integer k, we have

$$f\left(x_0/2^k\right)>2^{2^k-2k-1}x_0^2.$$

As k gets large, eventually we have $x_0/2^k$ is in $[0,1]$, but $f(x_0/2^k)>2016$. This contradicts (2). So for all $x\ge 0$, $f(x)\le x^2$.

<u>*Example 2 (2005 Russian Math Olympiad).*</u> Does there exist a *bounded* function $f:\mathbb{R}\to\mathbb{R}$ such that $f(1)>0$ and for all $x,y\in\mathbb{R}$, it satisfies the inequality

$$f^2(x+y)\ge f^2(x)+2f(xy)+f^2(y)\ ?$$

<u>*Solution.*</u> Assume such f exists. Let $a=2f(1)>0$. For $x_1\ne 0$, let $y_1=1/x_1$, then

$$f^2(x_1+y_1)\ge f^2(x_1)+2f(1)+f^2(y_1)\ge f^2(x_1)+a.$$

For $n>1$, let $x_n=x_{n-1}+y_{n-1}$, $y_n=1/x_n$. Then

$$f^2(x_n+y_n)\ge f^2(x_n)+a=f^2(x_{n-1}+y_{n-1})+a \ge f^2(x_{n-1})+2a\ge\cdots\ge f^2(x_1)+na.$$

As $n\to\infty$, f becomes unbounded, which is a contradiction.

<u>*Example 3 (2016 Ukranian Math Olympiad).*</u> Does there exist a function $f:\mathbb{R}\to\mathbb{R}$ such that for arbitrary real numbers x, y, we have

$$f(x-f(y))\le x-yf(x)\ ?$$

<u>*Solution.*</u> Assume such function exists. Let $y=0$. Then $f(x-f(0))\le x$. Replacing x by $x+f(0)$, we get $f(x)\le x+f(0)$. Then setting $x=f(y)$, we get

$$f(0)\le f(y)-yf(f(y))\le y+f(0)-yf(f(y)),$$

which implies $yf(f(y))\le y$. If $y<0$, then

$$1\le f(f(y))\le f(y)+f(0)\le y+2f(0).$$

The last inequality is satisfied for all $y<0$, which is a contradiction.

<u>*Example 4 (The Sixth IMAR Math Competition, 2008).*</u> Show that for any function $f:(0,+\infty)\to(0,+\infty)$ there exists real numbers $x>0$ and $y>0$ such that $f(x+y)<yf(f(x))$.

<u>*Solution.*</u> Assume $f(x+y)\ge yf(f(x))$ for all $x,y>0$. Let $a>1$, then $t=f(f(a))>0$. Now for $b\ge a(1+t^{-1}+t^{-2})>a$, we have

$$f(b)=f(a+(b-a))\ge(b-a)f(f(a))=(b-a)t\ge a(1+t^{-1})>a.$$

Then

$$f(f(b))=f(a+(f(b)-a))\ge(f(b)-a)t\ge a.$$

If we take $x\ge(ab+2)/(a-1)>b$, then

$$f(x)=f(b+(x-b))\ge(x-b)f(f(b))\ge(x-b)a\ge x+2.$$

Hence, $f(x)>x+1$ (*). However,

$$f(f(x))=f(x+(f(x)-x))\ge(f(x)-x)f(f(x)).$$

Cancelling $f(f(x))$ on both sides, we get $f(x)\le x+1$, which contradicts (*).

(continued on page 2)

Example 5 (2016 Romanian Math Olympiad). Determine all functions $f:\mathbb{R}\to\mathbb{R}$ satisfying for arbitrary $a, b\in\mathbb{R}$, we have

$$f(a^2)-f(b^2)\le(f(a)+b)(a-f(b)). \quad (1)$$

Solution. Let $a=b=0$, then $f^2(0)\le0$, so $f(0)=0$. Let $b=0$, then $f(a^2)\le af(a)$. Let $a=0$, then $f(b^2)\ge bf(b)$. So for all x, we have (2) $f(x^2)=xf(x)$. Using this on the left side of (1), we get (3) $f(a)f(b)\le ab$. Next, by (2), we have

$$-xf(-x)=f((-x)^2)=f(x^2)=xf(x).$$

So f is an odd function. This implies

$$f(a)f(b)=-f(a)f(-b)\ge-(-ab)=ab.$$

Using (3), we have $f(a)f(b)=ab$. Then $f^2(1)=1$. So $f(1)=\pm1$. Hence, for all x, $f(x)f(1)=x$, i.e. either $f(x)=x$ for all x or $f(x)=-x$ for all x. Simple checking shows both of these satisfy (1).

