Mathematical Excalibur

Volume 4, Number 2 March, 1998 - December, 1998

Olympiad Corner

Tenth Asian Pacific Mathematics Olympiad, March, 1998:

Each question is worth 7 points.

Problem 1. Let F be the set of all n-tuples $(A_1, A_2,\ldots, A_n)$ where each A_i, $i = 1, 2, \ldots, n$ is a subset of $\{1,2,\ldots,1998\}$. Let $|A|$ denote the number of elements of the set A. Find the number

$$\sum_{(A_1, A_2,\ldots,A_n)} \left|A_1 \cup A_2 \cup \ldots A_n\right|.$$

Problem 2. Show that for any positive integers a and b, $(36a+b)(a+36b)$ cannot be a power of 2.

Problem 3. Let a, b, c be positive real numbers. Prove that

$$\left(1+\frac{a}{b}\right)\left(1+\frac{b}{c}\right)\left(1+\frac{c}{a}\right) \ge 2\left(1+\frac{a+b+c}{\sqrt[3]{abc}}\right)$$

Problem 4. Let ABC be a triangle and D the foot of the altitude from A. Let E and F be on a line passing through D such that AE is perpendicular to BE, AF is perpendicular to CF, and E and F are different from D. Let M and N be the midpoints of the line segments BC and EF, respectively. Prove that AN is perpendicular to NM.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math Dept, Catherine NG, EEE Dept, HKUST and Tam Siu Lung for general assistance.

On-line: http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is December 31, 1998.

For individual subscription for the three remaining issues for the 98-99 academic year, send us three stamped self-addressed envelopes. Send all correspondence to:

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A Taste of Topology

Wing-Sum Chan

Beauty is the first test: there is no permanent place in the world for ugly mathematics.

(G. H. Hardy)

In topology, there are many abstractions of geometrical ideas, such as continuity and closeness. 'Topology' is derived from the Greek words $\tau o\pi o\sigma$, a place and $\lambda o\gamma o\sigma$, a discourse. It was introduced in 1847 by Johann Benedict Listing (1808-1882), who was a student of Carl Friedrich Gauss (1777-1855). In the early days, people called it *analysis situs*, that is, analysis of position. Rubber-sheet geometry is a rather descritpive term to say what it is. (Just think of properties of objects drawn on a sheet of rubber which are not changed when the sheet is being distorted.) Hence, topologists could not distinguish a triangle from a rectangle and they may even consider a basketball as a ping-pong ball.

Topologists consider two objects to be the same (homeomorphic) if one can be continuously deformed to look like the other. Continuous deformations include bending, stretching and squashing without gluing or tearing points.

Example 1. The following are homeomorphic: (See Figure 1.)

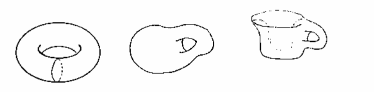


Fig 1.

Example 2. The following are non-homeomorphic: (See Figure 2.)

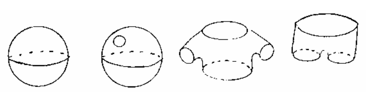


Fig 2.

In practise, continuous deformations may not be easy to carry out. In fact, there is a simple method to see two objects are non-homeomorphic, by seeking their Poincaré-Euler characteristics, (in short, Euler numbers). In order to see what the Euler number is, we need to introduce the concept of subdivision on an n-manifold (here $n \le 2$ throughout). (An n-manifold is roughly an n dimensional object in which each point has a neighborhood homeomorphic to an open interval (if $n = 1$) or an open disk (if $n = 2$). For example, a circle is a 1-manifold and a sphere is a 2-manifold.)

Basically, we start with an n manifold and subdividing it into a finite number of vertices, edges and faces. A vertex is a point. An edge is a curve with endpoints that are vertices. A face is a region with boundary that are edges.

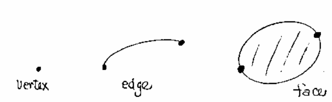


Fig 3

Here are typical pictures of vertex, edge and face, (see Figure 3.)

The Euler number (χ) of a compact (loosely speaking, bounded) 1-manifold is defined to be the number of vertices(v) minus the number of edges(e), and for a compact 2-manifold (surface), it is defined to be the number of vertices(v) minus the number of edges (e) plus the number of

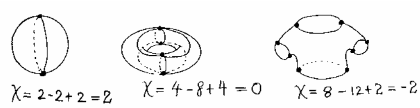


Fig 4

faces (f) (see Figure 4.) The following theorem is a test to distinguish non-homeomorphic objects.

