# Mathematical Excalibur

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## Olympiad Corner

Below were the problems of the 2017 Serbian IMO Team Selection Competition for high school students. The event was held in Belgrade on May 21 and 22, 2017.

Time allowed was 270 minutes per day.

**First Day**

**Problem 1.** (*Dušan Djukić*) Let $D$ be the midpoint of side $BC$ of a triangle $ABC$. Points $E$ and $F$ are taken on the respective sides $AC$ and $AB$ such that $DE=DF$ and $\angle EDF=\angle BAC$. Prove that

$$DE \geq \frac{AB+AC}{4}.$$

**Problem 2.** (*Bojan Bašić*) Given an ordered pair of positive integers $(x,y)$ with exactly one even coordinate, a *step* maps this pair to $(x/2, y+x/2)$ if $2|x$, and to $(x+y/2, y/2)$ if $2|y$. Prove that for every odd positive integer $n>1$ there exists an even positive integer $b$, $b<n$, such that after finitely many steps the pair $(n,b)$ maps to the pair $(b,n)$.

*(continued on page 4)*


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The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***August 31, 2018***.

For individual subscription for the next five issues for the 17-18 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:



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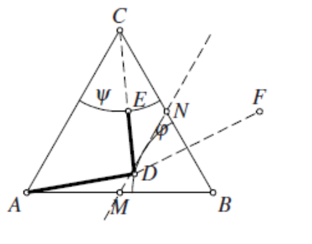
# Strategies and Plans

***Kin Y. Li***

In this article, we will be looking at some Math Olympiad problems from different countries and regions. Some require strategies or plans to perform certain tasks. We hope these arouse your interest. Here are the examples.

<u>Example 1 (1973 IMO).</u> A soldier has to investigate whether there are mines in an area that has the form of an equilateral triangle. The radius of his detector is equal to one-half of an altitude of the triangle. The soldier starts from one vertex of the triangle. Determine the shortest path that the soldier has to traverse in order to check the whole region.

<u>Solution.</u> Suppose that the soldier starts at the vertex $A$ of the equilateral triangle $ABC$ of side length $a$. Let $\varphi$ and $\psi$ be the arcs of circles with centers $B$ and $C$ and radii $a\sqrt{3}/4$ respectively, that lie inside the triangle. In order to check the vertices $B$ and $C$ he must visit some point $D$ in $\varphi$ and $E$ in $\psi$.




Thus his path cannot be shorter than the path $ADE$ (or $AED$) itself. The length of the path $ADE$ is $AD+DE\geq AD+DC-a\sqrt{3}/4$. Let $F$ be the reflection of $C$ across the line $MN$, where $M$ and $N$ are the midpoints of $AB$ and $BC$ respectively. Then $DC\geq DF$ and hence $AD+DC\geq AD+DF\geq AF$. So

$$AD+DE \geq AF - \frac{a\sqrt{3}}{4} = a\left(\frac{\sqrt{7}}{2}-\frac{\sqrt{3}}{4}\right)$$

with equality if and only if $D$ is the midpoint of arc $\varphi$ and $E$ is the intersection point of $CD$ and arc $\psi$. In following the path $ADE$, the soldier will check the whole region. Therefore, this path (as well as the one symmetric to it) is the shortest path the soldier can check the whole field.

<u>Example 2 (2011 Saudi Arabia Math Competition).</u> A Geostationary Earth Orbit is situated directly above the equator and has a period equal to the Earth's rotational period. It is at the precise distance of 22,236 miles above the Earth that a satellite can maintain an orbit with a period of rotation around the Earth exactly equal to 24 hours. Because the satellites revolve at the same rotational speed of the Earth, they appear stationary from the Earth surface. That is why most stationary antennas (satellite dishes) do not need to move once they have been properly aimed at a target satellite in the sky. In an international project, a total of ten stations were equally spaced on this orbit (at the precise distance of 22,236 miles above the equator). Given that the radius of the Earth is 3960 miles, find the exact straight distance between two neighboring stations. Write your answer in the form $a + b\sqrt{c}$, where $a, b, c$ are integers and $c>0$ is square-free.