Example 6 (1994 APMO). Let $f: \mathbb{R}\to\mathbb{R}$ be a function such that

(i) for all $x,y\in\mathbb{R}$

$$f(x)+f(y)+1\ge f(x+y)\ge f(x)+f(y),$$

(ii) for all $x\in[0,1)$, $f(0)\ge f(x)$,

(iii) $-f(-1)=f(1)=1$.

Find all such functions.

Solution. By (iii), $f(-1)=-1$, $f(1)=1$. So $f(0)=f(-1+1)\ge f(1)+f(-1)=0$. By (i), $f(1)=f(1+0)\ge f(1)+f(0)$. So $f(0)\le0$. Then $f(0)=0$.

Next we claim $f(x)=0$ for all x in $(0,1)$. Since $f(0)=0$, by (ii), $f(x)\le0$ for all x in $(0,1)$. By (i) and (ii), $f(x)+f(1-x)+1\ge f(1)=1$. So $f(x)\ge-f(1-x)$. If $x\in(0,1)$, then $1-x\in(0,1)$. So $f(1-x)\le0$ and $f(x)\ge-f(1-x)\ge0$. Then $f(x)=0$.

Next by (i) and (iii), we have $f(x+1)\ge f(x)+f(1)=f(x)+1$ and $f(x)\ge f(x+1)+f(-1)=f(x+1)-1$. These give $f(x+1)=f(x)+1$.

So $f(x)=0$ for $x\in[0,1)$ and $f(x+1)=f(x)+1$. Hence, $f(x)=[x]$. We can check directly $[x]$ satisfies (i), (ii) and (iii).

Example 7 (2007 Chinese IMO Team Training Test). Does there exist any function $f:\mathbb{R}\to\mathbb{R}$ satisfy $f(0)>0$ and

$$f(x+y)\ge f(x)+yf(f(x)) \text{ for all } x,y\in\mathbb{R}?$$

Solution. Assume such function exists. In that case, we claim there would exist real z such that $f(f(z))>0$. (Otherwise, for all x, $f(f(x))\le0$. So for all $y\le0$, we have $f(x+y)\ge f(x)+yf(f(x))\ge f(x)$. Then f is a decreasing function. So for all $x\in\mathbb{R}$, $f(0)>0\ge f(f(x))$, which implies $f(x)>0$. This contradicts $f(f(x))\le0$.)

From the claim, we see as $x\to+\infty$, $f(z+x)\ge f(z)+xf(f(z))\to+\infty$. So we get

$$f(x)\to+\infty \text{ as well as } f(f(x))\to+\infty.$$

Then there are $x, y>0$ such that $f(x)\ge0$, $f(f(x))>1$, $f(x+y)>0$, $f(f(x+y+1))>0$ and (*) $y\ge(x+1)/(f(f(x))-1)$. Define $A=x+y+1$, $B=f(x+y)-(x+y+1)$. Then $f(f(A))>0$ and

$$f(x+y)\ge f(x)+yf(f(x))\ge x+y+1 \text{ by (*)}.$$

So $B\ge0$. Next,

$$\begin{aligned} f(f(x+y)) &= f(A+B)\ge f(A)+Bf(f(A))\\ &\ge f(A)=f((x+y)+1)\\ &= f(x+y)+f(f(x+y))\\ &> f(f(x+y)), \end{aligned}$$

which is a contradiction.

