Mathematical Excalibur

Volume 21, Number 4 *February 2018 – March 2018*

Olympiad Corner

Below were the problems of the 2017 Serbian Mathematical Olympiad for high school students. The event was held in Belgrade on March 31 and April 1, 2017.

Time allowed was 270 minutes.

First Day

Problem 1. (*Nikola Petrović*) Let a, b and c be positive real numbers with $a+b+c=1$. Prove the inequality

$$a\sqrt{2b+1}+b\sqrt{2c+1}+c\sqrt{2a+1} \le \sqrt{2-(a^2+b^2+c^2)}.$$

Problem 2. (*Dušan Djukić*) A convex quadrilateral $ABCD$ is inscribed in a circle. The lines AD and BC meet at point E. Points M and N are taken on the sides AD, BC respectively, so that $AM:MD=BN:NC$. Let the circumcircles of triangle EMN and quadrilateral $ABCD$ intersect at points X and Y. Prove that either the lines AB, CD and XY have a common point or they are all parallel.

(continued on page 4)

Acknowledgment: Thanks to Sindy Ting, Math. Dept., HKUST for general assistance.

On-line: http://www.math.ust.hk/excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***April 21, 2018***.

For individual subscription for the next five issues for the 17-18 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

Dr. Kin-Yin LI, Math Dept., Hong Kong Univ. of Science and Technology, Clear Water Bay, Kowloon, Hong Kong

Fax: (852) 2358 1643
Email: makyli@ust.hk

Perfect Squares

Kin Y. Li

In this article, we will be looking at one particular type of number theory problems, namely problems on integers that have to do with the set of perfect squares 1, 4, 9, 16, 25, 36, …. This kind of problems have appeared in many Mathematical Olympiads from different countries for over 50 years. Here are some examples.

<u>Example 1 (1953 Kürschák Math Competition Problems).</u> Let n be a positive integer and let d be a positive divisor of $2n^2$. Prove that n^2+d is not a perfect square.

<u>Solution.</u> We have $2n^2=kd$ for some positive integer k. Suppose $n^2+d=m^2$ for some positive integer m. Then $m^2 = n^2+2n^2/k$ so that $(mk)^2=n^2(k^2+2k)$. Then k^2+2k must also be the square of a positive integer, but $k^2<k^2+2k<(k+1)^2$ leads to a contradiction.

<u>Example 2 (1980 Leningrad Math Olympiad).</u> Find all prime numbers p such that $2p^4-p^2+16$ is a perfect square.

<u>Solution.</u> For $p=2$, $2p^4-p^2+16=44$ is not a perfect square. For $p=3$, $2p^4-p^2+16 =169=13^2$. For prime $p>3$, $p\equiv 1$ or $2 \pmod 3$ and $2p^4-p^2+16\equiv 2 \pmod 3$. Assume $2p^4-p^2+16=k^2$. Then $k^2\equiv 0^2,1^2$ or $2^2\equiv 0$ or $1 \pmod 3$. So $2p^4-p^2+16\ne k^2$. Then $p=3$ is the only solution.

<u>Example 3 (2008 Singapore Math Olympiad).</u> Find all prime numbers p satisfying 5^p+4p^4 is a perfect square.

<u>Solution.</u> Suppose $5^p+4p^4=q^2$ for some integer q. Then

$$5^p = q^2-4p^4 = (q-2p^2)(q+2p^2).$$

Since 5 is a prime number, we have

$$q-2p^2 = 5^s \text{ and } q+2p^2 = 5^t$$

for some integers s, t with $t > s \ge 0$ and $s+t = p$. Eliminating q, we have

$$4p^2 = 5^s(5^{t-s}-1).$$

If $s>0$, then from 5 divides $4p^2$, we get $p=5$. So $5^p+4p^4=5625=75^2$ and $q=75$ is a solution. If $s=0$, then $t=p$. So $5^p=4p^2+1$. Now, for integer $k\ge 2$, we <u>*claim*</u> $5^k>4k^2+1$. The case $k=2$ is clear. Suppose the case k is true. Then

$$\frac{4(k+1)^2+1}{4k^2+1} = 1+\frac{8k}{4k^2+1}+\frac{4}{4k^2+1} < 1+1+1<5.$$

So $5^{k+1}=5\times 5^k > 5(4k^2+1) > 4(k+1)^2+1$. By mathematical induction, the claim is true. Therefore, $5^p = 4p^2+1$ has no prime solution p.

