Mathematical Excalibur

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Olympiad Corner

39th International Mathematical Olympiad, July 1998:

Each problem is worth 7 points.

Problem 1. In the convex quadrilateral *ABCD*, the diagonals *AC* and *BD* are perpendicular and the opposite sides *AB* and *DC* are not parallel. Suppose that the point *P*, where the perpendicular bisectors of *AB* and *DC* meet, is inside *ABCD*. Prove that *ABCD* is a cyclic quadrilateral if and only if the triangles *ABP* and *CDP* have equal areas.

Problem 2. In a competition, there are *a* contestants and *b* judges, where $b \geq 3$ is an odd integer. Each judge rates each contestant as either "pass" or "fail". Suppose *k* is a number such that, for any two judges, their ratings coincide for at most *k* contestants. Prove that

$$\frac{k}{a} \geq \frac{b-1}{2b}.$$

Problem 3. For any positive integer *n*, let *d(n)* denote the number of positive divisions of *n* (including 1 and *n* itself).

(continued on page 4)

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The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is April 30, 1999.

For individual subscription for the two remaining issues for the 98-99 academic year, send us two stamped self-addressed envelopes. Send all correspondence to:

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Rearrangement Inequality

Kin-Yin Li

The rearrangement inequality (or the permutation inequality) is an elementary inequality and at the same time a powerful inequality. Its statement is as follow. Suppose $a_1 \le a_2 \le \cdots \le a_n$ and $b_1 \le b_2 \le \cdots \le b_n$. Let us call

$$A = a_1b_1 + a_2b_2 + \cdots + a_nb_n$$

the *ordered sum* of the numbers and

$$B = a_1b_n + a_2b_{n-1} + \cdots + a_nb_1$$

the *reverse sum* of the numbers. If $x_1, x_2, ..., x_n$ is a rearrangement (or permutation) of the numbers $b_1, b_2, ..., b_n$ and if we form the *mixed sum*

$$X = a_1x_1 + a_2x_2 + \cdots + a_nx_n,$$

then the rearrangement inequality asserts that $A \ge X \ge B$. In the case the a_i's are strictly increasing, then equality holds if and only if the b_i's are all equal.

We will look at $A \ge X$ first. The proof is by mathematical induction. The case $n = 1$ is clear. Suppose the case $n = k$ is true. Then for the case $n = k + 1$, let $b_{k+1} = x_i$ and $x_{k+1} = b_j$. Observe that $(a_{k+1} - a_i)(b_{k+1} - b_j) \ge 0$. We get

$$a_ib_j + a_{k+1}b_{k+1} \ge a_ib_{k+1} + a_{k+1}b_j.$$

So in *X*, we may switch x_i and x_{k+1} to get a possibly larger sum. After switching, we can apply the case $n = k$ to the first *k* terms to conclude that $A \ge X$. The inequality $X \ge B$ follows from $A \ge X$ using $-b_n \le -b_{n-1} \le \cdots \le -b_1$ in place of $b_1 \le b_2 \le \cdots \le b_n$.

Now we will give some examples.

Example 1. (*Chebysev's Inequality*) Let *A* and *B* be as in the rearrangement inequality, then

$$A \ge \frac{(a_1 + \cdots + a_n)(b_1 + \cdots + b_n)}{n} \ge B.$$

Proof. Cyclically rotating the b_i's, we get *n* mixed sums

$$a_1b_1 + a_2b_2 + \cdots + a_nb_n,$$
$$a_1b_2 + a_2b_3 + \cdots + a_nb_1,$$
$$...,$$
$$a_1b_n + a_2b_1 + \cdots + a_nb_{n-1}.$$

By the re-arrangement inequality, each of these is between *A* and *B*, so their average is also between *A* and *B*. This average is just the expression given in the middle of Chebysev's inequality.

Example 2. (*RMS-AM-GM-HM In-equality*) Let $c_1, c_2, ..., c_n \ge 0$. The *root mean square* (RMS) of these numbers is $[(c_1^2 + \cdots + c_n^2)/n]^{1/2}$, the *arithmetic mean* (AM) is $(c_1 + c_2 + \cdots + c_n)/n$ and the *geometric mean* (GM) is $(c_1c_2\cdots c_n)^{1/n}$. We have $RMS \ge AM \ge GM$. If the numbers are positive, then the *harmonic mean* (HM) is $n/[(1/c_1) + \cdots + (1/c_n)]$. We have $GM \ge HM$.