Theorem 1. *If two n-manifolds are homeomorphic, then they have the same Euler number.*

So figure 4 and theorem 1 imply the sphere and the torus are not homeomorphic, i.e. the sphere cannot be continuously deformed to look like the torus and vice versa.

Here are two terms we need before we can state the next theorem. A connected manifold is one where any two points on the manifold can be connected by a curve on the manifold. The manifold is orientable if it has 2 sides, an inside and an outside.

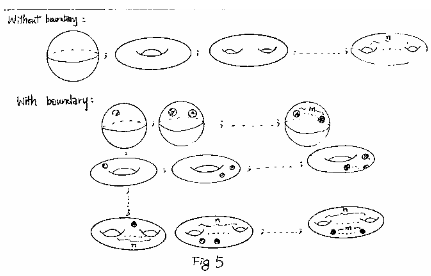
Theorem 2. *Two connected orientable n-manifolds* ($n \le 2$) *with the same number of boundary components are homeomorphic if and only if they have the same Euler number.*

Here are some important results that tell us the general pictures of one and two manifolds.

Classification I. *Any connected compact one-manifold is either homeomorphic to an open interval or a circle.*

Classification II. *Any connected, orientable and compact two-manifolds is homeomorphic to one of the followings:* (see Figure 5.)

Finally, we mention a famous open problem (the Poincaré conjecture), which is to show that every compact, simply connected three-manifold is homeomorphic to a three-sphere, where simply connected means any circle on the manifold can be

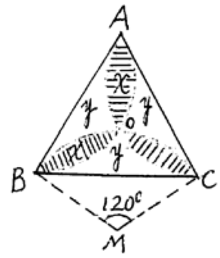


shrunk to a point on the manifold.

巧列一次方程組　妙解陰影面積題

于志洪
江蘇泰州橡膠總廠中學

有一類關於求陰影部分面積的問題，我們可根據題意適當設元，通過一次方程組求得結果。這種數形結合，將幾何面積問題轉化爲解一次方程組代數問題的方法，由於方法新穎、思路清晰，因而頗受師生重視。現舉三例分析說明如下：



一．列二元一次方程組求陰影面積

例一：如上圖，O 爲正三角形 ABC 的中心，$AB = 8\sqrt{3}\text{cm}$，則 $\overset{\frown}{AOB}$、$\overset{\frown}{BOC}$、$\overset{\frown}{COA}$ 所圍成的陰影部分的面積是____ cm^2。(1996 年陝西省中考題)

分析：上圖中含有形狀不同的兩類圖形，分別爲 x 和 y，由圓形特徵得知，2 個 x 和 1 個 y 組成一個圓心角爲 $120°$ 的弓形，而 3 個 x 和 3 個 y 組成一個正三角形 ABC。由於正三角形 ABC 的高 $= \frac{\sqrt{3}}{2} \times 8\sqrt{3} = 12$，又 O 爲正三角形 ABC 的中心，故 $BO = \frac{2}{3} \times 12 = 9 = MB$。

$$\therefore 2x + y = S_{扇形MBC} - S_{\Delta MBC}$$

$$= \frac{120\pi \cdot 8^2}{360} - \frac{1}{2} \times 8\sqrt{3} \times 4 = \frac{64\pi}{3} - 16\sqrt{3}$$

$$= \frac{1}{2} \times 8\sqrt{3} \times 12 = 48\sqrt{3}$$

解下列方程組

$$\begin{cases} 2x + y = \dfrac{64\pi}{3} - 16\sqrt{3} & \cdots\cdots\cdots(1) \\ 3x + 3y = 48\sqrt{3} & \cdots\cdots\cdots(2) \end{cases}$$

，得 $3x = 64\pi - 93\sqrt{3}$。這就是所求陰影部分的面積。

二．列三元一次方程組求陰影面積

例 2：如下圖，在正方形 $ABCD$ 中，有一個以正方形的中心爲圓心，以邊長一半爲半徑的圓。另分別以 A、B、C、D 爲圓心，以邊長一半爲半徑畫四條弧。若正方形的邊長爲 $2a$，求所圍成的陰影部分的面積。（1997 年泰州市中考模擬題）

分析：圖中含有形狀不同三類圖形，分別爲 x、y、z。由圖形特徵得知：4 個 x 和 1 個 y 組成一個圓；1 個 x 和 1 個 z 組成一個以 a 爲半徑、圓心角爲直角的的扇形；4 個 x、4 個 z 和 1 個 y 組成一個正方形。

