Mathematical Excalibur

Volume 2, Number 4 Sept-Oct, 1996

Olympiad Corner

37th International Mathematical Olympiad, July 5-17, 1996, Mumbai, India.

First Day (10 July, 1996)
Time: 4½ hours
(Each problem is worth 7 points.)

Problem 1. Let $ABCD$ be a rectangular board with $|AB| = 20$, $|BC| = 12$. The board is divided into 20×12 unit squares. Let r be a given positive integer. A coin can be moved from one square to another if and only if the distance between the centres of the two squares is $\sqrt{r}$. The task is to find a sequence of moves taking the coin from the square which has A as a vertex to the square which has B as a vertex.

(a) Show that the task cannot be done if r is divisible by 2 or 3.

(b) Prove that the task can be done if $r = 73$.

(c) Can the task be done when $r = 97$?

(*continued on page 4*)

Acknowledgment: Thanks to NG Hok Lin, POON Wing Chi and WAN Yung Chun for their help in typesetting.

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word are encouraged. The deadline for receiving material for the next issue is Nov 15, 1996.

For individual subscription for the remaining four issues for the 96-97 academic year, send us four stamped self-addressed envelopes. Send all correspondence to:

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Ptolemy's Theorem

Kin-Yin Li

If four points are chosen from a plane, the chance that they are collinear or concyclic is extremely small. So there should be some special conditions for this to happen. Such a condition is given by the famous theorem of Ptolemy.

Ptolemy's Theorem. *For distinct points A, B, C, D on a plane, we have $AB\cdot CD + AD\cdot BC \ge AC\cdot BD$. Equality happens if and only if A, B, C, D are collinear or concyclic with A, C separating B, D.*

A simple proof using complex numbers can be given as follows. Let a, b, c, d be the complex numbers corresponding to the points A, B, C, D respectively. Since

$$(b-a)(d-c) + (d-a)(c-b) = (c-a)(d-b),$$

taking absolute values and applying the triangle inequality, we get

$$\begin{aligned} AB\cdot CD + AD\cdot BC &= |b-a||d-c| + |d-a||c-b| \\ &\ge |c-a||d-b| = AC\cdot BD. \end{aligned}$$

From the triangle inequality, we have equality if and only if

$$(b-a)(d-c) = t(d-a)(c-b) \text{ for some } t>0.$$

In such case, $(d-a)/(b-a)$ is a positive multiple of $(d-c)/(c-b)$. So

$$\arg\{(d-a)/(b-a)\} = \arg\{(d-c)/(c-b)\},$$

i.e., $\angle DAB = 180° - \angle DCB$. This means A, B, C, D are collinear or concyclic with A, C separating B, D.

Next we will give two simple and useful corollaries.

Corollary 1. *For a cyclic quadrilateral $ABCD$ with ΔABC equilateral, we have $BD = AD + CD$.*

Corollary 2. *For a cyclic quadrilateral $ABCD$ with $\angle ABC = \angle ADC = 90°$, we have $BD = AC\sin\angle BAD$.*

The first corollary follows because $AB = BC = CA$ and thus

$$\begin{aligned} & AB\cdot CD + AD\cdot BC = AC\cdot BD \\ \Rightarrow\ & CA\cdot CD + AD\cdot CA = AC\cdot BD \\ \Rightarrow\ & CD + AD = BD. \end{aligned}$$

The second corollary also follows easily because

$$\begin{aligned} AC\sin\angle A &= AC\sin(\angle BAC+\angle DAC) \\ &= (BC\cdot AD + AB\cdot CD)/AC = BD \end{aligned}$$

using the compound angle formula. (Actually, corollary 2 is true if A, B, C, D are just concyclic, not necessarily in that order, and $\angle ABC = \angle ADC = 90°$, since by the sine law, $BD/\sin\angle BAD$ equals the diameter AC of the circumcircle of ΔBAD.)