<u>Example 4 (2009 Croatian Math Olympiad).</u> Find all positive integers m, n such that 6^m+2^n+2 is a perfect square.

<u>Solution.</u> If

$$6^m+2^n+2=2(3^m\times 2^{m-1}+2^{n-1}+1)$$

is a perfect square, then $3^m\times 2^{m-1}+2^{n-1}+1$ is even. So one of the integers $3^m\times 2^{m-1}$ and 2^{n-1} is odd and the other is even.

Suppose $3^m\times 2^{m-1}$ is odd, then $m = 1$ and $6^m+2^n+2 = 8+2^n = 4(2^{n-2}+2)$. So $2^{n-2}+2$ is a perfect square. Since every perfect square dived by 4 has remainder 0 or 1, so $2^{n-2}+2$ cannot be of the form 4k+2. Hence, $n-2=1$, i.e. $n=3$. So $(m,n)=(1,3)$ is a solution.

If 2^{n-1} is odd, then $n=1$ and

$$6^m+2^n+2 = 6^m+4 \equiv (-1)^m+4 \pmod 7.$$

This means 6^m+2^n+2 divided by 7 has remainder 3 or 5. However,

$(7k)^2\equiv 0 \pmod 7$, $(7k\pm 1)^2\equiv 1 \pmod 7$, $(7k\pm 2)^2\equiv 4 \pmod 7$, $(7k\pm 3)^2\equiv 2 \pmod 7$.

So every perfect square divided by 7 cannot have remainder 3, 5 or 6. Therefore, $(m,n) = (1,3)$ is the only solution.

(continued on page 2)

<u>Example 5 (2008 German Math Olympiad).</u> Determine all real numbers x such that $4x^5-7$ and $4x^{13}-7$ are perfect squares.

<u>Solution.</u> Suppose there are positive integers a and b such that

$$4x^5-7=a^2 \quad\text{and}\quad 4x^{13}-7=b^2.$$

Then $x^5=(a^2+7)/4>1$ is rational and $x^{13}=(b^2+7)/4>1$ is rational. So $x=(x^5)^8/(x^{13})^3$ is rational. Suppose $x=p/q$ with p and q positive relatively prime integers. Then from $(p/q)^5=(a^2+7)/4$, it follows q^5 divides $4p^5$ and so q=1. So x must be a positive integer and $x\ge 2$.

In the case x is an odd integer, we have $a^2\equiv 0, 1, 4 \pmod 8$, but $a^2=4x^5-7\equiv 5 \pmod 8$, contradiction. So x is even. In the case x=2, we have $4x^5-7=11^2$ and $4x^{13}-7=181^2$. For an even $x\ge 4$, $(ab)^2=(4x^5-7)(4x^{13}-7)=16x^{18}-28x^{13}-28x^7+49$. However, expanding $(4x^9-7x^4/2-1)^2$ and $(4x^9-7x^4/2)^2$ and using $x^9\ge 4x^8\ge 4^2x^7\ge 4^5x^4$, we see $(ab)^2$ is strictly between them. Then x=2 is the only solution.

<u>Example 6 (2011 Iranian Math Olympiad).</u> Integers a, b satisfy $a>b$. Also $ab-1$, $a+b$ are relatively prime and $ab+1$, $a-b$ are relatively prime. Prove that $(a+b)^2+(ab-1)^2$ is not a perfect square.