Proof. Setting $a_i = b_i = c_i$ in the left half of Chebysev's inequality, we easily get $RMS \ge AM$. Next we will show $AM \ge GM$. The case $GM = 0$ is clear. So suppose $GM > 0$. Let $a_1 = c_1/GM$, $a_2 = c_1c_2/GM^2$, ..., $a_n = c_1c_2\cdots c_n/GM^n = 1$ and $b_i = 1/a_{n-i+1}$ for $i = 1, 2, ... n$. (Note the a_i's may not be increasing, but the b_i's will be in the reverse order as the a_i's). So the mixed sum

$$a_1b_1 + a_2b_n + \cdots + a_nb_2 =$$
$$c_1/GM + c_2/GM + \cdots + c_n/GM$$

is greater than or equal to the reverse sum $a_1b_n + \cdots + a_nb_1 = n$. The AM-GM inequality follows easily. Finally $GM \ge HM$ follows by applying $AM \ge GM$ to the numbers $1/c_1, ..., 1/c_n$.

Example 3. (1974 *USA Math Olympiad*) If *a, b, c* > 0, then prove that

$$a^a b^b c^c \ge (abc)^{(a+b+c)/3}.$$

Solution. By symmetry, we may assume $a \le b \le c$, then $\ln a \le \ln b \le \ln c$. By Chebysev's inequality,

$$a\ln a + b\ln b + c\ln c$$
$$\ge \frac{(a+b+c)(\ln a + \ln b + \ln c)}{3}.$$

The desired inequality follows from exponentiation.

Example 4. *(1978 IMO)* Let c_1, c_2, ..., c_n be distinct positive integers. Prove that

$$c_1 + \frac{c_2}{4} + \cdots + \frac{c_n}{n^2} \ge 1 + \frac{1}{2} + \cdots + \frac{1}{n}.$$

Solution. Let a_1, a_2, ..., a_n be the c_i's arranged in increasing order. Since a_i's are distinct positive integers, $a_i \ge i$. Since $1 > 1/4 > ... > 1/n^2$, by the re-arrangement inequality,

$$c_1 + \frac{c_2}{4} + \cdots + \frac{c_n}{n^2}$$
$$\ge a_1 + \frac{a_2}{4} + \cdots + \frac{a_n}{n^2}$$
$$\ge 1 + \frac{1}{2} + \cdots + \frac{1}{n}.$$

Example 5. *(1995 IMO)* Let *a, b, c* > 0 and *abc* = 1. Prove that

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \ge \frac{3}{2}.$$

Solution. (HO Wing Yip, Hong Kong Team Member) Let *x* = *bc* = *1/a, y* = *ca* = *1/b, z* = *ab* = *1/c*. The required inequality is equivalent to

$$\frac{x^2}{z+y} + \frac{y^2}{x+z} + \frac{z^2}{y+x} \ge \frac{3}{2}.$$

By symmetry, we may assume $x \le y \le z$, then $x^2 \le y^2 \le z^2$ and $1/(z+y) \le 1/(x+z) \le 1/(y+x)$. The left side of the required inequality is just the ordered sum *A* of the numbers. By the rearrangement inequality,

$$A \ge \frac{x^2}{y+x} + \frac{y^2}{z+y} + \frac{z^2}{x+z},$$
$$A \ge \frac{x^2}{x+z} + \frac{y^2}{y+x} + \frac{z^2}{z+y}.$$

(continued on page 4)

Power of Points Respect to Circles

Kin-Yin Li

Intersecting Chords Theorem. *Let two lines through a point P not on a circle intersect the inside of the circle at chords AA' and BB', then PA × PA' = PB × PB'. (When P is outside the circle, the limiting case A = A' refers to PA tangent to the circle.)*

This theorem follows from the observation that triangles *ABP* and *A'B'P* are similar and the corresponding sides are in the same ratio. In the case *P* is inside the circle, the product *PA* × *PA*' can be determined by taking the case the chord *AA*' passes through *P* and the center *O*. This gives *PA* × *PA*' = $r^2 - d^2$, where *r* is the radius of the circle and *d* = *OP*. In the case *P* is outside the circle, the product *PA* × *PA*' can be determined by taking the limiting case *PA* is tangent to the circle. Then *PA* × *PA*' = $d^2 - r^2$.