Example 1. (*IMO 1995*) Let $ABCDEF$ be a convex hexagon with $AB=BC=CD$, $DE=EF=FA$ and $\angle BCD=\angle EFA=60°$. Let G and H be two points inside the hexagon such that $\angle AGB=\angle DHE=120°$. Show that

$$AG + GB + GH + DH + HE \ge CF.$$

Solution. Let X, Y be points outside the hexagon such that ΔABX and ΔDEY are equilateral. Then $ABCDEF$ is congruent to $DBXAEY$ and $CF = XY$. Now

$$\angle AXB+\angle AGB = \angle DYE+\angle DHE = 180°.$$

Thus $AXBG$ and $DHEY$ are cyclic quadrilaterals. By corollary 1, $XG = AG + GB$ and $HY = DH + HE$. So

$$\begin{aligned} & AG+GB+GH+DH+HE \\ &= XG+GH+HY \ge XY = CF. \end{aligned}$$

Example 2. (*IMO 1996*) Let P be a point inside ΔABC such that $\angle APB - \angle ACB = \angle APC - \angle ABC$. Let D, E be the incenters of ΔAPB, ΔAPC, respectively.

(*continued on page 2*)

Ptolemy's Theorem

(*continued from page 1*)

Show that AP, BD and CE meet at a point.

Solution. Equivalently we have to show the angle bisectors BD, CE of $\angle ABP$, $\angle ACP$, respectively, meet at the same point on AP. Let the feet of the perpendiculars from P to BC, CA, AB be X, Y, Z respectively. Then $AZPY$, $BXPZ$, $CYPX$ are cyclic quadrilaterals. Now

$$\begin{aligned}\angle APB - \angle ACB &= \angle YAP + \angle XBP \\ &= \angle YZP + \angle XZP \\ &= \angle YZX.\end{aligned}$$

Similarly $\angle APC - \angle ABC = \angle XYZ$. So $XZ = XY$. By corollary 2,

$$BP\sin\angle B = XZ = XY = CP\sin\angle C.$$

Then $BP/CP = \sin\angle C/\sin\angle B = AB/AC$. So $AB/BP = AC/CP$. By the angle bisector theorem, this implies BD and CE meet at the same point on AP.

Example 3. (*Erdös-Mordell Inequality*) Let P be a point inside ΔABC and let d_a, d_b, d_c be the distances from P to BC, CA, AB respectively. Show that

$$PA + PB + PC \geq 2(d_a + d_b + d_c)$$

with equality if and only if ΔABC is equilateral and P is the incenter.

Solution. Let X, Y, Z be the feet of perpendiculars from P to BC, CA, AB respectively. By corollary 2 or sine law and cosine law,

$$\begin{aligned}PA\sin\angle A &= YZ \\ &= \sqrt{d_b^2 + d_c^2 - 2d_b d_c \cos(180° - \angle A)}\,.\end{aligned}$$

Since $180° - \angle A = \angle B + \angle C$, expanding and regrouping, we get

$$\begin{aligned}PA\sin\angle A &= \{(d_b\sin\angle C + d_c\sin\angle B)^2 \\ &\quad + (d_b\sin\angle C + d_c\sin\angle B)^2\}^{1/2} \\ &\geq d_b\sin\angle C + d_c\sin\angle B.\end{aligned}$$

Using inequalities like the last one and the fact $x + 1/x \geq 2$, we have

$$\begin{aligned}PA + PB + PC &\geq \sum \frac{d_b \sin\angle C + d_c \sin\angle B}{\sin\angle A} \\ &= \sum d_a\left(\frac{\sin\angle B}{\sin\angle C} + \frac{\sin\angle C}{\sin\angle B}\right) \\ &\geq 2(d_a + d_b + d_c).\end{aligned}$$

