Mathematical Excalibur

Volume 2, Number 2 March-April, 1996

Olympiad Corner

Eighth Asian Pacific Mathematical Olympiad, March 19, 1996:

Time Allowed: Four hours.

Problem 1. Let $ABCD$ be a quadrilateral with $AB = BC = CD = DA$. Let MN and PQ be two segments perpendicular to the diagonal BD and such that the distance between them is $d > BD/2$, with $M \in AD$, $N \in DC$, $P \in AB$, and $Q \in BC$. Show that the perimeter of the hexagon $AMNCQP$ does not depend on the position of MN and PQ so long as the distance between them remains constant.

Problem 2. Let m and n be positive integers such that $n \le m$. Prove that

$$2^n n! \le \frac{(m+n)!}{(m-n)!} \le (m^2+m)^n.$$

Problem 3. Let P_1, P_2, P_3, P_4 be four points on a circle. and let I_1 be the incenter of the triangle $P_2P_3P_4$, I_2 be the incenter of the triangle $P_1P_3P_4$, I_3 be the

(*continued on page 2*)

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The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word are encouraged. The deadline for receiving material for the next issue is April 30, 1996.

For individual subscription for the remaining issue for the 95-96 academic year, send us a stamped self-addressed envelope. Send all correspondence to:

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秦九韶與「三斜求積術」

香港道教聯合會青松中學
梁子傑

周教六藝，數實成之，…大則可以通神明，順性命，小則可以經世務，類萬物，詎容以淺近窺哉？

——《數書九章序》秦九韶

前言

眾所周知，三角形面積公式是：面積＝底×高÷2。這數式雖然簡單，但是實際地使用起來，又似乎不大「方便」。試想想：如果我們在一個球場上繪畫了一個很大的三角形，要在地面上準確地定出某條底邊的高並不容易，而且過程亦很繁複。不過，明顯得很，這個三角形三條邊的邊長就不難求得，祇要用一把夠長的尺去量度就可以了。於是我們自然會問：知道三角形三條斜邊的長度，可以求到該三角形的面積嗎？

上述問題的答案是肯定的，而且該面積公式亦已被發現了好幾百年。現在就讓我為大家介紹中國南宋時期的數學家秦九韶，與及他的「三斜求積術」。

秦九韶

秦九韶生於1202年，卒於1261年，正是我國戰亂頻生的南宋時期，雖然秦九韶的父親是一名太守，但仍然逃不過需要四處遷徙逃避戰禍的命運。正因此，秦九韶自小就跟父親到過很多地方；此外，他自細就思想活躍，對天文、音律、算術、建築等學問，都有濃厚的興趣。在1247年，他從他以往曾研究過的數學問題中，精選了81道題目，將它們編寫成一本名叫《數書九章》的書。由於這本書的內容豐富，題目生動有趣，所以深受後世數學家的重視和喜愛，因此該書亦被認為是我國數學史上的巨著之一。

在《數書九章》的第三章中，秦九韶就提出了以下的問題：

問沙田一段，其小斜一十三里，中斜一十四里，大斜一十五里。…卻知為田幾何？

意思就是叫讀者求邊長分別為13里、14里和15里的一個三角形的面積。秦九韶稱這道題目為「三斜求積」，而後世人就稱書中求面積的方法為「三斜求積術」了。

三斜求積術

在《數書九章》中，秦九韶對求面積的方法有以下的闡釋：

以小斜冪，並大斜冪，減中斜冪，餘半之，自乘於上；以小斜冪乘大斜冪，減上，餘四約之為實，…開平方得積。

轉為現代的數學符號，就即是話：如果三角形三條斜邊的長度為a、b和c，則

$$\text{面積} = \frac{1}{2}\sqrt{a^2c^2-\left(\frac{a^2+c^2-b^2}{2}\right)^2}\ 。$$

依照此公式，代入$a = 13$, $b = 14$, $c = 15$，就得到三角形面積為84平方單位了。

此公式看來很複雜，而且正如中國其他古代數學書一樣，秦九韶並沒有在書中給出這公式的證明，但是祇要大家懂得「勾股定理」和代數式自乘的法則，就不難獲得此式，方法如下：

(*continued on page 4*)

Olympiad Corner:

(*continued from page 1*)

incenter of the triangle $P_1P_2P_4$, I_4 be the incenter of the triangle $P_2P_3P_1$. Prove that I_1, I_2, I_3, I_4 are the vertices of a rectangle.