The *power* of a point *P* with respect to a circle is the number $d^2 - r^2$ as mentioned above. (In case *P* is on the circle, we may define the power to be 0 for convenience.) For two circles C_1 and C_2 with different centers O_1 and O_2, the points whose power with respect to C_1 and C_2 are equal form a line perpendicular to line O_1O_2. (This can be shown by setting coordinates with line O_1O_2 as the *x*-axis.) This line is called the *radical axis* of the two circles. In the case of the three circles C_1, C_2, C_3 with noncollinear centers O_1, O_2, O_3, the three radical axes of the three pairs of circles intersect at a point called the *radical center* of the three circles. (This is because the intersection point of any two of these radical axes has equal power with respect to all three circles, hence it is on the third radical axis too.)

If two circles C_1 and C_2 intersect, their radical axis is the line through the intersection point(s) perpendicular to the line of the centers. (This is because the intersection point(s) have 0 power with respect to both circles, hence they are on the radical axis.) If the two circles do not intersect, their radical axis can be found by taking a third circle C_3 intersecting both C_1 and C_2. Let the radical axis of C_1, C_3 intersect the radical axis of C_2, C_3 at *P*. Then the radical axis of C_1, C_2 is the line through *P* perpendicular to the line of centers of C_1, C_2.

We will illustrate the usefulness of the intersecting chords theorem, the concepts of power of a point, radical axis and radical center in the following examples.

Example 1. (*1996 St. Petersburg City Math Olympiad*) Let *BD* be the angle bisector of angle *B* in triangle *ABC* with *D* on side *AC*. The circumcircle of triangle *BDC* meets *AB* at *E*, while the circumcircle of triangle *ABD* meets *BC* at *F*. Prove that *AE* = *CF*.

Solution. By the intersecting chords theorem, *AE* × *AB* = *AD* × *AC* and *CF* × *CB* = *CD* × *CA*, so *AE/CF* = (*AD/CD*)(*BC/AB*). However, *AB/CB* = *AD/CD* by the angle bisector theorem. So *AE* = *CF*.

Example 2. (*1997 USA Math Olympiad*) Let *ABC* be a triangle, and draw isosceles triangles *BCD, CAE, ABF* externally to *ABC*, with *BC, CA, AB* as their respective bases. Prove the lines through *A, B, C,* perpendicular to the lines *EF, FD, DE,* respectively, are concurrent.

Solution. Let C_1 be the circle with center *D* and radius *BD*, C_2 be the circle with center *E* and radius *CE*, and C_3 be the circle with center *F* and radius *AF*. The line through *A* perpendicular to *EF* is the radical axis of C_2, C_3, the line through *B* perpendicular to *FD* is the radical axis of C_3, C_1 and the line through *C* perpendicular to *DE* is the radical axis of C_1, C_2. These three lines concur at the radical center of the three circles.

Example 3. (*1985 IMO*) A circle with center *O* passes through vertices *A* and *C* of triangle *ABC* and intersects side *AB* at *K* and side *BC* at *N*. Let the circumcircles of triangles *ABC* and *KBN* intersect at *B* and *M*. Prove that *OM* is perpendicular to *BM*.

(continued on page 4)

Problem Corner

We welcome readers to submit solutions to the problems posed below for publication consideration. Solutions should be preceeded by the solver's name, home address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is *April 30, 1999.*

Problem 81. Show, with proof, how to dissect a square into at most five pieces in such a way that the pieces can be reassembled to form three squares no two of which have the same area. (*Source*: 1996 Irish Mathematical Olympiad)

Problem 82. Show that if n is an integer greater than 1, then n^4+4^n cannot be a prime number. (*Source:* 1977 Jozsef Kurschak Competition in Hungary)

Problem 83. Given an alphabet with three letters *a, b, c*, find the number of words of *n* letters which contain an even number of *a*'s. (*Source*: 1996 Italian Mathematical Olympiad)

Problem 84. Let M and N be the midpoints of sides AB and AC of ΔABC, respectively. Draw an arbitrary line through A. Let Q and R be the feet of the perpendiculars from B and C to this line, respectively. Find the locus of the intersection P of the lines QM and RN as the line rotates about A.