<u>Solution.</u> Assume $(a+b)^2+(ab-1)^2=c^2$ for some integer c. Then

$$c^2=a^2+b^2+a^2b^2+1=(a^2+1)(b^2+1).$$

Assume (*) there is a prime p such that $p \mid a^2+1$ and $p \mid b^2+1$, then $p \mid a^2+1-b^2+1=a^2-b^2$. So (**) $p \mid a-b$ or $p \mid a+b$.

Assume $p \mid a-b$. Then $p \mid ab-b^2$. Since $p \mid b^2+1$, so $p \mid ab-b^2+b^2+1=ab+1$, which contradicts $ab+1$, $a-b$ are relatively prime. Similarly, assume $p \mid a+b$. Then $p \mid ab+b^2$. Since $p \mid a^2+1$, so $p \mid ab+b^2+b^2-1=ab-1$, which contradicts $ab-1$, $a+b$ are relatively prime. So (**) as well as (*) are wrong.

Then a^2+1, b^2+1 are relatively prime. Since $a>b$, not both of them are 0. So $(a+b)^2+(ab-1)^2$ equals a^2+1 (if b=0) or b^2+1 (if a=0) or $(a^2+1)(b^2+1)$. Then $(a+b)^2+(ab-1)^2$ is not a perfect square.

<u>Example 7 (2000 Polish Math Olympiad).</u> Let m, n be positive integers such that m^2+n^2+m is divisible by mn. Prove that m is a perfect square.

<u>Solution.</u> Since m^2+n^2+m is divisible by mn, so for some positive integer k, $m^2+n^2+m=kmn$. Then $n^2-kmn+(m^2+m)=0$, which can be viewed as a quadratic equation in n. Then the discriminant $\Delta=k^2m^2-4m^2-4m$ is a perfect square. Suppose d is $\gcd(m, k^2m-4m-4)=1$. If d=1, then m (and $k^2m-4m-4$) are both perfect squares. If $d>1$, then

$$d=\gcd(m, k^2m-4m-4)=\gcd(m,4).$$

Since $d>1$ divides 4, so d is even. Then m is even. Also, $n^2\equiv m^2+n^2+m \pmod 2$. So n is even. Then mn, m^2+n^2 are divisible by 4.

As m^2+n^2+m is given to be divisible by mn, so m^2+n^2+m is divisible by 4. Then $m=m^2+n^2+m-(m^2+n^2)$ is divisible by 4. So we get $d=4$. Then

$$1=\gcd(m/4, k^2(m/4)-m-1).$$

Now $\Delta/16=(m/4)(k^2(m/4)-m-1)$ is a perfect square. So $m/4$ and $k^2(m/4)-m-1$ are perfect squares. Therefore, m is a perfect square.

<u>Example 8 (2006 British Math Olympiad).</u> Let n be an integer If $2+2\sqrt{1+12n^2}$ is an integer, then it is a perfect square.

<u>Solution.</u> If $2+2\sqrt{1+12n^2}$ is an integer, then $1+12n^2$ is a perfect square. Suppose $1+12n^2=m^2$ for some odd positive integer m. Then $12n^2=(m+1)(m-1)$. Let t be the integer $(m+1)/2$ and we have (*) $t(t-1)=3n^2$.

Now we <u>claim</u> $2+2\sqrt{1+12n^2}=2+2m=4t$ is a perfect square. By (*), we see $t-1$ or t is divisible by 3. Now $\gcd(t-1, t)=1$. Assume t is divisible by 3, then $(t/3)(t-1)=n^2$ and both $t/3$ and $t-1$ are perfect squares. Let $t/3=k^2$ for some integer k, Then $t-1=3k^2-1\equiv 2\not\equiv 0^2, 1^2$ or $2^2 \pmod 3$, contradiction. So $t-1$ is divisible by 3. Then we have $\gcd(t, (t-1)/3)=1$. From $t\times(t-1)/3=n^2$, we see t is a perfect squares. So the claim is true.