Problem 4. The National Marriage Council wishes to invite n couples to form 17 discussion groups under the following conditions:

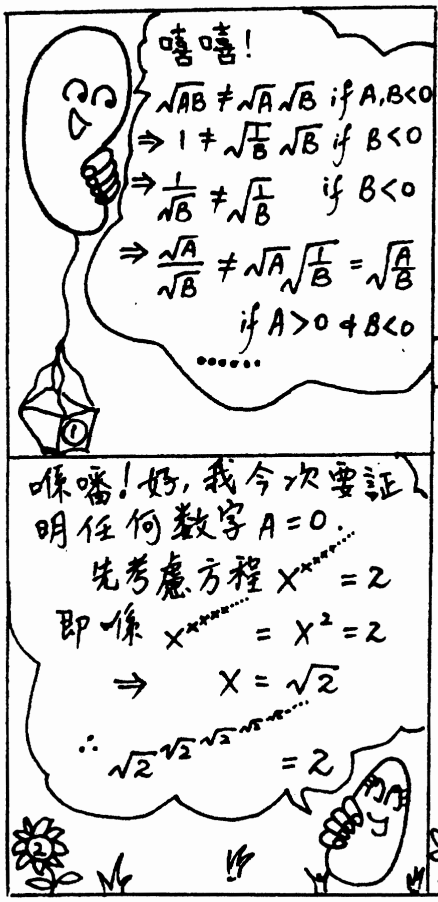
1) All members of a group must be the same sex, i.e., they are either all male or all female.
2) The difference in the size of any two groups is either 0 or 1.
3) All groups have at least one member.
4) Each person must belong to one and only one group.

Find all values of n, $n \le 1996$, for which this is possible. Justify your answer.

Problem 5. Let a, b, c be the lengths of the sides of a triangle. Prove that

$$\sqrt{a+b-c}+\sqrt{b+c-a}+\sqrt{c+a-b} \le \sqrt{a}+\sqrt{b}+\sqrt{c}$$

and determine when equality occurs.



Stirling's Inequality

Andy Liu
University of Alberta, Canada

It is useful to have a good approximation for $n!$, the factorial of a positive integer n. This is given by Stirling's Inequality which states that for $n \ge 2$,

$$\frac{n^{n+\frac{1}{2}}}{e^n} < n! < \frac{n^{n+\frac{1}{2}}}{e^{n-1}}.$$

This can be proved using elementary calculus.

We first deal with the upper bound. Consider the area under the curve $\ln x$ over the interval $[1, n]$. We divide it into $n-1$ subintervals of width 1. For $1 \le k \le n-1$, we approximate the area of the k-th strip over $[k, k+1]$ by replacing the curve with the chord joining the left endpoint $(k, \ln k)$ to the right endpoint $(k+1, \ln(k+1))$. The area of this trapezoid is $\frac{1}{2}(\ln k + \ln(k+1))$. Since $\ln x$ is concave down, it is less than the area of the strip. It follows that

$$\int_1^n \ln x dx > \frac{1}{2}(\ln 1 + 2\ln 2 + \cdots + 2\ln(n-1) + \ln n).$$

Using integration by parts, we have

$$n\ln n - n + 1 > \ln(n!) - \frac{1}{2}\ln n$$

or

$$\ln\left(\frac{n^n}{e^{n-1}}\right) > \ln\left(\frac{n!}{n^{\frac{1}{2}}}\right).$$

The desired upper bound follows from the fact that $\ln x$ is increasing.