故此，可列出方程組

$$\begin{cases} 4x + y = \pi a^2 & \cdots\cdots\cdots(1) \\ x + z = \frac{1}{4}\pi a^2 & \cdots\cdots\cdots(2) \\ 4x + y + 4z = 4a^2 & \cdots\cdots\cdots(3) \end{cases}$$

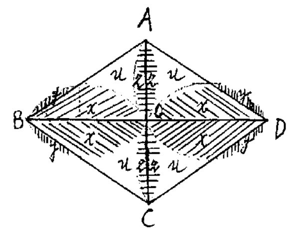
$(3)-(1)$ 得 $z = \frac{1}{4}(4-\pi)a^2$

再代入 (2) 得 $x = (\frac{\pi}{2} - 1)a^2$。

$\therefore S_{陰影} = 4x = (2\pi - 4)a^2$。

三．列四元一次方程組求陰影面積

例 3：如左圖，菱形 $ABCD$ 的兩條對角線長分別爲 a、b，分別以每邊爲直徑向形內作半圓。求 4 條半圓弧圍成的花瓣形面積（陰影部分的面積）。（人教版九年義務教材初中《幾何》第三冊 P. 212）



分析：圖中含有形狀不同的四類圖形，分別爲 x、y、z、u，則由圖形特徵得知：$2x$、$2y$、z、u 組成一個以邊長爲直徑的半圓；x、z、u 組成直角三角形 BOC。

解：設 x、y、z、u 如圖所示，則依題意得

$$\begin{cases} x + z + u = \dfrac{1}{2} \cdot \dfrac{a}{2} \cdot \dfrac{b}{2} & \cdots(1) \\ 2x + 2z + y + u = \dfrac{1}{2}\pi\left(\dfrac{1}{2}\sqrt{\dfrac{a^2+b^2}{4}}\right)^2 & \cdots(2) \end{cases}$$

$(2)-(1)$ 再乘 4 得

（續於第四頁）

Problem Corner

We welcome readers to submit solutions to the problems posed below for publication consideration. Solutions should be preceeded by the solver's name, home address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is *Dec 31, 1998.*

Problem 76. Find all positive integers N such that in base 10, the digits of $9N$ is the reverse of the digits of N and N has at most one digit equal 0. (*Source:* 1977 unused IMO problem proposed by Romania)

Problem 77. Show that if ΔABC satisfies

$$\frac{\sin^2 A+\sin^2 B+\sin^2 C}{\cos^2 A+\cos^2 B+\cos^2 C}=2,$$

then it must be a right triangle. (*Source:* 1967 unused IMO problem proposed by Poland)

Problem 78. If $c_1, c_2, \ldots, c_n$ ($n \geq 2$) are real numbers such that

$$(n-1)(c_1^2+c_2^2+\ldots+c_n^2) = (c_1+c_2+\ldots+c_n)^2,$$

show that either all of them are nonnegative or all of them are nonpositive. (*Source:* 1977 unused IMO problem proposed by Czechoslovakia)

Problem 79. Which regular polygons can be obtained (and how) by cutting a cube with a plane? (*Source:* 1967 unused IMO problem proposed by Italy)

Problem 80. Is it possible to cover a plane with (infinitely many) circles in such a way that exactly 1998 circles pass through each point? (*Source:* Spring 1988 Tournament of the Towns Problem)

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Solutions

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Problem 71. Find all real solutions of the system

$$x+\log(x+\sqrt{x^2+1})=y,$$
$$y+\log(y+\sqrt{y^2+1})=z,$$
$$z+\log(z+\sqrt{z^2+1})=x.$$

(*Source:* 1995 Israel Math Olympiad)

Solution: CHOI Fun Ieng (Pooi To Middle School (Macau), Form 5).

If $x < 0$, then $0 < x+\sqrt{x^2+1} < 1$. So $\log(x+\sqrt{x^2+1}) < 0$, which implies $y < x < 0$. Similarly, we get $z < y < 0$ and $x < z < 0$, yielding the contradiction $x < z < y < x$. If $x > 0$, then $x+\sqrt{x^2+1} > 1$. So $\log(x+\sqrt{x^2+1}) > 0$, which implies $y > x > 0$. Similarly, we get $z > y > 0$ and $x > z > 0$, yielding the contradiction $x > z > y > x$. If $x = 0$, then $x = y = z = 0$ is the only solution.