<u>Example 9 (2002 Australian Math Olympiad).</u> Find all prime numbers p, q, r such that p^q+p^r is a perfect square.

<u>Solution.</u> If $q=r$, then $p^q+p^r=2p^q$. So p=2 and q is an odd prime at least 3. All prime triples $(p,q,r)=(2,q,q)$ are solutions.

If $q\ne r$, then without loss of generality, let $q<r$ and so $p^q+p^r=p^q(1+p^s)$, where $s=r-q$ is at least 1. Since p^q and $1+p^s$ are relatively prime, so they are both perfect squares. Then, the prime q is 2. Also, since $1+p^s$ is a perfect square, $1+p^s=u^2$ for some positive integer u. Then

$$p^s=u^2-1=(u+1)(u-1).$$

Since $\gcd(u+1,u-1)$=1 or 2, so if it is 2, then u is odd and p is even. Hence, p=2 and both $u+1$ and $u-1$ are powers of 2. Then u can only be 3 and $1+p^s=3^2$ so that p=2, s=3, $r=q+s$=2+3=5. These lead to the solutions (p,q,r)=(2,2,5) or (2,5,2).

If $\gcd(u+1,u-1)$=1, then u is even and $u-1$ must be 1 (otherwise $u+1$ and $u-1$ have different odd prime factors and cannot be powers of the same prime). Then u=2, $p^s=(u-1)(u+1)$=3, p=3, s=1, $r=q+s$=3. The only such prime triples are (p,q,r) = (3,2,3) or (3,3,2).

Then all the solutions are (p,q,r) = (2,2,5), (2,5,2), (3,2,3), (3,3,2) and $(2,q,q)$ with q being a prime at least 3.

<u>Example 10 (2008 USA Team Selection Test).</u> Let n be a positive integer. Prove that n^7+7 is not a perfect square.

<u>Solution.</u> Assume $n^7+7=x^2$ for some positive integer x. Then

(1) n is odd (for otherwise $x^2\equiv 3 \pmod 4$, which is false).
(2) $n\equiv 1 \pmod 4$ (due to n odd and $x^2\not\equiv 2 \pmod 4$).

(3) $x^2+11^2=n^7+128=(n+2)N$, where N is $n^6-2n^5+4n^4-8n^3+16n^2-32n+64$.

(4) If $11\nmid x$, then every prime factor p of x^2+11^2 must be odd and $p\equiv 1 \pmod 4$ (for if $p=4k+3$, then $x^2\equiv -11^2 \pmod p$ and by Fermat's little theorem, $x^{p-1}\equiv -11^{p-1}\equiv -1 \pmod p$), contradiction).

From (3), we get $n+2 \mid x^2+11^2$, $n+2\equiv 3 \pmod 4$ implies x^2+11^2 has a prime factor congruent 3 (mod 4), which contradicts (4).

If $x=11y$ for some integer y, then (3) becomes $121(y^2+1)=(n+2)N$, but checking $n\equiv -5$ to 5 (mod 11), we see N is not a multiple of 11. So n+2 is a multiple of 121, say $M=(n+2)/121$. Then $y^2+1=MN$. Similarly, it can be checked that every prime factor of y^2+1 is congruent to 1 (mod 4). Hence, every odd factor of y^2+1 is congruent to 1 (mod 4). However, $M\equiv 3 \pmod 4$, so $y^2+1=MN$ cannot be true. Therefore, n^7+7 is not a perfect square.

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***April 21, 2018.***

Problem 511. Let $x_1,x_2,\ldots,x_{40}$ be positive integers with sum equal to 58. Find the maximum and minimum possible value of $x_1^2+x_2^2+\cdots+x_{40}^2$.