We now turn our attention to the lower bound. Consider the area under the curve $\ln x$ over the interval $[\frac{3}{2}, n]$. We divide it into $n-2$ subintervals of width 1 and a final interval $[n-\frac{1}{2}, n]$. For $1 \le k \le n-2$, we approximate the area of the k-th strip over $[k+\frac{1}{2}, k+\frac{3}{2}]$ by replacing the curve with its tangent at the midpoint $(k+1, \ln(k+1))$. The area of this trapezoid is $\ln(k+1)$. Since $\ln x$ is concave down, it is greater than the area of the strip. For the last strip, we replace the curve with a horizontal line through the right endpoint $(n, \ln n)$. The area of this rectangle is $\frac{1}{2}\ln n$. Since $\ln x$ is increasing, it is greater than the area of the strip. It follows that

$$\int_{\frac{3}{2}}^n \ln x dx < \ln 2 + \ln 3 + \cdots + \ln(n-1) + \frac{1}{2}\ln n.$$

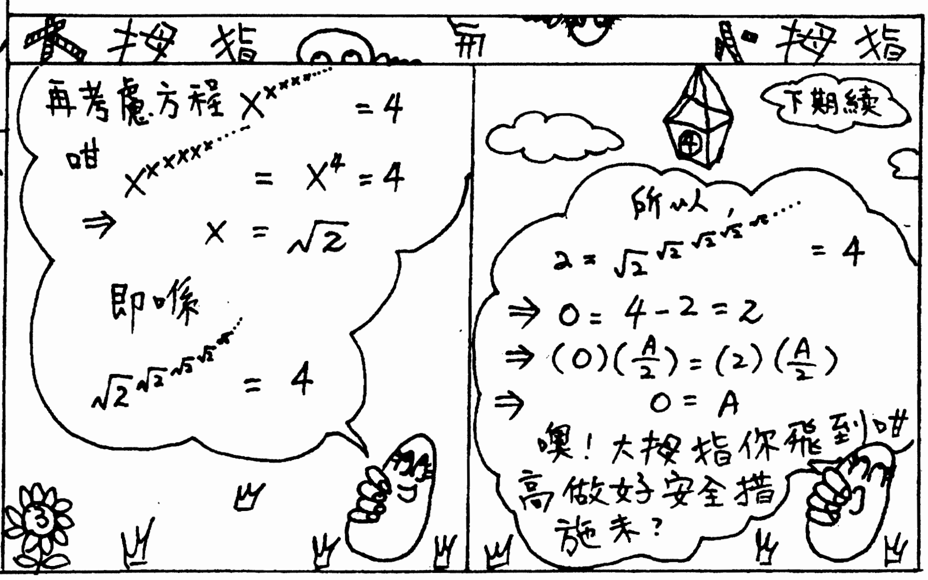
Using integration by parts, we have

$$n\ln n - n + \frac{3}{2}(1 - \ln\frac{3}{2}) < \ln(n!) - \frac{1}{2}\ln n.$$

We can drop the term $\frac{3}{2}(1-\ln\frac{3}{2})$ since $1 > \ln\frac{3}{2}$. Hence

$$\ln\left(\frac{n^n}{e^n}\right) < \ln\left(\frac{n!}{n^{\frac{1}{2}}}\right).$$

The desired lower bound follows from the fact that $\ln x$ is increasing.



Problem Corner

We welcome readers to submit solutions to the problems posed below for publication consideration. Solutions should be preceded by the solver's name, address, school affiliation and grade level. Please send submissions to *Dr. Kin-Yin Li, Dept of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is April 30, 1996.

Problem 31. Show that for any three given odd integers, there is an odd integer such that the sum of the squares of these four integers is also a square.

Problem 32. Let $a_0 = 1996$ and $a_{n+1} = a_n^2/(a_n + 1)$ for $n = 0, 1, 2, \ldots$. Prove that $[a_n] = 1996 - n$ for $n = 0, 1, 2, \ldots, 999$, where $[x]$ is the greatest integer less than or equal to x.

Problem 33. Let A, B, C be noncollinear points. Prove that there is a unique point X in the plane of ABC such that $XA^2 + XB^2 + AB^2 = XB^2 + XC^2 + BC^2 = XC^2 + XA^2 + CA^2$. (A problem proposed by Germany in the last IMO.)

Problem 34. Let $n > 2$ be an integer, c be a nonzero real number and z be a nonreal root of $X^n + cX + 1$. Show that

$$|z| \geq \frac{1}{\sqrt[n]{n-1}}.$$

Problem 35. On a blackboard, nine 0's and one 1 are written. If any two of the numbers on the board may both be replaced by their average in one operation, what is the least *positive* number that can appear on the board after a finite number of such operations?