Problem 85. Starting at (1, 1), a stone is moved in the coordinate plane according to the following rules:

(a) From any point (a, b), the stone can be moved to $(2a, b)$ or $(a, 2b)$.

(b) From any point (a, b), the stone can be moved to $(a - b, b)$ if $a > b$, or to $(a, b - a)$ if $a < b$.

For which positive integers *x, y*, can the stone be moved to (x, y)? (*Source:* 1996 German Mathematical Olympiad)

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Solutions

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Problem 76. Find all positive integers N such that in base 10, the digits of $9N$ is the reverse of the digits of N and N has at most one digit equal 0. (*Source:* 1977 unused IMO problem proposed by Romania)

Solution. LAW Ka Ho (Queen Elizabeth School, Form 6) and **Gary NG Ka Wing** (STFA Leung Kau Kui College, Form 6).

Let $[a_1a_2 \ldots a_n]$ denote N in base 10 with $a_1 \neq 0$. Since $9N$ has the same number of digits as N, we get $a_1 = 1$ and $a_n = 9$. Since $9 \times 19 \neq 91$, $n > 2$. Now $9[a_2 \ldots a_{n-1}] + 8 = [a_{n-1} \ldots a_2]$. Again from the number of digits of both sides, we get $a_2 \leq 1$. The case $a_2 = 1$ implies $9a_{n-1} + 8$ ends in a_2 and so $a_{n-1} = 7$, which is not possible because $9[1 \ldots 7] + 8 > [7 \ldots 1]$. So $a_2 = 0$ and $a_{n-1} = 8$. Indeed, 1089 is a solution by direct checking. For $n > 4$, we now get $9[a_3 \ldots a_{n-2}] + 8 = [8\, a_{n-2} \ldots a_3]$. Then $a_3 \geq 8$. Since $9a_{n-2} + 8$ ends in a_3, $a_3 = 8$ will imply $a_{n-2} = 0$, causing another 0 digit. So $a_3 = 9$ and $a_{n-2} = 9$. Indeed, 10989 and 109989 are solutions by direct checking. For $n > 6$, we again get $9[a_4 \ldots a_{n-3}] + 8 = [8\, a_{n-3} \ldots a_4]$. So $a_4 = \ldots = a_{n-3} = 9$. Finally direct checking shows these numbers are solutions.

Other recommended solvers: **CHAN Siu Man** (Ming Kei College, Form 6), **CHING Wai Hung** (STFA Leung Kau Kui College, Form 7), **FANG Wai Tong Louis** (St. Mark's School, Form 6), **KEE Wing Tao Wilton** (PLK Centenary Li Shiu Chung Memorial College, Form 7), **KWOK Chi Hang** (Valtorta College, Form 7), **TAM Siu Lung** (Queen Elizabeth School, Form 6**), WONG Chi Man** (Valtorta College, Form 4), **WONG Hau Lun** (STFA Leung Kau Kui College, Form 7) and **WONG Shu Fai** (Valtorta College, Form 7).

Problem 77. Show that if $\Delta\, ABC$ satisfies

$$\frac{\sin^2 A+\sin^2 B+\sin^2 C}{\cos^2 A+\cos^2 B+\cos^2 C}=2,$$

then it must be a right triangle. (*Source*: 1967 unused IMO problem proposed by Poland)

Solution. (All solutions received are essentially the same.)

Using $\sin^2 x = (1-\cos 2x)/2$ and $\cos^2 x = (1 + \cos 2x)/2$, the equation is equivalent to

$$\cos 2A + \cos 2B + \cos 2C + 1 = 0.$$

This yields $\cos(A + B)\cos(A - B) + \cos^2 C = 0$. Since $\cos(A + B) = -\cos C$, we get $\cos C\,(\cos(A - B) + \cos(A + B)) = 0$. This simplifies to $\cos C \cos A \cos B = 0$. So one of the angles A, B, C is 90^0.