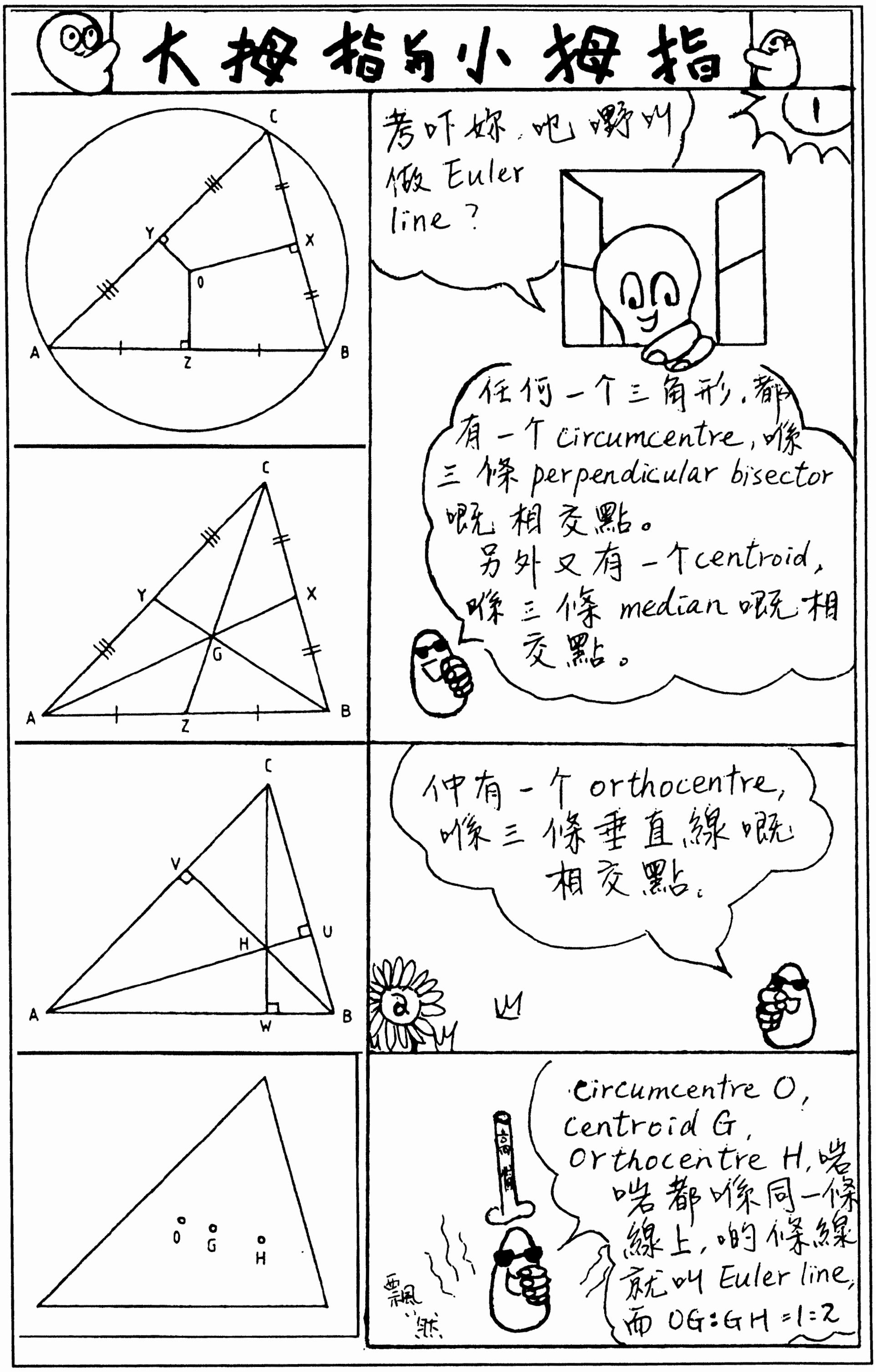
where the middle equation was obtained by rearranging terms. Finally, equality occurs if and only if $\angle A = \angle B = \angle C$ and $d_a = d_b = d_c$, i.e., ΔABC is equilateral and P is the incenter.

Example 4. (*IMO 1991*) Let ABC be a triangle and P an interior point in ABC. Show that at least one of the angles $\angle PAB$, $\angle PBC$, $\angle PCA$ is less than or equal to 30°.