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Solutions

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Problem 26. Show that the solutions of the equation $\cos \pi x = \frac{1}{3}$ are all irrational numbers. (*Source:* 1974 Putnam Exam.)

Solution: Official Solution.

Assume $x = m/n$ (where m, n are nonzero integers and n positive) is a solution of $\cos \pi x = 1/3$. Consider $a_k = \cos k\pi x = \cos km\pi/n$ for positive integer k. Since cosine is 2π-periodic,

$$a_{k+2n} = \cos (km\pi/n + 2m\pi) = a_k,$$

so there are at most $2n$ different possible values of a_k. Using $\cos 2\theta = 2\cos^2\theta - 1$, we have

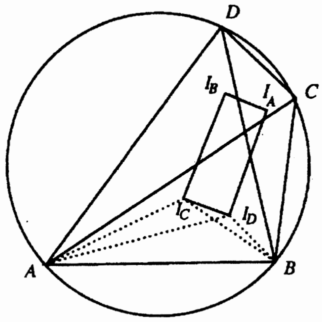
$$a_2 = -\tfrac{7}{9},\ a_4 = \tfrac{17}{81},\ \ldots,\ a_{2^p} = \frac{c_p}{3^{2^p}},\ \ldots$$

where the numerators

$$c_1 = -7,\ \ldots,\ c_p = 2c_{p-1}^2 - 3^{2^p},\ \ldots$$

are integers not divisible by 3 via mathematical induction. So the numbers $a_2, a_4, a_8, a_{16}, \ldots$ are all different, a contradiction.

Problem 27. Let $ABCD$ be a cyclic quadrilateral and let I_A, I_B, I_C, I_D be the incenters of ΔBCD, ΔACD, ΔABD, ΔABC, respectively. Show that $I_AI_BI_CI_D$ is a rectangle.



Solution: Independent solution by **CHEUNG Cheuk Lun** (S.T.F.A. Leung Kau Kui College, Form 4) and **Henry NG Ka Man** (S.T.F.A. Leung Kau Kui College, Form 5).

Draw segments AI_C, AI_D, BI_C, BI_D. Since $\angle ADB = \angle ACB$, we get

$$\angle DAB + \angle DBA = \angle CAB + \angle CBA.$$

Then

$$\begin{aligned} \angle I_CAI_D &= \angle I_CAB - \angle I_DAB \\ &= \tfrac{1}{2}\angle DAB - \tfrac{1}{2}\angle CAB \\ &= \tfrac{1}{2}\angle CBA - \tfrac{1}{2}\angle DBA \\ &= \angle I_DBA - \angle I_CBA \\ &= \angle I_CBI_D. \end{aligned}$$

So A, B, I_D, I_C are concyclic. Similarly, A, D, I_B, I_C are concyclic. Now

$$\begin{aligned} \angle I_BI_CI_D &= 360° - (\angle I_DI_CA + \angle I_BI_CA) \\ &= \angle I_DBA + \angle I_BDA \\ &= \tfrac{1}{2}\angle CBA + \tfrac{1}{2}\angle ADC \\ &= 90°. \end{aligned}$$

Similarly, the other three angles of $I_AI_BI_CI_D$ are right angles.

Comments: Surprisingly, this problem is the same as Problem 3 of the recently held APMO (c.f. Olympiad Corner on page 1).

Other commended solver: **William CHEUNG Pok Man** (S.T.F.A. Leung Kau Kui College).

Problem 28. The positive integers are separated into two subsets with no common elements. Show that one of these two subsets must contain a three term arithmetic progression.

Solution: William CHEUNG Pok Man (S.T.F.A. Leung Kau Kui College).

Let x be an integer greater than 6. If $x + 2$, $x + 4$, $x + 6$ are in the same subset, then we found a three term arithmetic progression there. Otherwise, x and (at least) one of $x + 2, x + 4, x + 6$ (call it $x + 2y$) are in the same subset. If this subset also contains one of $x - 2y, x + y, x + 4y$, then again there is a three term arithmetic progression. If not, then $x - 2y$, $x + y$, $x + 4y$ are in the other subset and they form a three term arithmetic progression there.