Example 8 (2015 Greek IMO Team Selection Test). Determine all functions $f:\mathbb{R}\to\mathbb{R}$ such that for arbitrary $x, y\in\mathbb{R}$, we have

$$f(xy)\le yf(x)+f(y). \quad (1)$$

Solution. In (1), using $-y$ to replace y, we get

$$f(-xy)\le-yf(x)+f(-y). \quad (2)$$

Adding (1) and (2), we get

$$f(xy)+f(-xy)\le f(y)+f(-y). \quad (3)$$

Setting $y=1$, we get

$$f(x)+f(-x)\le f(1)+f(-1). \quad (4)$$

In (3), using $1/y$ with $y\neq0$ to replace x, we get

$$f(1)+f(-1)\le f(y)+f(-y). \quad (5)$$

By (4) and (5), for $y\neq0$, we have

$$f(y)+f(-y)=f(1)+f(-1).$$

Let $c=f(1)+f(-1)$. Then (2) becomes

$$c-f(xy)\le-yf(x)+c-f(y).$$

Then

$$f(xy)\ge yf(x)+f(y). \quad (6)$$

By (1) and (6), for all $x,y\neq0$,

$$f(xy)=yf(x)+f(y). \quad (7)$$

Setting $x=y=1$, we get $f(1)=0$. In (7), interchanging x and y, we get

$$f(yx)=xf(y)+f(x). \quad (8)$$

Subtracting (7) and (8), we get

$$(y-1)f(x)=(x-1)f(y).$$

Then for $x, y\neq0,1$, we get $\dfrac{f(x)}{x-1}=\dfrac{f(y)}{y-1}$.

Since $f(1)=0$, we see there exists a such that $f(x)=a(x-1)$ for all $x\neq0$. Setting $x=0$ in (1), we get $f(y)\ge(1-y)f(0)$. Then for $y\neq0$, we get $a(y-1)\ge(1-y)f(0)$, which is $(y-1)(a+f(0))\ge0$. Then $a=-f(0)$ and we get for all real x, $f(x)=f(0)(1-x)$. Setting $f(0)$ to be any real constant, we can check all such functions satisfy (1).

Example 9 (2013 Croatian IMO Team Selection Test). Determine all functions $f:\mathbb{R}\to\mathbb{R}$ such that for all real numbers x, y, we have $f(1)\ge0$ and

$$f(x)-f(y)\ge(x-y)f(x-y). \quad (*)$$

Solution. Setting $y=x-1$, we get $f(x)-f(x-1)\ge f(1)\ge0$. So

$$f(x)\ge f(x-1). \quad (1)$$

Setting $y=0$, we get

$$f(x)-f(0)\ge xf(x). \quad (2)$$

Replacing y by x and x by 0, we get

$$f(0)-f(x)\ge-xf(-x). \quad (3)$$

Adding (2) and (3), we get

$$0\ge xf(x)-xf(-x).$$

Then for every $x>0$, we get

$$f(-x)\ge f(x). \quad (4)$$

Setting $x=1$, $y=0$ in (0), we get

$$f(0)\le0. \quad (5)$$

By (5), (1), (4), we get $0\ge f(0)\ge f(-1)\ge f(1)\ge0$. So $f(0)=f(-1)=f(1)=0$. Using (1) repeatedly, we get

$$f(x)\ge f(x-1)\ge f(x-2)\ge\cdots, \quad (6)$$

i.e. $f(x)\ge f(x-k)$ for all real x, positive integer k. Using (6), (1) and replacing x by $x-1$ and y by -1 in (*), we get

$$f(x)\ge f(x-1)=f(x-1)-f(-1)\ge xf(x).$$

Then $f(x)(x-1)\le0$. So if $x>1$, then $f(x)\le0$. If $x<1$, then $f(x)\ge0$.

For $x>1$, there is $y<1$ such that $k=x-y$ is a positive integer. Then

$$0\ge f(x)\ge f(x-k)=f(y)\ge0.$$

So for $x>1$, $f(x)=0$. Similarly, for $x<1$, there is $y>1$ such that $k=y-x$ is a positive integer. Then as above, all $f(x)=0$. We can check directly $f(x)=0$ satisfies (*).

