Olympiad Corner

25th United States of America Mathematical Olympiad:

Part I (9am-noon, May 2, 1996)

Problem 1. Prove that the average of the numbers \( \pi \sin n^\circ \) (\( n = 2, 4, 6, \ldots, 180 \)) is \( \cot 1^\circ \).

Problem 2. For any nonempty set \( S \) of real numbers, let \( \sigma(S) \) denote the sum of the elements of \( S \). Given a set \( A \) of \( n \) positive integers, consider the collection of all distinct sums \( \sigma(S) \) as \( S \) ranges over the nonempty subsets of \( A \). Prove that this collection of sums can be partitioned into \( n \) classes so that in each class, the ratio of the largest sum to the smallest sum does not exceed 2.

Problem 3. Let \( ABC \) be a triangle. Prove that there is a line \( l \) (in the plane of triangle \( ABC \)) such that the intersection of the interior of triangle \( ABC \) and the interior of its reflection \( A'B'C' \) in \( l \) has area more than \( 2/3 \) the area of triangle \( ABC \).

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Error Correcting Codes (Part I)

Tsz-Mei Ko

Suppose one would like to transmit a message, say "HELLO...", from one computer to another. One possible way is to use a table to encode the message into binary digits. Then the receiver would be able to decode the message with a similar table. One such table is the American Standard Code for Information Interchange (ASCII) shown in Figure 1. The letter H would be encoded as 1001000, the letter E would be encoded as 1000101, etc. (Figure 2).

Figure 1. ASCII code

Figure 2. Two computers talking

The receiver will be able to decode the message correctly if there is no error during transmission. However, if there are transmission errors, the receiver may decode the message incorrectly. For example, the letter H (1001000) would be received as J (1001010) if there is an error at position 6.

Figure 3. Error at position 6.

One possible way to detect transmission errors is to add redundant bits, i.e., append extra bits to the original message. For an even parity code, a 0 or 1 is appended so that the total number of 1's is an even number. The letters H and E would be represented by 10010000 and 10001011 respectively. With an even parity code, the receiver can detect one transmission error, but unable to correct it. For example, if 10010000 (for the letter H) is received as 10010101, the receiver knows that there is at least one error during transmission since the received bit sequence has an odd parity, i.e., the total number of 1's is an odd number.

Figure 4. Even parity code

Is there an encoding method so that the receiver would be able to correct transmission errors? Figure 5 shows one such method by arranging the bit sequence (e.g., 1001) into a rectangular block and add parity bits to both rows and columns. For the example shown, 1001 would be encoded as 10011111 (by first appending the row parities and then the column parities). If there is an error during transmission, say at position 2, the receiver can similarly arrange the received sequence 11011111 into a rectangular block and detect that there is an error in row 1 and column 2.

Figure 5. A code that can correct 1 error.

The above method can be used to correct one error but rather costly. For every four bits, one would need to transmit an extra four redundant bits. Is there a better way to do the encoding? In 1950, Hamming found an ingenious method to

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Problem Corner

We welcome readers to submit solutions to the problems posed below for publication consideration. Solutions should be preceded by the solver’s name, address, school affiliation and grade level. Please send submissions to Dr. Kin-Yin Li, Dept. of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon. The deadline for submitting solutions is Jan 31, 1997.

Problem 46. For what integer a does $x^2 - x + a$ divide $x^{13} + x + 90$?

Problem 47. If $x$, $y$, $z$ are real numbers such that $x^2 + y^2 + z^2 = 2$, then show that $x + y + z \leq xyz + 2$.

Problem 48. Squares $ABDE$ and $BCFG$ are drawn outside of triangle $ABC$. Prove that triangle $ABC$ is isosceles if $DG$ is parallel to $AC$.

Problem 49. Let $u_1$, $u_2$, $u_3$, ..., be a sequence of integers such that $u_1 = 29$, $u_2 = 45$ and $u_{n+2} = u_{n+1} - u_n$ for $n = 1, 2, 3, ...$. Show that 1996 divides infinitely many terms of this sequence. (Source: 1986 Canadian Mathematical Olympiad with modification)

Problem 50. Four integers are marked on a circle. On each step we simultaneously replace each number by the difference between this number and the next number on the circle in a given direction (that is, the numbers are either all odd (there are $2^{10}$ such subsets) or two odd and one even, but the even one is not divisible by 4, the numbers have product not divisible by 4, the numbers are either all odd (there are $2^{10}$ such subsets) or two odd and one even, but the even one is not divisible by 4).

Suppose $x$, $y$ are nonnegative integers such that $(xy - 7)^2 = x^2 + y^2$. Then $(xy - 6)^2 + 13 = (x + y)^2$ by algebra. So

$$13 = [(x+y) + (xy-6)][(x+y) - (xy-6)].$$

Since 13 is prime, the factors on the right side can only be $\pm 1$ or $\pm 13$. There are four possibilities yielding $(x,y) = (0,7), (7,0), (3,4), (4,3)$.

Other commended solvers: CHAN Ming Chiu (La Salle College, Form 6), CHENG Wing Kin (S.K.H. Lam Woo Memorial Secondary School, Form 5), LIU Wai Kwong (Pui Tak Canossian College), Henry NG Ka Man (STFA Leung Kau Kui College, Form 5), POON Wing Chi (La Salle College, Form 7), TSANG Sai Wing (Valtorta College, Form 6), YU Chon Ling (HKU), YUEN Chu Ming (Kiangsu-Chekiang College (Shatin), Form 6) and YUNG Fai (CUHK).