Comments: This problem is a special case of Van der Waerden's Theorem, which asserts that for every $m > 1$ and $n > 2$, there is a least integer $w(m, n)$ such that no matter how the numbers 1, 2, 3, …, $w(m, n)$ are separated into m subsets with no pairs having any common element, there will be at least one subset having an n term arithmetic progression. Two solvers, Chan Wing Sum and Alan Leung Wing Lun, independently pointed out that $w(2, 3) = 9$.

Other commended solvers: **CHAN Wing Sum** (HKUST), **KU Yuk Lun** (HKUST), **LEUNG Wing Lun** (S.T.F.A. Leung Kau Kui College, Form 4) and **Henry NG Ka Man** (S.T.F.A. Leung Kau Kui College, Form 5) and **POON Wing Chi** (La Salle College).

Problem 29. Suppose $P(x)$ is a nonconstant polynomial with integer coefficients and all coefficients are

(continued on page 4)

Problem Corner

(*continued from page 3*)

greater than or equal to -1. If $P(2) = 0$, show that $P(1) \neq 0$.

Solution: Independent solution by **William CHEUNG Pok Man** (S.T.F.A. Leung Kau Kui College), **Bobby POON Wai Hoi** (St Paul's College) and **WONG Him Ting** (HKU).

Since $P(2) = 0$, $P(x) = (x-2)(a_n x^n + \cdots + a_0)$, where $n \geq 0$, $a_n \neq 0$, $a_n, \ldots, a_0$ are integers. We may assume $n > 0$ as the case $n = 0$ is easy. Since the coefficients of $P(x)$ are at least -1, we have $-2a_0 \geq -1$, $a_{i-1} - 2a_i \geq -1$ for $i = 1, \ldots, n$ and $a_n \geq -1$. So $a_0 (\leq 1/2)$ is 0 or a negative integer. Inductively, if $a_{i-1} \leq 0$, then $a_i \leq (a_{i-1} + 1)/2 \leq 1/2$ will also be 0 or negative. Hence, $a_0, \ldots, a_n \leq 0$. Then $a_n = -1$ and $P(1) = -(a_n + \ldots + a_0) \geq -a_n = 1$.

Comments: This is a variation of a problem on the 1988 Tournament of the Towns.

Problem 30. For positive integer $n > 1$, define $f(n)$ to be 1 plus the sum of all prime numbers dividing n multiplied by their exponents, *e.g.*, $f(40) = f(2^3 \times 5^1) = 1 + (2\times3 + 5\times1) = 12$. Show that if $n > 6$, the sequence $n, f(n), f(f(n)), f(f(f(n))), \ldots$ must eventually be repeating 8, 7, 8, 7, 8, 7, ….

Solution: Independent solution by **Bobby POON Wai Hoi** (St Paul's College) and **WONG Him Ting** (HKU).

Considering the factorizations of n, we see that $f(n) \leq 6$ if and only if $n \leq 6$. Clearly, $f(7) = 8$, $f(8) = 7$. For $n \geq 9$ and not prime, we will first show $f(n) \leq n - 2$ by induction.

We have $f(9) = 7$. Suppose it is true for 9 to $n - 1$. For $n > 9$ and not prime, there are positive integers $r, s > 1$ such that $n = rs$ and $(r-1)(s-1) \geq 4$. (This is because $(r-1)(s-1) \leq 3$ implies $rs \leq 2 \times 4 = 8$.) If $2 \leq r \leq 8$ or r prime, then $f(r) \leq r + 1$. Otherwise, $9 \leq r < n$ and r is not prime, which imply by the induction step that $f(r) \leq r - 2 < r + 1$. Similarly, $f(s) \leq s + 1$. From the definition of f, we get

$$\begin{aligned} f(n) &= f(r) + f(s) - 1 \\ &\leq (r+1) + (s+1) - 1 \\ &= n + 2 - (r-1)(s-1) \\ &\leq n - 2, \end{aligned}$$

which completes the induction.