<u>Solution.</u> Let $A$ and $B$ be neighboring stations and $O$ be the center of the Earth. Now $\angle AOB=36°$. Let $\theta=18°$. Then $AB=2R\sin\theta$, where $R = 22236 + 3960 = 26196$. Since we have $\sin 36°=\cos 54°$, so $\sin 2\theta=\cos 3\theta$. That is, $2\cos\theta\sin\theta = 4\cos^3\theta-3\cos\theta$. Dividing by $\cos\theta$ and expressing in terms of $\sin\theta$, we get $4\sin^2\theta+2\sin\theta-1=0$. Using the quadratic formula, we have $\sin\theta=(\sqrt{5}-1)/4$. Then $AB=2R\sin\theta=13098(\sqrt{5}-1)$. So $a=-13098$, $b = 13098$ and $c = 5$.

<u>Example 3 (2008 German National Math Competition).</u> On a bookshelf, there are $n$ books ($n\geq 3$) from different authors standing side by side. A librarian inspects the two leftmost books and changes their places if and

*(continued on page 2)*

only if they are not in alphabetical order. Afterward, he does the same to the second and the third book from the left and so on. Acting this way, he passes the whole row of books three times in total. Determine the number of different starting arrangements for which the books will finally be ordered alphabetically.

Solution. There are exactly $6\cdot 4^{n-3}$ arrangements for which the books are in order after 3 runs. For a proof, we number the positions and the books in alphabetical order from 1 to $n$. Obviously, for the position of $p(k)$ of book number $k$ at the beginning it is necessary that $p(k)-k\le 3$. Now this condition is also sufficient: At every ordering run, all of the books standing right to their correct place are shifted one place to the left. On the other hand, no book can be shifted to the right beyond its correct place because if there is a book at position $p(k)$ with $p(k)>k$, there must be at least one book on the left side of $p(k)$ with its number larger than $p(k)$. Such a book takes over any book with a number smaller than $p(k)$.

The number given in the answer is then calculated by regarding that each of the books with numbers $1,2,\ldots,n-4$ that is not occupied by a book with a smaller number. For the last three books there are only 3, 2 and 1 places left. Hence the result follows.

Example 4 (2000 Russian Math Olympiad). Two pirates divide their loot, consisting of two sacks of coins and one diamond. They decide to use the following rules. On each turn, one pirate chooses a sack and takes $2m$ coin from it, keeping $m$ for himself and putting the rest into the other sack. The pirates alternatively taking turns until no more moves are possible; the first pirate unable to make a move loses the diamond, and the other pirate takes it. For which initial numbers of coins can the first pirate guarantee that he will obtain the diamond?

Solution. We claim that if there are $x$ and $y$ coins left in the two sacks, respectively, then the next player $P_1$ to move has a winning strategy if and only if $|x-y|>1$. Otherwise, the other player $P_2$ has a winning strategy.

We prove the claim by induction on the total numbers of coins, $x+y$. If $x+y=0$, then no moves are possible and the next player does not have a winning strategy. Now assuming that the claim is true when $x+y\le n$ for some nonnegative $n$, we prove that it is true when $x+y=n+1$.

First consider the case $|x-y|\le 1$. Assume that a move is possible. Otherwise, the next player $P_1$ automatically loses, in accordance with our claim. The next player must take $2m$ coins from one sack, say the one containing $x$ coins, and put m coins into the sack containing $y$ coins. Hence the new difference between the number of coins in the sacks is

$$|(x-2m)-(y+m)|\ge|-3m|-|y-x|\ge 3-1=2.$$

At this point, there are now a total of $x+y-m$ coins in the sacks, and the difference between the numbers of coins in the two sacks is at least 2. Thus, by induction hypothesis, $P_2$ has a winning strategy. This proves the claim when $|x-y|\le 1$.