Solution. Suppose none of the three angles is less than or equal to 30°. If one of them is at least 150°, then the other two will be at most 30°, a contradiction. So we may assume the three angles are greater than 30° and less than 150°. Let d_a be the distance from P to BC, then

$$\begin{aligned}2d_a &= 2PB\sin\angle PBC \\ &> (2\sin 30°)PB = PB.\end{aligned}$$

Three such inequalities added together will yield $2(d_a + d_b + d_c) > PA + PB + PC$, contradicting the Erdös-Mordell inequality. So one of the three angles is at most 30°.



Problem Corner

We welcome readers to submit solutions to the problems posed below for publication consideration. Solutions should be preceded by the solver's name, address, school affiliation and grade level. Please send submissions to *Dr. Kin-Yin Li, Dept of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is Nov 15, 1996.

Problem 41. Find all nonnegative integers x, y satisfying $(xy-7)^2 = x^2 + y^2$.

Problem 42. What are the possible values of $\sqrt{x^2+x+1}-\sqrt{x^2-x+1}$ as x ranges over all real numbers?

Problem 43. How many 3-element subsets of the set $X = \{1, 2, 3, ..., 20\}$ are there such that the product of the 3 numbers in the subset is divisible by 4?

Problem 44. For an acute triangle ABC, let H be the foot of the perpendicular from A to BC. Let M, N be the feet of the perpendiculars from H to AB, AC, respectively. Define L_A to be the line through A perpendicular to MN and similarly define L_B and L_C. Show that L_A, L_B and L_C pass through a common point O. (This was an unused problem proposed by Iceland in a past IMO.)

Problem 45. Let $a, b, c > 0$ and $abc=1$. Show that

$$\frac{ab}{a^5+b^5+ab}+\frac{bc}{b^5+c^5+bc}+\frac{ca}{c^5+a^5+ca}\le 1$$

(This was an unused problem in IMO96.)

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Solutions

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Problem 36. Let a, b and c be positive numbers such that $a^2 + b^2 - ab = c^2$. Prove that $(a-c)(b-c) \le 0$.

Solution: POON Wing Chi (La Salle College, Form 6).

Without loss of generality, we may assume $a \le b$. Since $a, b > 0$, so

$$a \le \sqrt{a^2+b(b-a)} = c = \sqrt{b^2-a(b-a)} \le b.$$

Therefore $(a-c)(b-c) \le 0$.

Other commended solvers: **CHAN Ming Chiu** (La Salle College, Form 5), **CHAN Wing Sum** (HKUST), **CHENG Wing Kin** (S.K.H. Lam Woo Memorial Secondary School, Form 4), **Calvin CHEUNG Cheuk Lun** (S.T.F.A. Leung Kau Kui College, Form 54), **KWOK Wing Yin** (St. Clare's Girls' School), **LEE Ho Fai Vincent** (Queen's College, Form 6), **Alan LEUNG Wing Lun** (S.T.F.A. Leung Kau Kui College, Form 4), **LIU Wai Kwong** (Pui Tak Canossian College), **Gary NG Ka Wing** (S.T.F.A. Leung Kau Kui College, Form 3), **Henry NG Ka Man** (S.T.F.A. Leung Kau Kui College, Form 5), **NG Pui Keung** (St. Paul's Co-educational College, From 5), **PAI Hung Ming Tedward** (S.K.H. Tang Shiu Kin Secondary School, Form 6), **SZE Hoi Wing Holman** (St. Paul's Co-educational College , Form 5) and **YU Kit Wing** (HKUST).

Problem 37. Two non-intersecting circles λ_1 and λ_2 have centres O_1 and O_2 respectively. A_1 and A_2 are points on λ_1 and λ_2 respectively, such that A_1A_2 is an external common tangent of the circles. The segment O_1O_2 intersects λ_1 and λ_2 at B_1 and B_2 respectively. The lines A_1B_1 and A_2B_2 intersect at C, and the line through C perpendicular to B_1B_2 intersects A_1A_2 at D. Prove that D is the midpoint of A_1A_2.