Now suppose the problem is true for $n = 7, 8, \ldots, m-1$, i.e., the sequence $n, f(n), f(f(n)), f(f(f(n))), \ldots$ eventually repeats 8, 7, 8, 7, 8, 7, …. For the case $n = m$, if m is not prime, then $7 \leq f(m) \leq m - 2$. By the induction step, the case $f(m)$ is true, so the case m will also be true. If m is prime, then $f(m) = 1 + m$ is not prime and so $7 \leq f(f(m)) \leq f(m) - 2 = m - 1$. By the induction step, the case $f(f(m))$ is true, so the case m will also be true.

秦九韶與「三斜求積術」：

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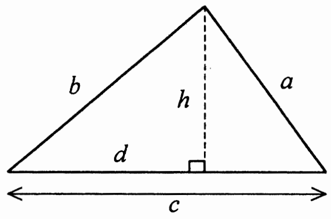
如下圖，



$$d = c - \sqrt{a^2 - h^2}$$

$$\therefore \quad b^2 = \left(c - \sqrt{a^2 - h^2}\right)^2 + h^2$$

$$= a^2 + c^2 - 2c\sqrt{a^2 - h^2}$$

$$\therefore \quad c\sqrt{a^2 - h^2} = ½(a^2 + c^2 - b^2)$$

兩端平方得

$$a^2c^2 - c^2h^2 = \left(\frac{a^2 + c^2 - b^2}{2}\right)^2$$

但，

$$面積 = ½ch$$

$$= \frac{1}{2}\sqrt{a^2c^2 - \left(\frac{a^2 + c^2 - b^2}{2}\right)^2}$$

（證畢）

秦氏三角

後世人對《數書九章》都有不少的批評，其中一項就是指它「脫離現實」。好似上面討論過的例子，我們跟本不可能會有一塊田，它的一邊會長「一十五里」。這個數字實在大了一點。不過，批評秦九韶的人似乎忘記了，秦九韶所引用的例子，其實有一個非常特別的特性：這個三角形的邊長是三個連續整數，而且面積亦剛好又是一個整數！祇要大家細心想想就會瞭解，這是一個很難得的「巧合」。秦九韶一定是經過細心選擇，才引用這個例子的。

因此有數學家就稱具有上述特性的三角形為「秦氏三角」。除了(13, 14, 15)可以組成「秦氏三角」之外，還有(3, 4, 5)、(51, 52, 53)、(193, 194, 195)、(723, 724, 725)等等。現在更有數學家發現：如果k是一個正整數，祇要$\sqrt{3(k^2-1)}$亦是一個整數，則$(2k-1, 2k, 2k+1)$就可組成一個「秦氏三角」了。

希羅公式 (Heron's formula)

最後一提的是，秦九韶並不是歷史上第一位懂得「三斜求積」方法的人。大約在公元100年左右，希臘數學家希羅(Heron)早已提出了一個計算面積而且更簡單的公式。不過，他當時提出的證明就非常複雜，如果從秦九韶的公式出發，我們會更容易獲得那「希羅公式」：

$$面積 = \frac{1}{2}\sqrt{a^2c^2 - \left(\frac{a^2 + c^2 - b^2}{2}\right)^2}$$

$$= \sqrt{\frac{1}{16}\left(4a^2c^2 - (a^2 + c^2 - b^2)^2\right)}$$

$$= \sqrt{\frac{1}{16}(2ac + a^2 + c^2 - b^2)(2ac - a^2 - c^2 + b^2)}$$

$$= \sqrt{\frac{1}{16}\left((a+c)^2 - b^2\right)\left(b^2 - (a-c)^2\right)}$$

$$= \sqrt{\frac{1}{16}(a+c+b)(a+c-b)(b+a-c)(b-a+c)}$$

$$= \sqrt{\left(\frac{a+b+c}{2}\right)\left(\frac{a+b+c}{2} - a\right)\left(\frac{a+b+c}{2} - b\right)\left(\frac{a+b+c}{2} - c\right)}$$

最後設 $s = ½(a + b + c)$，則

$$面積 = \sqrt{s(s-a)(s-b)(s-c)}\ ,$$

這就是「希羅公式」了。

參考書目

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王守義遺著：數書九章新釋，安徽科學技術出版社。