Now consider the case $|x-y|\ge 2$. Without loss of generality, let $x>y$. $P_1$ would like to find a $m$ such that $2m\le x$, $m\ge 1$ and

$$|(x-2m)-(y+m)|\le 1.$$

The number $m=[(x-y-1)/3]$ satisfies the last two inequalities above and we claim $2m\le x$ as well. Indeed, $x-2m$ is nonnegative because it differs by at most 1 from the positive number $y+m$. After taking $2m$ coins from the sack with $x$ coins, $P_1$ leaves a total of $x+y-m$ coins, where the difference between the numbers of coins in the sacks is at most 1. Hence, by the induction hypothesis, the other player $P_2$ has no winning strategy. It follows that $P_1$ has a winning strategy, as desired.

This completes the proof of the induction and of the claim. It follows that the first pirate can guarantee that he will obtain the diamond if and only if the number of coins initially in the sacks differs by at least 2.

Example 5 (2015 Croatian National Math Competition). In a country between every two cities there is a direct bus or a direct train line (all lines are two-way and they don't pass through any other city). Prove that all cities in that country can be arranged in two disjoint sets so that all cities in one set can be visited using only train so that no city is visited twice, and all cities in the other set can be visited using only bus so that no city is visited twice.

Solution. Let $G$ be the set of all cities in the country. For disjoint subsets $A$, $Z$ of $G$, we call a pair $(A,Z)$ *good* if all cities in the set $A$ can be visited using only bus such that no city is visited twice and all cities in the set $Z$ can be visited using only train such that no city is visited twice.

Let $(A,Z)$ be a good pair such that $A\cup Z$ has the maximum number of elements. If we prove $A\cup Z=G$, then the statement of the problem will follow.

Let us assume the opposite, i.e. there is a city $g$ which is not from $A$ nor $Z$. Without loss of generality we can assume that $A$ and $Z$ are non-empty because otherwise we can transfer any city from a non-empty set to an empty one.

Let $n$ be the number of cities in the set $A$ and $m$ be the number of cities in the set $Z$. Let us arrange the cities from $A$ in the series $a_1,\ldots,a_n$ such that every two consecutive cities in that series are connected by a direct bus line. Also, let us arrange the cities from $Z$ in the series $z_1,\ldots,z_m$ such that every two consecutive cities in that series are connected by a direct train line.

Since we assumed that the pair $(A,Z)$ is maximum, the cities $g$ and $a_1$ have to be connected by train (otherwise the pair $(A\cup\{g\},Z)$ would be a good pair whose union would have more elements than $A\cup Z$, and $g$ and $z_1$ have to be connected by bus (otherwise the pair $(A,Z\cup\{g\})$ would be a good pair whose union would have more element than $A\cup Z$).

The cities $a_1$ and $z_1$ have to be connected by bus or by train. If $a_1$ and $z_1$ are connected by bus, let us put $A'=\{z_1,g,a_1,\ldots,a_n\}$ and $Z'=\{z_2,\ldots,z_m\}$. Then $(A',Z')$ is a good pair and the number of elements of $A'\cup Z'$ is greater than the number of elements of $A\cup Z$, which contradicts the assumption.

If $a_1$ and $z_1$ are connected by train, let us put $A''=\{a_2,\ldots,a_n\}$ and $Z''=\{a_1,g,z_1,\ldots,z_m\}$. Then $(A'',Z'')$ is a good pair and the number of elements of $A''\cup Z''$ is greater than the number of elements of $A\cup Z$, which contradicts the assumption.

Since all cases lead to contradiction, we conclude that the assumption was wrong and that every city is either in the set $A$ or in the set $Z$.

# Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for sending solutions is ***August 31, 2018.***

**Problem 516.** Determine all triples $(p,m,n)$ of positive integers such that $p$ is prime and $2^m p^2+1=n^5$ holds.

**Problem 517.** For all positive $x$ and $y$, prove that

$$x^2y^2(x^2+y^2-2) \ge (xy-1)(x+y).$$

**Problem 518.** Let $I$ be the incenter and $AD$ be a diameter of the circumcircle of $\Delta ABC$. Let point $E$ be on the ray $BA$ and point $F$ be on the ray $CA$. If the lengths of $BE$ and $CF$ are both equal to the semiperimeter of $\Delta ABC$, then prove that lines $EF$ and $DI$ are perpendicular.