Solution: Independent solution by **CHAN Ming Chiu** (La Salle College, Form 5), **CHENG Wing Kin** (S.K.H. Lam Woo Memorial Secondary School, Form 4), **Calvin CHEUNG Cheuk Lun** (S.T.F.A. Leung Kau Kui College, Form 4), **LIU Wai Kwong** (Pui Tak Canossian College), **Gary NG Ka Wing** (S.T.F.A. Leung Kau Kui College, Form 3), **Henry NG Ka Man** (S.T.F.A. Leung Kau Kui College, Form 5) and **PAI Hung Ming Tedward** (S.K.H. Tang Shiu Kin Secondary School, Form 6).

We have $\angle DA_1C = 90° - \angle O_1A_1B_1 = 90° - \angle O_1B_1A_1 = 90° - \angle CB_1O_2 = \angle A_1CD$, which implies $A_1D = CD$. Similarly, $A_2D = CD$. So $A_1D = A_2D$.

Problem 38. Prove that from any sequence of 1996 real numbers, one can choose a block of consecutive terms whose sum differs from an integer by at most 0.001.

Solution: LIU Wai Kwong (Pui Tak Canossian College).

Let the numbers be $x_1, x_2, ..., x_{1996}$ and let $s_i = x_1 + x_2 + ... + x_i$ for $i = 1, 2, ..., 1996$. Define $\{x\} = x - [x]$, where $[x]$ is the greatest integer less than or equal to x. Consider the 1995 intervals $[0, \frac{1}{1995})$, $[\frac{1}{1995}, \frac{2}{1995})$, ..., $[\frac{1994}{1995}, 1)$ and the 1996 numbers $\{s_1\}, \{s_2\}, ..., \{s_{1996}\}$. By the pigeon-hole principle, there is a pair s_i, s_j with $\{s_i\}$, $\{s_j\}$ in the same interval. By cancelling the common terms in s_i, s_j, we get a block of consecutive terms whose sum differ from an integer by at most $\frac{1}{1995} < 0.001$.

Problem 39. Eight students took part in a contest with eight problems.

(a) Each problem was solved by 5 students. Prove that there were two students who between them solved all eight problems.

(b) Prove that this is not necessarily the case if 5 is replaced by 4. (A counterexample is enough.)

Solution: Independent solution by **Gary NG Ka Wing** (S.T.F.A. Leung Kau Kui College, Form 3), **Henry NG Ka Man** (S.T.F.A. Leung Kau Kui College, Form 5) and **POON Wing Chi** (La Salle College, Form 6).

(a) If a pair of students together did not solve all 8 problems, then there was at least 1 problem they both missed. Among 8 students, there are 28 pairs. However, for each problem, there were only 3 students (which give 3 pairs) missed the problem. For the 8 problems, there were at most 24 pairs missing at least 1 problem. Since 28 > 24, by the pigeonhole principle, there was a pair together solved all 8 problems. (In fact, there were at least 28 – 24 = 4 such pairs!)

(*continued on page 4*)

Problem Corner

(*continued from page 3*)

(b) Here is a counter example:

students 1, 2 solved problems 1, 2, 3, 4
students 3, 4 solved problems 3, 4, 5, 6
students 5, 6 solved problems 1, 6, 7, 8
students 7, 8 solved problems 2, 5, 7, 8

Other commended solvers: **CHAN Wing Sum** (HKUST) and **LIU Wai Kwong** (Pui Tak Canossian College).

Problem 40. ABC is an equilateral triangle. For a positive integer $n \geq 2$, D is the point on AB such that $AD = \frac{1}{n}AB$. $P_1, P_2, \ldots, P_{n-1}$ are points on BC which divide it into n equal segments. Prove that $\angle AP_1D + \angle AP_2D + \ldots + \angle AP_{n-1}D = 30^{\circ}$.
[*Hint*: Consider Q_i such that ADP_iQ_i is a parallelogram.]