Problem 512. Let *AD, BE, CF* be the altitudes of acute ΔABC. Points *P* and *Q* are on segments *DF* and *EF* respectively. If $\angle PAQ=\angle DAC$, then prove that *AP* bisects $\angle FPQ$.

Problem 513. Let $a_0,a_1,a_2,\ldots$ be a sequence of nonnegative integers satisfying the conditions:

(1) $a_{n+1}=3a_n-3a_{n-1}+a_{n-2}$ for $n>1$,
(2) $2a_1=a_0+a_2-2$,
(3) for every positive integer *m*, in the sequence $a_0,a_1,a_2,\ldots$, there exist m terms $a_k,a_{k+1},\ldots,a_{k+m-1}$, which are perfect squares.

Prove that every term in $a_0,a_1,a_2,\ldots$ is a perfect square.

Problem 514. Let *n* be a positive integer and let $p(x)$ be a polynomial with real coefficients on the interval $[0,n]$ such that $p(0)=p(n)$. Prove that there are *n* distinct ordered pairs (a_i, b_i) with $i=1,2,\ldots,n$ such that $0\le a_i<b_i\le n$, b_i-a_i is an integer and $p(a_i)=p(b_i)$.

Problem 515. There are ten distinct nonzero real numbers. It is known that for every two of the numbers, either the sum or the product of them is rational. Prove that the square of each of the ten numbers is rational.

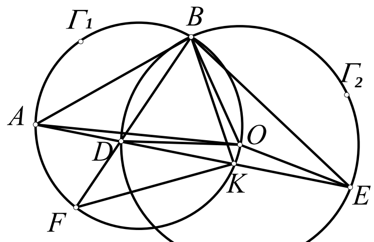
****

Solutions

****

Problem 506. Points *A* and *B* are on a circle Γ_1. Line *AB* is tangent to another circle Γ_2 at *B* and the center *O* of Γ_2 is on Γ_1. A line through *A* intersects Γ_2 at points *D* and *E* (with *D* between *A* and *E*). Line *BD* intersects Γ_1 at a point *F*, different from *B*. Prove that *D* is the midpoint of *BF* if and only if *BE* is tangent to Γ_1.

Solution. **FONG Tsz Lo** (SKH Lam Woo Memorial Secondary School) and **George SHEN**.



Let point *K* be the intersection of Γ_1 with line *DE*. Then $\Delta KFD \sim \Delta ABD$. Since $\angle ABD = \angle AEB$, so $\Delta ABD \sim \Delta AEB$. Then $\Delta KFD \sim \Delta AEB$. Hence, $FD/DK = AB/BE$.

Let *O* be the center of Γ_2. Since $OB\perp AB$, *AO* is a diameter of Γ_1. So $AK\perp OK$. Then $\angle DKO = \angle AKO = 90^\circ$. So $DK=EK$. Now *BE* is tangent to $\Gamma_1 \Leftrightarrow \angle EBK = \angle BAD \Leftrightarrow \Delta EBK \sim \Delta BAD \Leftrightarrow AB/BE=DB/KE \Leftrightarrow FD/DK=DB/KE \Leftrightarrow FD=DB$ (i.e. *D* is the midpoint of *BF*).

Other commended solvers: **DBS Maths Solving Team** (Diocesan Boys' School), **Jae Woo LEE** (Hamyang High School, South Korea), **LIN Meng Fei**, **Akash Singha ROY** (West Bengal, India) and **Toshihiro SHIMIZU** (Kawasaki, Japan).

Problem 507. Determine all functions $f:\mathbb{R}\to\mathbb{R}$ such that for all $x, y \in\mathbb{R}$,

$$(x-2)f(y) + f(y+2f(x)) = f(x+yf(x)). \quad (*)$$

Solution. **DBS Maths Solving Team** (Diocesan Boys' School), **FONG Tsz Lo** (SKH Lam Woo Memorial Secondary School), **Jae Woo LEE** (Hamyang High School, South Korea) and **Toshihiro SHIMIZU** (Kawasaki, Japan).