**Problem 519.** Let $A$ and $B$ be subsets of the positive integers with 10 and 9 elements respectively. Suppose for every $x,y,u,v\in A$ satisfying $x+y=u+v$, we have $\{x,y\}=\{u,v\}$. Prove that the set $A+B=\{a+b: a\in A, b\in B\}$ has at least 50 elements.

**Problem 520.** Let $P$ be the set of all polynomials $f(x)=ax^2+bx$,where $a,b$ are nonnegative integers less than $2010^{18}$. Find the number of polynomials $f$ in $P$ for which there is a polynomial $g$ in $P$ such that $g(f(k))\equiv k \pmod{2010^{18}}$ for all integers $k$.

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## *Solutions*

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**Problem 511.** Let $x_1,x_2,\ldots,x_{40}$ be positive integers with sum equal to 58. Find the maximum and minimum possible value of $x_1^2+x_2^2+\cdots+x_{40}^2$.

***Solution.*** **Arpon BASU** (AECS-4, Mumbai, India), **CHUI Tsz Fung** (Ma Tau Chung Government Primary School, P4), **William KAHN** (Sidney, Australia), **LAI Wai Lok** (La Salle Primary School), **LEUNG Hei Chun**, **LUI On Ki**, **George SHEN**, **Toshihiro SHIMIZU** (Kawasaki, Japan) and **ZHANG Yupei** (HKUST).

If there exist $x_m, x_n\ge 2$, then we can replace them by $x_m+x_n-1$, 1 due to

$$(x_m+x_n-1)^2+1^2-(x_m^2+x_n^2) = 2(x_m-1)(x_n-1) \ge 0.$$

So the maximum case can be attained by one 19 and thirty-nine 1's. This gives the maximum value $39\times1^2+1\times19^2=400$.

For the minimum case, there exists at least one 1, otherwise $58=x_1+x_2+\cdots+x_{40}\ge 2\times40=80$, contradiction. Let $x_k$ be a largest term. If $x_k\ge3$, then we can replace $x_k$ and 1 by $x_k-1$ and 2 to lower the square sums since

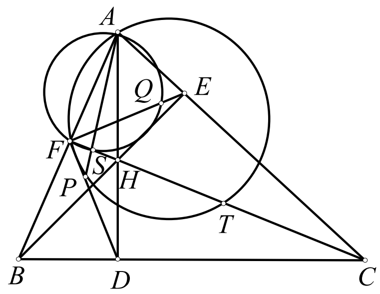
$$(x_k^2+1^2)-[(x_k-1)^2+2^2] = 2(x_k-2) > 0.$$

So in the minimum case, there are twenty-two 1's and eighteen 2's yielding $22\times1^2+18\times2^2=94$.

*Other commended solvers:* **George SHEN** and **Nicuşor ZLOTA** ("Traian Vuia" Technical College, Focşani, Romania).

**Problem 512.** Let $AD$, $BE$, $CF$ be the altitudes of acute $\Delta ABC$. Points $P$ and $Q$ are on segments $DF$ and $EF$ respectively. If $\angle PAQ=\angle DAC$, then prove that $AP$ bisects $\angle FPQ$.

***Solution.*** **George SHEN** and **Toshihiro SHIMIZU** (Kawasaki, Japan).




Let $H$ be the orthocenter of $\Delta ABC$. Let $S$ be the intersection of $AP$ and $CF$. Let $T$ be the intersection of $AQ$ and $CF$. Now $\angle AFC=90°=\angle ADC$. As $AFDC$ is cyclic,

$$\angle PAT=\angle PAQ=\angle DAC=\angle DFC=\angle PFT,$$

points $A, T, F, P$ are concyclic. Also, since

$$\angle SFQ=\angle HFE=\angle HAE =\angle DAC=\angle PAQ=\angle SAQ,$$

points $A, F, S, Q$ are concyclic. Then since

$$\angle SQT=\angle SFA=90° =\angle AFT=\angle APT=\angle SPT=\angle SAQ,$$

points $S,P,T,Q$ are concyclic. Therefore, we have

$$\angle FPA=\angle FTA=\angle STQ=\angle SPQ,$$

which implies $AP$ bisects $\angle FPQ$.