Example 10 (2011 IMO Problem 3 proposed by Belarus). Let $f:\mathbb{R}\to\mathbb{R}$ be a real-valued function defined on the set of real numbers that satisfies

$$f(x+y)\le yf(x)+f(f(x)) \quad (1)$$

for all real numbers x and y. Prove that $f(x)=0$ for all $x\le0$.

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Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong*. The deadline for sending solutions is ***February 10, 2018.***

Problem 506. Points A and B are on a circle Γ_1. Line AB is tangent to another circle Γ_2 at B and the center O of Γ_2 is on Γ_1. A line through A intersects Γ_2 at points D and E (with D between A and E). Line BD intersects Γ_1 at a point F, different from B. Prove that D is the midpoint of BF if and only if BE is tangent to Γ_1.

Problem 507. Determine all functions $f:\mathbb{R}\rightarrow\mathbb{R}$ such that for all $x, y \in \mathbb{R}$,

$$(x-2)f(y) + f(y+2f(x)) = f(x+yf(x)).$$

Problem 508. Determine the largest integer k such that for all integers x,y, if $xy+1$ is divisible by k, then $x+y$ is also divisibly by k.

Problem 509. In $\triangle ABC$, the angle bisector of $\angle CAB$ intersects BC at a point L. On sides AC, AB, there are points M, N respectively such that lines AL, BM, CN are concurrent and $\angle AMN=\angle ALB$. Prove that $\angle NML= 90°$.

Problem 510. Numbers 1 to 20 are written on a board. A person randomly chooses two of these numbers with a difference of at least 2. He adds 1 to the smaller one and subtracts 1 from the larger one. Then he performs an operation by replacing the original two chosen numbers on the board with the two new numbers. Determine the maximum number of times he can do this operation.

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Solutions

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Problem 501. Let x, y, s, m, n be positive integers such that $x+y=s^m$ and $x^2+y^2=s^n$. Determine the number of digits s^{300} has in base 10.

Solution. **CHUI Tsz Fung** (Ma Tau Chung Government Primary School, P4), **Soham GHOSH** (RKMRC Narendrapur, Kalkata, India), **Mark LAU, LEE Jae Woo** (Hamyang High School, South Korea), **Toshihiro SHIMIZU** (Kawasaki, Japan).

Since $s^{2m} = (x+y)^2 > x^2+y^2 = s^n$, so $2m > n$. Then

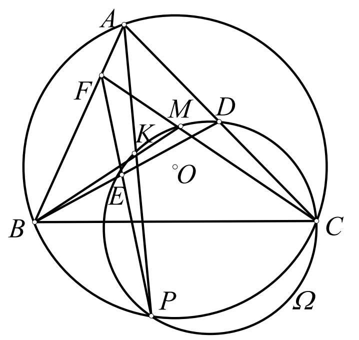
$$0 \le (x-y)^2 = 2(x^2+y^2)-(x-y)^2 = 2s^n-s^{2m} = s^n(2-s^{2m-n}).$$

If $s\ge3$, then we have $2-s^{2m-n}\le2-s<0$, a contradiction. If $s=1$, then we have $1+1\le x+y=s^m=1$, a contradiction. So s must be 2. Since $\log_{10}2^{300} = 300\log_{10}2 = 0.3010\ldots\times300= 90.3\ldots$, 2^{300} has 91 digits.

Other commended solvers: **DBS Maths Solving Team** (Diocesan Boys' School), **Akash Singha ROY** (Hariyana Vidya Mandir High School, India) and **George SHEN**.

Problem 502. Let O be the center of the circumcircle of acute $\triangle ABC$. Let P be a point on arc BC so that A, P are on opposite sides of side BC. Point K is on chord AP such that BK bisects $\angle ABC$ and $\angle AKB > 90°$. The circle Ω passing through C, K, P intersect side AC at D. Line BD meets Ω at E and line PE meets side AB at F. Prove that $\angle ABC = 2\angle FCB$.

Solution. **George SHEN** and **Toshihiro SHIMIZU** (Kawasaki, Japan).