Solution: CHAN Ming Chiu (La Salle College, Form 5).

Let P_0 be B and P_n be C. Let Q_0 be the point on AB such that $Q_0B = \frac{1}{n}AB$ and Q_i (i = 1, 2, ..., $n-1$) be such that ADP_iQ_i forms a parallelogram. Then $\angle AP_iD = \angle Q_iAP_i$. Now P_iQ_i is parallel to and has the same length as AD and P_0Q_0 (i = 1, 2, ..., $n-1$). Also, $P_iQ_i = P_iP_{i+1}$ (i = 0, 1, ..., $n-1$). These imply $\Delta P_iQ_iP_{i+1}$ is equilateral (i = 0, 1, ..., $n-1$). By symmetry with respect to the perpendicular bisector of BC, we get $\angle Q_iAP_i = \angle Q_{n-i-1}AP_{n-i}$. Thus

$$\begin{aligned}&(\angle Q_1AP_1 + \angle Q_{n-2}AP_{n-1})\\ &\quad + (\angle Q_2AP_2 + \angle Q_{n-3}AP_{n-2})\\ &\quad + \ldots + (\angle Q_{n-1}AP_{n-1} + \angle Q_0AP_1)\\ &\quad = \angle BAC\end{aligned}$$

and

$$\angle AP_1D + \angle AP_2D + \ldots + \angle AP_{n-1}D = \frac{1}{2}\angle BAC = 30^{\circ}.$$

Other commended solvers: **LIU Wai Kwong** (Pui Tak Canossian College), **Gary NG Ka Wing** (S.T.F.A. Leung Kau Kui College, Form 3) and **Henry NG Ka Man** (S.T.F.A. Leung Kau Kui College, Form 5).

Olympiad Corner

(*continued from page 1*)

Problem 2. Let P be a point inside triangle ABC such that

$$\angle APB - \angle ACB = \angle APC - \angle ABC.$$

Let D, E be the incentres of triangles APB, APC respectively. Show that AP, BD and CE meet at a point.

Problem 3. Let S = {0, 1, 2, 3, …} be the set of non-negative integers. Find all functions f defined on S and taking their values in S such that

$$f(m + f(n)) = f(f(m)) + f(n)$$

for all m, n in S.

Second Day (11 July, 1996)
Time: 4½ hours
(Each problem is worth 7 points.)

Problem 4. The positive integers a and b are such that the numbers $15a+16b$ and $16a-15b$ are both squares of positive integers. Find the least possible value that can be taken by the minimum of these two squares.

Problem 5. Let $ABCDEF$ be a convex hexagon such that AB is parallel to ED, BC is parallel to FE and CD is parallel to AF. Let R_A, R_C, R_E denote the circumradii of triangles FAB, BCD, DEF respectively, and let p denote the perimeter of the hexagon. Prove that

$$R_A + R_C + R_E \geq \frac{p}{2}.$$

Problem 6. Let n, p, q be positive integers with $n > p + q$. Let $x_0, x_1, \ldots, x_n$ be integers satisfying the following conditions:

(a) $x_0 = x_n = 0$;

(b) for each integer i with $1 \leq i \leq n$, either $x_i - x_{i-1} = p$ or $x_i - x_{i-1} = -q$.

Show that there exists a pair (i,j) of indices with $i < j$ and $(i,j) \neq (0,n)$ such that $x_i = x_j$.

IMO96, Mumbai, India
Facts and Statistics

Number of Participating Teams: 75
Informal Rank for the Hong Kong Team: 25
Medals for the Hong Kong Team: 1 silver and 4 bronze medals.

Below: A photo of the Hong Kong Team taken at the Kai Tak Airport before departure. From left to right are: Bobby POON Wai Hoi, MOK Tze Tao, HO Wing Yip, Roger NG Keng Po (observer), TSE Shan Shan, LAM Sze Ho (Deputy Leader), YU Chun Ling, LAW Siu Lung.