*Other commended solvers:* **Andrea FANCHINI** (Cantù, Italy), **William KAHN** (Sidney, Australia), **LEUNG Hei Chun**, **George SHEN** and **ZHANG Yupei** (HKUST).

**Problem 513.** Let $a_0$, $a_1$, $a_2$, … be a sequence of nonnegative integers satisfying the conditions:

(1) $a_{n+1}=3a_n-3a_{n-1}+a_{n-2}$ for $n>1$,

(2) $2a_1=a_0+a_2-2$,

(3) for every positive integer $m$, in the sequence $a_0$, $a_1$, $a_2$, …, there exist $m$ terms $a_k$, $a_{k+1}$, … , $a_{k+m-1}$, which are perfect squares.

Prove that every term in $a_0,a_1,a_2,\ldots$ is a perfect square.

***Solution.*** **William KAHN** (Sidney, Australia), **LEUNG Hei Chun**, **George SHEN** and **Toshihiro SHIMIZU** (Kawasaki, Japan).

We show we can select integers $\alpha$, $\beta$, $\gamma$ such that $a_n=n(n-1)\alpha/2+n\beta+\gamma$. For $n=0$, we must have $\gamma=a_0$. For $n=1$, we must have $a_1=\beta+\gamma$ and we can set integer $\beta$ as $a_1-\gamma = a_1-a_0$. Finally for $n=2$, we must have $a_2=\alpha+2\beta+\gamma$ and we can set integer $\alpha = a_2-2\beta-\gamma = a_2-2(a_1-a_0) - a_0$. Then since all three sequences $b_n=n^2$, $b_n=n$ and $b_n=1$ satisfy the relation $b_{n+1}=3b_n-3b_{n-1}+b_{n-2}$, we also have $a_n= n(n-1)\alpha/2 + n\beta + \gamma = n^2\alpha/2+n(\beta-\alpha/2)+\gamma$ satisfies the relation.

From (2), we get $2(\beta+\gamma) = \gamma + \alpha + 2\beta + \gamma - 2$ or $\alpha = 2$. Therefore, we have $a_n= n(n-1)+n\beta+\gamma$, which can be put in the form $[(2n+t)^2+s]/4$ for some integers $s$ and $t$.

Assume $s\ne0$. If

$$(2n+t-1)^2<(2n+t)^2+s<(2n+t+1)^2 \quad (*),$$

then $a_n$ cannot be a perfect square. However, (*) is equivalent to

$$-(4n+2t-1) < s < 4n+2t+1$$

or $-2t+1-s<4n$ and $s-2t-1<4n$, which is valid for sufficiently large $n$. Therefore, (3) would lead to $s=0$.

Since $a_0 = t^2/4$ must be an integer, so $t$ must be even. Let $t=2t'$, then

$$a_n = \frac{(2n+2t')^2}{4} = (n+t')^2,$$

which implies that every term in $a_n$ is a perfect square.

*Other commended solvers:* **Arpon BASU** (AECS-4, Mumbai, India), **George SHEN** and **ZHANG Yupei** (HKUST).

**Problem 514.** Let $n$ be a positive integer and let $p(x)$ be a polynomial with real coefficients on the interval $[0,n]$ such that $p(0)=p(n)$. Prove that there are $n$ distinct ordered pairs $(a_i, b_i)$ with $i=1,2,\ldots,n$ such that $0\le a_i<b_i\le n$, $b_i-a_i$ is an integer and $p(a_i)=p(b_i)$.