There are $C_3^{20} = 1140$ 3-element subsets of $X$. For a 3-element subset whose 3 numbers have product not divisible by 4, the numbers are either all odd (there are $2^{10}$ such subsets) or two odd and one even, but the even one is not divisible by 4, the numbers have product not divisible by 4, the numbers are either all odd (there are $2^{10}$ such subsets) or two odd and one even, but the even one is not divisible by 4).

Solution: William CHEUNG Pok-man (STFA Leung Kau Kui College, Form 6).

Let $A=(x,0), B=(-\frac{1}{2}, \frac{\sqrt{3}}{2}), C=(\frac{1}{2}, \frac{\sqrt{3}}{2})$. The expression $\sqrt{x^2 + x + 1} - \sqrt{x^2 - x + 1}$ is just $AB - AC$. As $x$ ranges over all real numbers, $A$ moves along the real axis and the triangle inequality yields


All numbers on the interval $(-1, 1)$ are possible.

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All numbers on the interval $(-1, 1)$ are possible.

Problem 44. For an acute triangle $ABC$, let $H$ be the foot of the perpendicular from $A$ to $BC$. Let $M, N$ be the feet of the perpendiculars from $H$ to $AB, AC$, respectively. Define $L_A$ to be the line through $A$ perpendicular to $MN$ and similarly define $L_B$ and $L_C$. Show that $L_A, L_B$ and $L_C$ pass through a common point $O$. (This was an unused problem proposed by Iceland in a past IMO.)

Solution: William CHEUNG Pok-man (STFA Leung Kau Kui College, Form 6).

Let $L_A$ intersect the circumcircle of $\Delta ABC$ at $A$ and $E$. Since $\angle AMH = 90^\circ = \angle ANH, A, M, H, N$ are concyclic. So $\angle MAH = \angle MNH = 90^\circ - \angle ANM = \angle NAE = \angle CBE$. Now $\angle ABE = \angle CBE + \angle ABC = \angle MAH + \angle ABC = 90^\circ$. So $AE$ is a diameter of the circumcircle and

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Point P such that $\angle PAB = 10^\circ$, $\angle PBA = 20^\circ$, $\angle PCA = 30^\circ$, $\angle PAC = 40^\circ$. Prove that triangle $ABC$ is isosceles.

Problem 6. Determine (with proof) whether there is a subset $X$ of the integers with the following property: for any integer $n$ there is exactly one solution of $a + 2b = n$ with $a, b \in X$.

Problem 45. Let $a$, $b$, $c > 0$ and $abc=1$. Show that

$$\frac{ab}{a^2+b^2} + \frac{bc}{b^2+c^2} + \frac{ca}{c^2+a^2} \leq 1$$

(This was an unused problem in IMO 96.)

Solution: YUNG Fai (CUHK)

Expanding $(a^2 - b^2)(a^2 - b^2) \geq 0$, we get $a^6 + b^6 \geq a^2b^2(a+b)$. So using this and $abc = 1$, we get

$$\frac{ab}{a^2+b^2+ab} = \frac{ab}{a+b+c}$$

Adding 3 such inequalities, we get the desired inequality. In fact, equality can occur if and only if $a = b = c = 1$.

Other commended solvers: POON Wing Chi (La Salle College, Form 7) and YU Chon Ling (HKU).

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Part II (1pm-4pm, May 2, 1996)

Problem 4. An $n$-term sequence $(x_1, x_2, \ldots, x_n)$ in which each term is either 0 or 1 is called a binary sequence of length $n$. Let $a_n$ be the number of binary sequences of length $n$ containing no three consecutive terms equal to 0, 1, 0 in that order. Let $b_n$ be the number of binary sequences of length $n$ that contain no four consecutive terms equal to 0, 0, 1, 1 or 1, 1, 0, 0 in that order. Prove that $b_{n+1} = 2a_n$ for all positive integers $n$.

Problem 5. Triangle $ABC$ has the following property: there is an interior point $P$ such that $\angle PAB = 10^\circ$, $\angle PBA = 20^\circ$, $\angle PCA = 30^\circ$, $\angle PAC = 40^\circ$. Prove that triangle $ABC$ is isosceles.

Problem 6. Determine (with proof) whether there is a subset $X$ of the integers with the following property: for any integer $n$ there is exactly one solution of $a + 2b = n$ with $a, b \in X$.

Error Correcting Codes (Part I)
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add the redundancy. To encode a four-bit sequence $p_1p_2p_3p_4$ (say 1001), one would first draw three intersecting circles $A$, $B$, $C$ and put the information bits $p_1$, $p_2$, $p_3$, $p_4$ into the four overlapping regions $A_1B_1$, $A_1C_1$, $B_1C_1$ and $A_1B_1C_1$ (Figure 6). Then three parity bits $p_5$, $p_6$, $p_7$ are generated so that the total number of 1's in each circle is an even number. For the example shown, 1001 would be encoded as 1001001.

Other commended solvers: Calvin CHEUNG Cheuk Lon (STFA Leung Kau Kui College, Form 5), LIU Wai Kwong (Pui Tak Canossian College), POON Wing Chi (La Salle College, Form 7) and YU Chon Ling (HKU).

Figure 6. Hamming code

If there is one error during transmission, say 1001001 received as 1011001, the receiver can check the parities of the three circles to find that the error is in circles $B$ and $C$ but not in $A$. This (7,4) Hamming code (the notation (7,4) means that every 4 information bits are encoded as a 7 bit sequence) can be generalized. For example, one may draw 4 intersecting spheres in a three-dimensional space to obtain a (15,11) Hamming code. Hamming has also proved that his coding method is optimum for single error correction.

(... to be continued)