Solvers: **CHAN Lai Yin, CHAN Man Wai**, **CHAN Siu Man**, **CHAN Suen On**, **CHEUNG Kin Ho**, **CHING Wai Hung**, **CHOI Ching Yu**, **CHOI Fun Ieng**, **CHOI Yuet Kei**, **FANG Wai Tong Louis**, **FUNG Siu Piu, HUNG Kit**, **KEE Wing Tao Wilton**, **KO Tsz Wan**, **KWOK Chi Hang, LAM Tung Man**, **LAM Wai Hung**, **LAM Yee**, **LAW Ka Ho**, **LI Ka Ho**, **LING Hoi Sheung**, **LOK Chan Fai, LUNG Chun Yan**, **MAK Wing Hang**, **MARK Kai Pan**, **Gary NG Ka Wing**, **OR Kin**, **TAM Kwok Cheong**, **TAM Siu Lung**, **TSANG Kam Wing**, **TSANG Pui Man**, **TSANG Wing Kei**, **WONG Chi Man**, **WONG Hau Lun**, **YIM Ka Wing** and **YU Tin Wai**.

Problem 78. If $c_1, c_2, \ldots, c_n\ (n \geq 2)$ are real numbers such that

$$(n-1)(c_1^2+c_2^2+\cdots+c_n^2) = (c_1+c_2+\cdots+c_n)^2,$$

show that either all of them are non-negative or all of them are non-positive. (*Source*: 1977 unused IMO problem proposed by Czechoslovakia)

Solution. CHOY Ting Pong (Ming Kei College, Form 6).

Assume the conclusion is false. Then there are at lease one negative and one positive numbers, say $c_1 \leq c_2 \leq \cdots \leq c_k \leq 0 < c_{k+1} \leq \cdots \leq c_n$ with $1 \leq k < n$, satisfying the condition. Let $w = c_1 + \ldots + c_k$, $x = c_{k+1}+\cdots+c_n$, $y = c_1^2+\cdots+c_k^2$ and $z = c_{k+1}^2+\cdots+c_n^2$. Expanding w^2 and x^2 and applying the inequality $a^2+b^2 \geq 2ab$, we get $ky \geq w^2$ and $(n - k)\, z \geq x^2$. So

$$(w+x)^2 = (n-1)(y+z) \geq ky + (n-k)z \geq w^2+x^2.$$

Simiplifying, we get $wx \geq 0$, contradicting $w < 0 < x$.

Other commended solvers: **CHAN Siu Man** (Ming Kei College, Form 6), **FANG Wai Tong Louis** (St. Mark's School, Form 6), **KEE Wing Tao Wilton** (PLK Centenary Li Shiu Chung Memorial College, Form 7), **Gary NG Ka Wing** (STFA Leung Kau Kui College, Form 6), **TAM Siu Lung** (Queen Elizabeth School, Form 6), **WONG Hau Lun** (STFA Leung Kau Kui College, Form 7) and **YEUNG Kam Wah** (Valtorta College, Form 7).

Problem 79. Which regular polygons can be obtained (and how) by cutting a cube with a plane? (*Source*: 1967 unused IMO problem proposed by Italy)

Solution. FANG Wai Tong Louis (St. Mark's school, Form 6), **KEE Wing Tao** (PLK Centenary Li Shiu Chung Memorial School, Form 7), **TAM Siu Lung** (Queen Elizabeth School, Form 6) and **YEUNG Kam Wah** (Valtorta College, Form 7).

Observe that if two sides of a polygon is on a face of the cube, then the whole polygon lies on the face. Since a cube has 6 faces, only regular polygon with 3, 4, 5 or 6 sides are possible. Let the vertices of the bottom face of the cube be *A, B, C, D* and the vertices on the top face be *A', B', C', D'* with *A'* on top of *A, B'* on top of *B* and so on. Then the plane through *A, B', D'* cuts an equilateral triangle. The perpendicular bisecting plane to edge *AA'* cuts a square. The plane through the mid-points of edges *AB, BC, CC', C'D', D'A', A'A* cuts a regular hexagon. Finally, a regular pentagon is impossible, otherwise the five sides will be on five faces of the cube implying two of the sides are on parallel planes, but no two sides of a regular pentagon are parallel.