Other commended solvers: **AU Cheuk Yin** (Ming Kei College, Form 5). **CHEUNG Kwok Koon** (HKUST), **CHING Wai Hung** (STFA Leung Kau Kui College, Form 6), **HO Chung Yu** (Ming Kei College, Form 6), **KEE Wing Tao Wilton** (PLK Centenary Li Shiu Chung Memorial College, Form 6), **KU Wah Kwan** (Heep Woh College, Form 7), **KWOK Chi Hang** (Valtorta College, Form 6), **LAM Yee** (Valtorta College, Form 6), **LAW Ka Ho** (Queen Elizabeth School, Form 5), **Gary NG Ka Wing** (STFA Leung Kau Kui College, Form 5), **TAM Siu Lung** (Queen Elizabeth School, Form 5), **WONG Chi Man** (Valtoria College, Form 3) and **WONG Hau Lun** (STFA Leung Kau Kui College, Form 6).

Problem 72. Is it possible to write the numbers 1,2,…,121 in an 11x11 table so that any two consecutive numbers be written in cells with a common side and all perfect squares lie in a single column? (*Source:* 1995 Russian Math Olympiad)

Solution: Gary NG Ka Wing (STFA Leung Kau Kui College, Form 5).

Suppose such a table exists. The table would be divided into 2 parts by the single column of perfect squares, with one side $11n$ $(0 \le n \le 5)$ cells and the other side $110 - 11n$ cells. Note that numbers between 2 successive perfect squares, say a^2, $(a+1)^2$, lie on one side since they cannot cross over the perfect square column, and those between $(a+1)^2$, $(a+2)^2$ lie on opposite side. Now the number of integers (strictly) between 1, 4, 9, 16, …, 100, 121 is 2, 4, 6, 8, …, 20, respectively. So one side has 2 + 6 + 10 + 14 + 18 = 50 numbers while the other side has 4 + 8 + 12 + 16 + 20 = 60 numbers. Both 50 and 60 are not multiple of 11, a contradiction.

Other commended solvers: **CHEUNG Kwok Koon** (HKUST), **HO Chung Yu** (Ming Kei College, Form 6), **LAI Chi Fung Brian** (Queen Elizabeth School, Form 4), **LAW Ka Ho** (Queen Elizabeth School, Form 5), **TAM Siu Lung** (Queen Elizabeth School, Form 5), **WONG Hau Lun** (STFA Leung Kau Kui College, Form 6) and **WONG Shu Fai** (Valtorta College, Form 6).

Problem 73. Prove that if a and b are rational numbers satisfying the equation $a^5+b^5=2a^2b^2$, then $1-ab$ is the square of a rational number. (*Source:* 26th British Math Olympiad)

Solution: CHAN Wing Sum (City U).

If $b=0$, then $1-ab=1^2$. If $b\neq 0$, then $a^6+ab^5=2a^3b^2$. So $a^6-2a^3b^2+b^4 = b^4-ab^5 = b^4(1-ab)$. Therefore, $1-ab = (a^6-2a^3b^2+b^4)/b^4$ is the square of the rational number $(a^3-b^2)/b^2$.

Other recommended solvers: **CHING Wai Hung** (STFA Leung Kau Kui College, Form 6), **CHOI Fun Ieng** (Pooi To Middle School (Macau), Form 5), **KU Wah Kwan** (Heep Woh College, Form 7) and **Gary NG Ka Wing** (STFA Leung Kau Kui College, Form 5).

Problem 74. Points A_2, B_2, C_2 are the midpoints of the altitudes AA_1, BB_1, CC_1 of acute triangle ABC, respectively. Find the sum of $\angle B_2A_1C_2$, $\angle C_2B_1A_2$ and $\angle A_2C_1B_2$. (*Source:* 1995 Russian Math Olympiad)

Solution: LAM Po Leung (Ming Kei College, Form 5)

Let A_3, B_3, C_3 be the midpoints of BC, CA, AB, respectively, and H be the orthocenter of ΔABC. Since C_3A_3 is parallel to AC, so $\angle HB_2A_3 = 90° = \angle HA_1A_3$, which implies H, B_2, A_3, A_1 are concyclic. So $\angle B_2A_1H = \angle B_2A_3H$. Since B_3A_3 is parallel to AB, so

$\angle HC_2A_3 = 90° = \angle HA_1A_3$, which implies H, C_2, A_3, A_1 are concyclic. So $\angle C_2A_1H = \angle C_2A_3H$. Then $\angle B_2A_1C_2 = \angle B_2A_1H + \angle C_2A_1H = \angle B_2A_3H + \angle C_2A_3H = \angle C_3A_3B_3 = \angle BAC$ (because $\Delta A_3B_3C_3$ is similar to ΔABC). Similarly, $\angle B_2C_1A_2 = \angle BCA$ and $\angle A_2B_1C_2 = \angle ABC$. Therefore, the sum of $\angle B_2A_1C_2$, $\angle C_2B_1A_2$, $\angle A_2B_1C_2$ is 180^o.