***Solution.*** **Toshihiro SHIMIZU** (Kawasaki, Japan) and **ZHANG Yupei** (HKUST).

We can solve the problem with continuous functions in place of polynomials. We will prove this by using mathematical induction. The case $n=1$ is trivial. Suppose the case $n-1$ is true. Define $f(x)=p(x+1)-p(x)$. Then

$f(0)+f(1)+\cdots+f(n-1)=p(n)-p(0)=0.$ (*)

First we show there exists $w\in[0,n-1]$ such that $p(w)=p(w+1)$. In fact, if there exists $k\in\{0,1,2,\ldots,n-1\}$ such that $f(k)=0$, then taking $w=k$, we are done. Otherwise, from (*), we know there exists $j\in\{0,1,2,\ldots,n-1\}$ such that $f(j)f(j+1)<0$. Then there is $w\in(j, j+1)$ such that $f(w)=0$. So $p(w) = p(w+1)$.

Next, define $g(x)=p(x)$ for $x\in[0,w]$ and $g(x)=p(x+1)$ for $x\in[w,n-1]$. Then $g(x)$ is continuous on $[0,n-1]$ and $g(0) = g(n-1)$. From induction hypothesis, there exist $x_i$ and $y_i$ with $y_i-x_i\in\mathbb{N}$ satisfying $g(x_i)=g(y_i)$ for $i=1,2,\ldots,n-1$. Then there are three cases:

(1) for $y_i<w$, $0 = g(y_i) - g(x_i) = p(y_i) - p(x_i)$,

(2) for $x_i \le w \le y_i$, $0 = g(y_i) - g(x_i) = p(y_i+1) - p(x_i)$ and

(3) for $w < x_i$, $0 = g(y_i) - g(x_i) = p(y_i+1) - p(x_i+1)$.

Together with $p(0) = p(n)$, we get the case $n$ completing the induction step.

*Other commended solvers:* **William KAHN** (Sidney, Australia) and **George SHEN**.

**Problem 515.** There are ten distinct nonzero real numbers. It is known that for every two of the numbers, either the sum or the product of them is rational. Prove that the square of each of the ten numbers is rational.

***Solution.*** **Toshihiro SHIMIZU** (Kawasaki, Japan) and **ZHANG Yupei** (HKUST).

Pick six of the nonzero distinct real numbers, say $A_1$, $A_2$, $\cdots$, $A_6$ (with the property that for $i\neq j$, either $A_iA_j\in\mathbb{Q}$ or $A_i+A_j\in\mathbb{Q}$). Consider a graph with $A_1$, $A_2$, $\cdots$, $A_6$ as vertices and color the edge with vertices $A_i$, $A_j$ blue if $A_i+A_j\in\mathbb{Q}$, otherwise red for $A_iA_j\in\mathbb{Q}$. By Ramsay's Theorem, there is a red or a blue triangle in the complete graph with $A_1$, $A_2$, $\cdots$, $A_6$ as vertices.

There are two cases. In case 1, there is a blue triangle with vertices, say $A_1$, $A_2$ and $A_3$. Then $A_1+A_2$, $A_2+A_3$, $A_3+A_1\in\mathbb{Q}$. So $2A_1=(A_1+A_2)+(A_3+A_1)-(A_2+A_3)\in\mathbb{Q}$. Then $A_1\in\mathbb{Q}$ and similarly $A_2,A_3\in\mathbb{Q}$.

Next, for any $B\in\{A_4, A_5, \ldots, A_{10}\}$, we see $A_1+B\in\mathbb{Q}$ or $A_1B\in\mathbb{Q}$. So $B=(A_1+B)-A_1\in\mathbb{Q}$ or $B=(A_1B)/A_1\in\mathbb{Q}$. Then all ten $A_i\in\mathbb{Q}$.