Take point M on line KB such that $MB=MC$. Then we have $\triangle BMC$ is isosceles and

$$\angle KPC = \angle APC=\angle ABC = \angle MBC+\angle MCB = 180°-\angle BMC = 180°-\angle KMC.$$

This implies M is on the circle Ω. Applying Pascal's theorem to the points P, E, D, C, M, K on Ω, we have $PE\cap CM$, $ED\cap MK=B$ and $DC\cap KP=A$ are collinear. Since this line coincides with line AB, so $PE\cap CM=F$. Then

$$2\angle FCB=2\angle MCB=2\angle MBC=\angle ABC.$$

Other commended solvers: **LEE Jae Woo** (Hamyang High School, South Korea), **Vijaya Prasad NALLURI** (Retd Principal APES, Rajahmundry, India) and **Akash Singha ROY** (Hariyana Vidya Mandir High School, India).

Problem 503. Let S be a subset of $\{1,2,\ldots,2015\}$ with 68 elements. Prove that S has three pairwise disjoint subsets A, B, C such that they have the same number of elements and the sums of the elements in A, B, C are the same.

Solution. **Mark LAU** and **George SHEN**.

There are totally $(68\times67\times66)/6=50116$ 3-element subsets of S. The possible sums of the three elements in these subsets of S are from $1+2+3=6$ to $2013+2014+2015=6042$. Now $50116 > 8\times(6042-6+1)$. So by the pigeonhole principle, there are 9 distinct 3-element subsets $A_1, A_2,\ldots, A_9$ of S with the same sum of elements.

Assume $x\in S$ appears in $A_1, A_2,\ldots, A_9$ at least 3 times, say in A_1, A_2, A_3. Then no two of the sets $U=A_1\setminus\{x\}$, $V=A_2\setminus\{x\}$, $W=A_3\setminus\{x\}$ are the same. Otherwise say $U=V$, then $A_1=A_2$, contradiction.

So every $x\in S$ appear at most twice among $A_1, A_2,\ldots, A_9$. Then there can only be at most 3 of $A_2,\ldots, A_9$ (say A_2, A_3, A_4) having an element in common with A_1 (as every element of A_1 can only appear in at most one of $A_2,\ldots, A_9$). Without loss of generality, say each of $A_5,\ldots, A_9$ is disjoint with A_1. Similarly, among $A_6,\ldots, A_9$, there are at most three of them (say A_6, A_7, A_8) have a common element with A_5. Then A_9 and A_5 are disjoint. So the pairwise disjoint sets $A=A_1$, $B=A_5$, $C=A_9$ have the same sum of elements.

Other commended solvers: **LEE Jae Woo** (Hamyang High School, South Korea), **Akash Singha ROY** (Hariyana Vidya Mandir High School, India), and **Toshihiro SHIMIZU** (Kawasaki, Japan).

Problem 504. Let $p>3$ be a prime number. Prove that there are infinitely many positive integers n such that the sum of k^n for $k=1,2,\ldots,p-1$ is divisible by p^3.

Solution. **CHUI Tsz Fung** (Ma Tau Chung Government Primary School, P4), **DBS Maths Solving Team**

(Diocesan Boys' School), **Mark LAU**, **LEE Jae Woo** (Hamyang High School, South Korea), **LEUNG Hei Chun** (SKH Tang Shiu Kin Secondary School), **Akash Singha ROY** (Hariyana Vidya Mandir High School, India) and **Toshihiro SHIMIZU** (Kawasaki, Japan).

As $\varphi(p^3)=p^2(p-1)$, by Euler's theorem, for all positive integers r,s, we have

$$\sum_{k=1}^{p-1} k^{r+p^2(p-1)s} \equiv \sum_{k=1}^{p-1} k^r \pmod{p^3}.$$

In the case $r=p^2$, we have

$$\sum_{k=1}^{p-1} k^{p^2} = \sum_{k=1}^{(p-1)/2} (k^{p^2}+(p-k)^{p^2})$$

$$= \sum_{k=1}^{(p-1)/2} \left(k^{p^2} + \sum_{t=0}^{p^2} \binom{p^2}{t} p^t (-k)^{p^2-t} \right)$$

$$\equiv \sum_{k=1}^{(p-1)/2} \left(p^3 k^{p^2-1} - \frac{p^4(p^2-1)}{2} k^{p^2-2} \right)$$

$$\equiv 0 \pmod{p^3}.$$

So all cases $n=p^2+p^2(p-1)s$ works.