If $f(0)=0$, then setting $x=0$ in (*) yields $f(y)=0$ for all $y \in\mathbb{R}$, i.e. *f* is the zero function, which is a solution of (*).

If $f(0)\ne0$, then setting $y=0$ in (*) yields $(x-2)f(0) + f(2f(x)) = f(x)$ for all $x \in\mathbb{R}$. Now $f(x) = f(y)$ implies $(x-2)f(0) + f(2f(x)) = f(x) = f(y) = (y-2)f(0) + f(2f(y)) = (y-2)f(0) + f(2f(x))$ yielding $x=y$. So *f* is injective.

Setting $x=2$ in (*) yields $f(y+2f(2)) = f(2+yf(2))$ for all $y\in\mathbb{R}$. Since *f* is injective, $y+2f(2)=2+yf(2)$ for all $y\in\mathbb{R}$. Setting $y=0$, we get $f(2)=1$. Since *f* is injective, $f(3)\ne1$. Setting $x=3$, $y=3/(1-f(3))$ in (*), we get $f(3/(1-f(3))+2f(3))=0$. Thus, *f* has a root at $a=3/(1-f(3))+2f(3)$. Setting $y=a$ in (*), we get $f(a+2f(x))=f(x+af(x))$ for all $x\in\mathbb{R}$. Since *f* is injective, we get $a+2f(x) = x+af(x)$. Now $a\ne2$. So $f(x) = (x-a)/(2-a)$. Putting this in (*), we get $a=1$. Then the function can only be (1) $f(x)=0$ for all $x\in\mathbb{R}$ or (2) $f(x)=x-1$ for all $x\in\mathbb{R}$. Putting these in (*) show they are in fact solutions of (*).

Other commended solvers: **Yagub N. ALIYEV** (Problem Solving Group of ADA University, Baku, Azerbaijan) and **Akash Singha ROY** (West Bengal, India).

Problem 508. Determine the largest integer *k* such that for all integers *x,y*, if $xy+1$ is divisible by *k*, then $x+y$ is also divisibly by *k*.

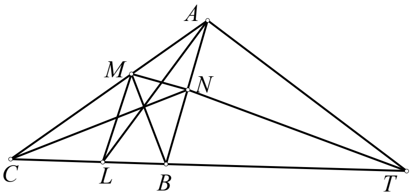
Solution. **George SHEN.**

Let *k* be such an integer. Let *S* be the set of all integers *x* such that $\gcd(x,k)=1$. For *x* in *S,* choose integer *m* in $[1,k-1]$ such that $mx^2 \equiv -1 \pmod k$. Let $y=mx$, then $k\mid xy+1$. So $k\mid x+y$ and $k\mid (x+y)x-(xy+1) = x^2-1$. Then for every *x* in *S*, every prime factor *p* of *k* satisfies $x^2\equiv1 \pmod p$. If all prime factors *p* of *k* are at least 5, then $x=2, 3$ are in *S*, but $x^2\equiv1 \pmod p$ fails due to $p\nmid 2^2-1, 3^2-1$. So the prime factors of *k* can only be 2 or 3. So *k* is of the form 2^r3^s and $S=\{x: \gcd(x,2)=1=\gcd(x,3)\}$ Then for $x=5$ in *S*, $x^2\equiv1 \pmod{2^r}$ implies $2^r\mid 24$ and so $r\le3$. Also, for $x=5$ in *S*, $x^2\equiv1 \pmod{3^s}$ implies $3^s\mid 24$ and so $s\le1$. Then $k \le 2^33=24$.

Finally, for $k=24$, $xy\equiv-1 \pmod{24}$ implies $\gcd(x,24) = 1 = \gcd(y,24)$. Then $x,y \equiv 1, 5, 7, 11, 13, 17, 19$ or $23 \pmod{24}$. The only possible cases for $xy\equiv-1 \pmod{24}$ are $\{x,y\} = \{1,23\}, \{5,19\}, \{7,17\}, \{11,23\}$. Then $24\mid x+y$. So $k=24$ is the required largest integer.