In case 2, there is a red triangle with vertices, say $A_1$, $A_2$ and $A_3$. Then $A_1A_2$, $A_2A_3$, $A_3A_1\in\mathbb{Q}$. Now

$$A_1^2=(A_1A_2)(A_3A_1)/(A_2A_3)\in\mathbb{Q}$$

and similarly $A_2^2$, $A_3^2\in\mathbb{Q}$. If at least one of $A_1$, $A_2$, $A_3\in\mathbb{Q}$, say $A_1\in\mathbb{Q}$, then pick any $C\in\{A_2, A_3, \ldots, A_{10}\}$. Observe that $A_1+C\in\mathbb{Q}$ or $A_1C\in\mathbb{Q}$. It follows that we get $C=(A_1+C)-A_1\in\mathbb{Q}$ or $C=(A_1C)/A_1\in\mathbb{Q}$. Then all ten $A_i\in\mathbb{Q}$.

Otherwise, if $A_1^2\in\mathbb{Q}$, but $A_1\notin\mathbb{Q}$, then $A_1=m\sqrt{x}$, where $m=1$ or $m=-1$ and $x\in\mathbb{Q}$. Since $A_1A_2\in\mathbb{Q}$, we get $A_1A_2=(m\sqrt{x})A_2=b$ for some $b\in\mathbb{Q}$. Then we get $A_2=b/(m\sqrt{x})=r\sqrt{x}$, where $r=b/(mx)\in\mathbb{Q}$ and $m\neq r$ due to $A_1\neq A_2$. For $A_i\neq A_1$, $A_2$, if $A_1+A_i\in\mathbb{Q}$ and $A_2+A_i\in\mathbb{Q}$, then $(A_1+A_i)-(A_2+A_i)\in\mathbb{Q}$, but $(A_1+A_i)-(A_2+A_i)= A_1- A_2 = (m-r)\sqrt{x}\notin\mathbb{Q}$. Finally, if $A_1A_i\in\mathbb{Q}$ or $A_2A_i\in\mathbb{Q}$, then as above we get $A_i= s_i\sqrt{x}$ for some $s_i\in\mathbb{Q}$ with $s_i\neq m,r$. Then we have $A_i^2=s_i^2x\in\mathbb{Q}$.

*Other commended solvers:* **Arpon BASU** (AECS-4, Mumbai, India), **CHUI Tsz Fung** (Ma Tau Chung Government Primary School, P4), **William KAHN** (Sidney, Australia), **LUO On Ki** and **George SHEN**.

## Olympiad Corner

***(Continued from page 1)***

**Problem 3.** (*Marko Radovanović*) Call a function $f:\mathbb{N}\to\mathbb{N}$ *lively* if

$f(a+b-1)=f(f(\cdots f(b)\cdots))$ for all $a,b\in\mathbb{N}$, where $f$ appears $a$ times on the right side.

Suppose that $g$ is a lively function such that $g(A+2018)=g(A)+1$ holds for some $A\ge 2$.

(a) Prove that $g(n+2017^{2017})=g(n)$ for all $n\ge A+2$.

(b) If $g(A+2017^{2017})\neq g(A)$, determine $g(n)$ for $n\le A-1$.

**Second Day**

**Problem 4.** (*Dušan Djukić*) An $n\times n$ square is divided into unit squares. One needs to place a number of isosceles right triangles with hypotenuse 2, with vertices at grid points, in such a way that every side of every unit square belongs to exactly one triangle (i.e. lies inside it or on its boundary). Determine all numbers $n$ for which this is possible.

**Problem 5.** (*Dušan Djukić*) For a positive integer $n\ge 2$, let $C(n)$ be the smallest positive real constant such that there is a sequence of $n$ real numbers $x_1,x_2,\ldots,x_n$, not all zero, satisfying the following conditions:

(i) $x_1+x_2+\cdots+x_n=0$;

(ii) for each $i=1,2,\ldots,n$, it holds that $x_i\le x_{i+1}$ or $x_i\le x_{i+1}+C(n)x_{i+2}$ (the indices are taken modulo $n$).

Prove that:

(a) $C(n)\ge 2$ for all $n$;

(b) $C(n)=2$ if and only if $n$ is even.

**Problem 6.** (*Bojan Bašić*) Let $k$ be a positive integer and let $n$ be the smallest positive integer having exactly $k$ divisors. If $n$ is a perfect cube, can the number $k$ have a prime divisor of the form $3j+2$?