Other commended solvers: **Soham GHOSH** (RKMRC Narendrapur, Kalkata, India) and **George SHEN.**

Problem 505. Determine (with proof) the least positive real number r such that if z_1, z_2, z_3 are complex numbers having absolute values less than 1 and sum 0, then

$$|z_1z_2+z_2z_3+z_3z_1|^2 + |z_1z_2z_3|^2 < r.$$

Solution. **Akash Singha ROY** (Hariyana Vidya Mandir High School, India) and **George SHEN**.

For i=1,2,3, let $a_i=|z_i|^2$, then $0\le a_i<1$. Since $z_1+z_2+z_3=0$, we have

$$\begin{aligned} & z_2\overline{z_3}+\overline{z_2}z_3 \\ &= (z_2+z_3)(\overline{z_2}+\overline{z_3}) - |z_2|^2 - |z_3|^2 \\ &= (-z_1)(-\overline{z_1}) - a_2 - a_3 \\ &= a_1 - a_2 - a_3. \end{aligned}$$

Let $b= z_1z_2+z_2z_3+z_3z_1$ and $c= z_1z_2z_3$. Let the notation $\Sigma f(u,v,w)$ denote the sum of $f(u,v,w)$, $f(v,w,u)$ and $f(w,u,v)$. We have

$$\begin{aligned} & |z_1z_2+z_2z_3+z_3z_1|^2 + |z_1z_2z_3|^2 \\ &= b\overline{b} + c\overline{c} \\ &= \sum |z_1z_2|^2 + \sum |z_1|^2 (z_2\overline{z_3}+\overline{z_2}z_3) + |z_1z_2z_3|^2 \\ &= \sum a_1a_2 + \sum a_1(z_2\overline{z_3}+\overline{z_2}z_3) + a_1a_2a_3 \\ &= \sum a_1a_2 + \sum a_1(a_1-a_2-a_3) + a_1a_2a_3 \\ &= a_1^2+a_2^2+a_3^2 - a_1a_2 - a_2a_3 - a_3a_1 + a_1a_2a_3 \\ &\le a_1+a_2+a_3 - a_1a_2 - a_2a_3 - a_3a_1 + a_1a_2a_3 \\ &= 1-(1-a_1)(1-a_2)(1-a_3) < 1. \end{aligned}$$

Next, for $0<x<1$, consider $z_1=x$, $z_2=-x$ and $z_3=0$. Then $|z_1z_2+z_2z_3+z_3z_1|^2 + |z_1z_2z_3|^2 = x^4 < r$. Letting x tend to 1, we get $1\le r$. Therefore, the least positive r is 1.

Other commended solvers: **DBS Maths Solving Team** (Diocesan Boys' School), **LEE Jae Woo** (Hamyang High School, South Korea) and **Toshihiro SHIMIZU** (Kawasaki, Japan).

Functional Inequalities

(Continued from page 2)

Solution. In (1), let $y=t-x$, then

$$f(t)\le tf(x)-xf(x)+f(f(x)). \qquad (2)$$

Consider $a,b\in\mathbb{R}$. Using (2) to $t=f(a)$, $x=b$ and $t=f(b)$, $x=a$, we get

$$f(f(a))-f(f(b)) \le f(a)f(b) - bf(b),$$
$$f(f(b))-f(f(a)) \le f(b)f(a) - af(a).$$

Adding these, we get

$$2f(a)f(b)\ge af(a)+bf(b).$$

Setting $b=2f(a)$, we get

$$2f(a)f(b)\ge af(a)+2f(a)f(b) \text{ or } af(a)\le 0.$$

Then for $a<0, f(a)\ge 0. \qquad (3)$

Now suppose $f(x)>0$ for some x. By (2), we see for every $t<(xf(x)-f(f(x)))/f(x)$, we have $f(t)<0$. This contradicts (3). So

$$f(x)\le 0 \quad \text{for all real } x. \qquad (4)$$

By (3) again, we get $f(x)=0$ for all $x<0$. Finally setting $t=x<0$ in (2), we get $f(x) \le f(f(x))$. As $f(x)=0$, this implies $0\le f(0)$. This together with (4) give $f(0)=0$.