Problem 80. Is it possible to cover a plane with (infinitely many) circles in such a way that exactly 1998 circles pass through each point? (*Source*: Spring 1988 Tournament of the Towns Problem)

Solution. Since no solution is received, we will present the modified solution of Professor Andy Liu (University of Alberta, Canada) to the problem.

First we solve the simpler problem where 1998 is replaced by 2. Consider the lines $y = k$, where k is an integer, on the coordinate plane. Consider every circle of diameter 1 tangent to a pair of these lines. Every point (x, y) lies on exactly two of these circles. (If y is an integer, then (x, y) lies on one circle on top of it and one below it. If y is not an integer, then (x, y) lies on the right half of one circle and on the left half of another.) Now for the case 1998, repeat the argument above 998 times (using lines of the form $y = k + (j/999)$ in the j-th time, $j = 1, 2, ..., 998$.)

Olympiad Corner

(continued from page 1)

Determine all positive integers k such that

$$\frac{d(n^2)}{d(n)} = k$$

for some n.

Problem 4. Determine all pairs (a, b) of positive integers such that $ab^2 + b + 7$ divides $a^2b + a + b$.

Problem 5. Let I be the incentre of triangle ABC. Let the incircle of ABC touch the sides BC, CA and AB at K, L and M, respectively. The line through B parallel to MK meets the lines LM and LK at R and S, respectively. Prove that $\angle RIS$ is acute.

Problem 6. Consider all functions f from the set $\mathbb{N}$ of all positive integers into itself satisfying

$$f(t^2 f(s)) = s(f(t))^2,$$

for all s and t in $\mathbb{N}$. Determine the least possible value of $f(1998)$.

Rearrangement Inequality

(continued from page 2)

So

$$A \geq \frac{1}{2}\left(\frac{y^2+x^2}{y+x} + \frac{z^2+y^2}{z+y} + \frac{x^2+z^2}{x+z}\right).$$

Applying the RMS-AM inequality $r^2 + s^2 \geq (r+s)^2/2$, the right side is at least $(x+y+z)/2$, which is at least $3(xyz)^{1/3}/2 = 3/2$ by the AM-GM inequality.

Power of Points Respect to Circles

(continued from page 2)

Solution. For the three circles mentioned, the radical axes of the three pairs are lines AC, KN and BM. (The centers are noncollinear because two of them are on the perpendicular bisector of AC, but not the third.) So the axes will concur at the radical center P. Since $\angle PMN = \angle BKN = \angle NCA$, it follows that P, M, N, C are concyclic. By power of a point, $BM \times BP = BN \times BC = BO^2 - r^2$ and $PM \times PB = PN \times PK = PO^2 - r^2$, where r is the radius of the circle through A, C, N, K. Then $PO^2 - BO^2 = BP(PM - BM) = PM^2 - BM^2$. This implies OM is perpendicular to BM. (See remarks below.)

Remarks. By coordinate geometry, it can be shown that the locus of points X such that $PO^2 - BO^2 = PX^2 - BX^2$ is the line through O perpendicular to line BP. This is a useful fact.

Example 4. (*1997 Chinese Math Olympiad*) Let quadrilateral $ABCD$ be inscribed in a circle. Suppose lines AB and DC intersect at P and lines AD and BC intersect at Q. From Q, construct the tangents QE and QF to the circle, where E and F are the points of tangency. Prove that P, E, F are collinear.

Solution. Let M be a point on PQ such that $\angle CMP = \angle ADC$. Then D, C, M, Q are concyclic and also, B, C, M, P are concyclic. Let r_1 be the radius of the circumcircle C_1 of $ABCD$ and O_1 be the center of C_1. By power of a point, $PO_1^2 - r_1^2 = PC \times PD = PM \times PQ$ and $QO_1^2 - r_1^2 = QC \times QB = QM \times PQ$. Then $PO_1^2 - QO_1^2 = (PM - QM)PQ = PM^2 - QM^2$, which implies $O_1M \perp PQ$. The circle C_2 with QO_1 as diameter passes through M, E, F and intersects C_1 at E, F. If r_2 is the radius of C_2 and O_2 is the center of C_2, then $PO_1^2 - r_1^2 = PM \times PQ = PO_2^2 - r_2^2$. So P lies on the radical axis of C_1, C_2, which is the line EF.