Other commended solvers: **HO Chung Yu** (Ming Kei College, Form 6).

Problem 75. Let $P(x)$ be any polynomial with integer coefficients such that $P(21) = 17$, $P(32) = -247$, $P(37) = 33$. Prove that if $P(N) = N + 51$ for some integer N, then $N = 26$. (*Source:* 23rd British Math Olympiad)

Solutions: HO Chung Yu (Ming Kei College, Form 6).

If $P(N) = N + 51$ for some integer N, then $P(x) – x – 51 = (x - N)Q(x)$ for some polynomial $Q(x)$ by the factor theorem. Note $Q(x)$ has integer coefficients because $P(x) – x – 51 = P(x) – P(N) – (x – N)$ is a sum of $a_i(x^i – N^i)$ terms (with a_i's integer). Since $Q(21)$ and $Q(37)$ are integers, $P(21) – 21 – 51 = -55$ is divisible by $21 – N$ and $P(3) – 37 – 51 = -55$ is divisible by $37 – N$ is 16, we must have $N = 26$ or 32. However, if $N = 32$, then we get $-247 = P(32) = 32 + 51$, a contradiction. Therefore $N = 26$.

Other commended solvers: **CHEUNG Kwok Koon** (HKUST), **KU Wah Kwan** (Heep Who College, Form 7), **TAM Siu Lung** (Queen Elizabeth School, Form 5) and **WONG Shu Fai** (Valtorta College, Form 6).

Olympiad Corner

(continued from page 1)

Problem 5. Determine the largest of all integers n with the property that n is divisible by all positive integers that are less than $\sqrt[3]{n}$.

（續第二頁）

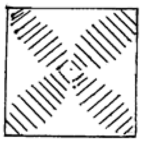
$$4(x+y+z)=\frac{\pi a^2+\pi b^2-4ab}{8}$$ 。

這就是所求陰影部分的面積。

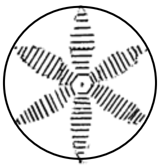
綜上所述可知：一般的陰影圖形大多是由多種規則圖形組成的，所以利用方程式組解決這類問題時，首先要根據圖形的特徵（尤其是對稱性）把圖形分成幾類，用字母表示各類圖形的面積；其次要仔細觀察圖形的組成，分析圖形中各部分之間及各部分與整體圖形的關係，通過規則圖形面積公式列出方程組；最後解方程求出陰影面積。

附練習題

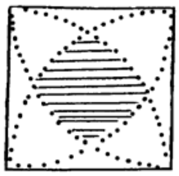
１．如右圖，已知一塊正方形的地瓷磚邊長爲 a，瓷磚上的圖案是以各邊爲直徑在正方形內畫圓所圍成的（陰影部分），那麼陰影部分的面積是多少？（1997 年寮夏回族自治區中考題）



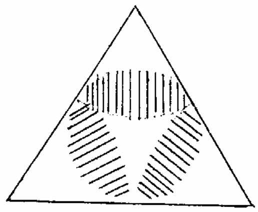
２．如右圖，已知圓形 O 的半徑爲 R，求圖中陰影部分的面積。（1998 年泰州市中考模擬題）



３．如右圖，正方形的邊長爲 a，分別以正方形的四個頂點爲圓心，邊長爲半徑，在正方形內畫弧，那麼這四條弧所圍成的陰影部分的面積是多少？（1994 年安徽省中考題）



４．如右圖，圓 O 內切於邊長爲 a 的正三角形，分別以三角形的三頂點爲圓心，$\frac{a}{2}$ 爲半徑畫弧相交成圓中所示的陰影，求陰影部分的面積。（1996 年泰州市中考模擬題）



參考資料

1．《陰影部分面積的幾種解法》
《初中生數學園地》
安義人(華南師大主辦) 1997 年 3 月
2．《列一次方程組解陰影面積題》
《中小學數學》
于志洪(中國教育學會主辦)1997年11月

練習題答案:

1. $(\frac{\pi}{2}-1)a^2$
2. $2\pi R^2-3\sqrt{3}R^2$
3. $(1-\sqrt{3}+\frac{\pi}{3})a^2$
4. $\frac{5\pi-6\sqrt{3}}{24}a^2$

(Hong Kong team to IMO 98: (from left to right) Lau Wai Tong (Deputy Leader), Law Ka Ho, Chan Kin Hang, Choi Ming Cheung, Lau Lap Ming, Cheung Pok Man, Leung Wing Chung, Liu Kam Moon (Leader).)