Other commended solvers: **CHUI Tsz Fung** (Ma Tau Chung Government Primary School, P4) and **DBS Maths Solving Team** (Diocesan Boys' School), **Jae Woo LEE** (Hamyang High School, South Korea), **Akash Singha ROY** (West Bengal, India) and **Toshihiro SHIMIZU** (Kawasaki, Japan).

Problem 509. In ΔABC, the angle bisector of $\angle CAB$ intersects *BC* at a point *L*. On sides *AC, AB*, there are points *M, N* respectively such that lines *AL, BM, CN* are concurrent and $\angle AMN = \angle ALB$. Prove that $\angle NML = 90^\circ$.



Solution 1. **Apostolis MANOLOUDIS** and **George SHEN**.

Let $T=MN\cap BC$. From $\angle AMT=\angle AMN =\angle ALB = \angle ALT$, we get A, M, L, T are concyclic. So $\angle NML=\angle TML=\angle TAL$. To get $\angle TAL=90°$, it suffices to show AT is the exterior bisector of $\angle CAB$.

By Menelaos' theorem, as M,N,T are collinear, $(AM/MC)(CT/TB)(BN/NA) = 1$. By Ceva's theorem, as AL, BM, CN concur, $(AM/MC)(CL/LB)(BN/NA) =1$. Then $CL/LB=CT/TB$. By the angle bisector theorem, $CA/AB=CL/LB= CT/TB$. So AT is the external bisector of $\angle CAB$.

Solution 2. **FONG Tsz Lo** (SKH Lam Woo Memorial Secondary School), **Akash Singha ROY** (West Bengal, India) and **Toshihiro SHIMIZU** (Kawasaki, Japan).

AL, BM, CN concurrent implies T, B, L, C is a harmonic range of points. Then $\angle AMT =\angle AMN =\angle ALB = \angle ALT$ led to T, A, M, L concyclic. By Apollonius' Theorem, $90° =\angle TAL =\angle NML$.

Other commended solvers: **Jae Woo LEE** (Hamyang High School, South Korea), **LEUNG Hei Chun** (SKH Tang Shiu Kin Secondary School), **Titu ZVONARU** (Comănești, Romania) and **Neculai STANCIU** ("George Emil Palade" School, Buzău, Romania).

Problem 510. Numbers 1 to 20 are written on a board. A person randomly chooses two of these numbers with a difference of at least 2. He adds 1 to the smaller one and subtracts 1 from the larger one. Then he performs an operation by replacing the original two chosen numbers on the board with the two new numbers. Determine the maximum number of times he can do this operation.

Solution. **CHUI Tsz Fung** (Ma Tau Chung Government Primary School, P4), **FONG Tsz Lo** (SKH Lam Woo Memorial Secondary School), **Akash Singha ROY** (West Bengal, India) and **Toshihiro SHIMIZU** (Kawasaki, Japan).

Note after each operation, the sum of the numbers is always 210. Suppose the person chooses m,n with $m-n\ge 2$, then $(m-1)^2 + (n+1)^2 = n^2 + m^2 + 2 - 2(m-n) \le n^2+m^2-2$ with equality only for $m-n=2$. If the absolute value of the difference of the two numbers is 1, then the operation does not change anything. At the end, the board has ten 10's and ten 11's.

In the beginning, the sum of the squares is $1^2+2^2+\cdots+20^2=2870$ and at the end, it is $10\times(10^2+11^2)=2210$. After each operation, the sum of squares reduces by at least 2, so the number of operation that can be done is at most $(2870-2210)/2=330$. Below we will show the person can do 330 operations with the absolute values of the difference of the two numbers is 2.