Example 11 (2009 IMO Shortlisted Problem proposed by Belarus). Let f be any function that maps the set of real numbers into the set of real numbers. Prove that there exist real numbers x and y such that

$$f(x-f(y)) > yf(x)+x. \qquad (1)$$

Solution. Assume the contrary, i.e. $f(x-f(y)) \le yf(x)+x$ for all real x and y. Let $a=f(0)$. Setting y=0 in (1) gives $f(x-a)\le x$ for all real x. This is equivalent to

$$f(y)\le y+a \quad \text{for all real } y. \qquad (2)$$

Setting $x=f(y)$ in (1) and using (2), we get

$$a=f(0)\le yf(f(y))+f(y)\le yf(f(y))+y+a.$$

This implies $0\le y(f(f(y))+1)$ and so

$$f(f(y))\ge -1 \text{ for all } y>0. \qquad (3)$$

By (2) and (3), we get $-1\le f(f(y))\le f(y)+a$ for all $y>0$. So

$$f(y)\ge -a-1 \text{ for all } y>0. \qquad (4)$$

Next, we claim $f(x)\le 0$ for all real x. (5) Assume the contrary, i.e. there is some $f(x)>0$. Now take y such that $y<x-a$ and

$$y<(-a-x-1)/f(x). \qquad (6)$$

By (2), we get $x-f(y) \ge x-(y+a) > 0$. By (1) and (4), we get

$$yf(x)+x \ge f(x-f(y)) \ge -a-1.$$

Then $y \ge (-a-x-1)/f(x)$, contradicting (6). So (5) is true.

Now setting y=0 in (5) leads to $a=f(0)\le 0$ and using (2), we get

$$f(x)\le x \text{ for all real } x. \qquad (7)$$

Now choose $y>0$, $y>-f(-1)-1$ and set $x=f(y)-1$. By (1),(5) and (7), we get

$$\begin{aligned} f(-1) &= f(x-f(y)) \\ &\le yf(x)+x = yf(f(y)-1)+f(y)-1 \\ &\le y(f(y)-1)-1 \le -y-1. \end{aligned}$$

Then $y\le -f(-1)-1$, which contradicts the choice of y.

Example 12 (64th Bulgarian Math Olympiad in 2015). Determine all functions $f:(0,+\infty)\to(0,+\infty)$ such that for arbitrary positive real numbers x, y, we have

(1) $f(x+y)\ge f(x)+y$;
(2) $f(f(x))\le x$.

Solution. As $y>0$, (1) implies f is strictly increasing on $(0,+\infty)$. By (2) and (1), we have

$$x+y\ge f(f(x+y))\ge f(f(x)+y). \qquad (*)$$

Using (*) and in (1), replacing x by y and y by $f(x)$, we get

$$x+y \ge f(f(x)+y) \ge f(x)+f(y). \qquad (**)$$

Since f is strictly increasing and $f(x)>0$, so the limit of $f(x)$ as $x\to 0^+$ is a nonnegative number c. By (2), the limit of $f(f(x))$ as $x\to 0^+$ is 0.

If $c>0$, then since f is strictly increasing, $f(f(x)) \ge f(c) > 0$. Taking the limit of $f(f(x))$ as $x\to 0^+$ leads to $0 \ge f(c) > 0$, contradiction. So c=0.

Now taking limit as $y\to 0^+$ in (**), we get $x\ge f(x)$ for all $x>0$. This and (1) lead to

$$x+y\ge f(x+y)\ge f(x)+y. \qquad (***)$$

Subtracting $f(x)+y$ in (***), we get $x-f(x) \ge f(x+y)-f(x)-y \ge 0$. Letting $w=x+y$ in (***) and taking limit of $w\ge f(w)\ge f(x)+w-x$ as $x\to 0^+$, we get $w=f(w)$. So $f(x+y)=f(w)=w=x+y$. Then f is the identity function on $(0,+\infty)$, which certainly satisfy (1) and (2).