The plan is to eliminate the minimum and the maximum of the remaining numbers until we get only 10's and 11's. In round 1, we eliminate 1's and 20's by operating on pairs (1,3), (2,4), …, (18,20) one time for every pair. In round 2, we eliminate 2's and 19's by operating on pairs (2,4), (3,5), …, (17,19) two times for every pair. Keep on eliminating in this way until we have only 9's, 10's, 11's and 12's. In round 9, we eliminate 9's and 12's by operating on pairs (9,11) and (10,12) nine times. The total number of operations is $18\times1+16\times2+\cdots+2\times9=330$.

Olympiad Corner

(Continued from page 1)

Problem 3. (*Dušan Djukić*) There are $2n-1$ bulbs in a line. Initially, the central (n-th) bulb is on, whereas all others are off. A step consists of choosing a string of at least three (consecutive) bulbs, the leftmost and rightmost ones being off and all between them being on, and changing the states of all bulbs in the string (for instance, the configuration ●○○○● will turn into ○●●●○). At most how many steps can be performed?

Second Day

Problem 4. (*Dušan Djukić*) Suppose that a positive integer a is such that, for any positive integer n, the number n^2a-1 has a divisor greater than 1 and congruent to 1 modulo n. Prove that a is a perfect square.

Problem 5. (*Bojan Bašić* and *PSC*) Determine the maximum number of queens that can be placed on a 2017×2017 chessboard so that each queen attacks at most one of the others.

Problem 6. (*Dušan Djukić*) Let k be the circumcircle of triangle ABC, and let k_a be its excircle opposite to A. The two common tangents of k and k_a meet the line BC at points P and Q. Prove that $\angle PAB=\angle QAC$.

Perfect Squares

(Continued from page 2)

Example 11 (2006 Thai Math Olympiad). Determine all prime numbers p such that $(2^{p-1}-1)/p$ are perfect squares.

Solution. For every prime number p, let $f(p)= (2^{p-1}-1)/p$. We will show for $p>7$, $f(p)$ is not a perfect square.

Assume there is a prime $p>7$ such that $2^{p-1}-1=pm^2$ for some positive integer m. Then m must be odd. Now there are two cases, (1) p is of the form $4k+1$ with $k>1$ or (2) p is of the form $4k+3$ with $k>1$.

In case (1), we have $2^{p-1} -1 = pm^2 = (4k+1)m^2 \equiv 1 \pmod 4$, but also $2^{p-1} -1 = 2^{4k} -1 \equiv 3 \pmod 4$, which is a contradiction.

In case (2), we have $2^{p-1}-1= 2^{4k+2}-1 = (2^{2k+1}-1)(2^{2k+1}+1) = pm^2$.

Since $\gcd(2^{2k+1}-1,2^{2k+1}+1)=1$, again we have two subcases:

(a) $2^{2k+1}-1=u^2$, $2^{2k+1}+1=pv^2$ for some positive integers u, v;

(b) $2^{2k+1}-1=pu^2$, $2^{2k+1}+1=v^2$ for some positive integers u, v.

In subcase (a), since $k > 1$, $2^{2k+1}+1\equiv1 \pmod 4$, but $pv^2\equiv3\times1=4 \pmod 4$, which is a contradiction.

In subcase (b), we have $2^{2k+1} = v^2-1 = (v-1)(v+1)$. Then $v-1=2^s$, $v+1=2^t$ for some positive integers $s<t$. Observe that $2^{t-s}=(v+1)/(v-1)=2/(v-1)+1$. Then $v=2$ or 3. If $v=2$, then $2^{2k+1}+1=v^2=4$, which is a contradiction. If $v=3$, then $2^{2k+1}=v^2-1=8$ leads to $k=1$, which is a contradiction as $k>1$.

Finally, checking the cases $p=2,3,5,7$, we see only cases $p=3$ and 7 have solutions $(2^{3-1}-1)/3=1^2$ and $(2^{7-1}-1)/7 = 3^2$.