Problem 1. Determine all solutions \((x, y, z)\) of positive integers such that
\[(x + 1)^{y+1} + 1 = (x + 2)^{z+1}.
\]

Problem 2. Let \(a_1, a_2, \ldots, a_{1999}\) be a sequence of nonnegative integers such that for any integers \(i, j\), with \(i + j \leq 1999\),
\[a_i + a_j \leq a_{i+j} \leq a_i + a_j + 1.
\]
Prove that there exists a real number \(x\) such that \(a_n = \lfloor nx \rfloor\) for each \(n = 1, 2, \ldots, 1999\), where \(\lfloor nx \rfloor\) denotes the largest integer less than or equal to \(nx\).

Problem 3. There are 1999 people participating in an exhibition. Two of any 50 people do not know each other. Prove that there are at least 41 people, and each of them knows at most 1958 people.

(continued on page 4)
condition on the set implies \( P_i \) is on the perpendicular bisector of \( P_{i-1} \) \( P_{i+1} \). So \( P_{i-1} P_i = P_i P_{i+1} \). Considering the perpendicular bisector of \( P_{i-1} \) \( P_i \) \( P_{i+2} \), we see that \( \angle P_{i-1} P_i P_{i+1} = \angle P_i P_{i+1} P_{i+2} \). So the boundary of \( H \) is a regular polygon.

Next, there cannot be any point \( P \) inside the regular polygon. (To see this, assume such a \( P \) exists. Place it at the origin and the furthest point \( Q \) of \( S \) from \( P \) on the positive real axis. Since the origin \( P \) is in the interior of the convex polygon, not all the vertices can lie on or to the right of the \( y \)-axis. So there exists a vertex \( P_j \) to the left of the \( y \)-axis. Since the perpendicular bisector of \( PQ \) is an axis of symmetry, the mirror image of \( P_j \) will be a point in \( S \) further than \( Q \) from \( P \), a contradiction.) So \( S \) is the set of vertices of some regular polygon. Conversely, such a set clearly has the required property.

Next we look at the official solution, which is shorter and goes as follows: Suppose \( S = \{ X_1, \ldots, X_n \} \) is such a set. Consider the barycenter of \( S \), which is the point \( G \) such that
\[
\overrightarrow{OG} = \frac{OX_1 + \cdots + OX_n}{n}.
\]

Note the barycenter does not depend on the origin. To see this, suppose we get a point \( G' \) using another origin \( O' \), i.e. \( O'G' \) is the average of \( O'X_i \) for \( i = 1, \ldots, n \). Subtracting the two averages, we get \( \overrightarrow{OG} - O'G' = 0O' \). Adding \( O'G' \) to both sides, \( \overrightarrow{OG} = O'G' \), so \( G = G' \).

By the condition on \( S \), after reflection with respect to the perpendicular bisector of every segment \( X_iX_j \), the points of \( S \) are permuted only. So \( G \) is unchanged, which implies \( G \) is on every such perpendicular bisector. Hence, \( G \) is equidistant from all \( X_i \)'s. Therefore, the \( X_i \)'s are concyclic. For three consecutive points of \( S \), say \( X_i, X_j, X_k \), on the circle, considering the perpendicular bisector of segment \( X_iX_k \), we have \( X_iX_j = X_jX_k \). It follows that the points of \( S \) are the vertices of a regular polygon and the converse is clear.

Have you ever wondered why the volume of a sphere of radius \( r \) is given by the formula \( \frac{4}{3}\pi r^3 \)? The \( r^3 \) factor can be easily accepted because volume is a three dimensional measurement. The \( \pi \) factor is probably because the sphere is round. Why then is there \( \frac{4}{3} \) in the formula?

In school, most people told you it came from calculus. Then, how did people get the formula before calculus was invented? In particular, how did the early Egyptian or Greek geometers get it thousands of years ago?

Those who studied the history of mathematics will be able to tell us more of the discovery. Below we will look at one way of getting the formula, which may not be historically the first way, but it has another interesting application as we will see. First, let us introduce

**Cavalieri’s Principle**: Two objects having the same height and the same cross sectional area at each level must have the same volume.

To understand this, imagine the two objects are very large, like pyramids that are built by piling bricks one level on top of another. By definition, the volume of the objects are the numbers of 1x1x1 bricks used to build the objects. If at each level of the construction, the number of bricks used (which equals the cross sectional area numerically) is the same for the two objects, then the volume (which equals the total number of bricks used) would be the same for both objects.

To get the volume of a sphere, let us apply Cavalieri’s principle to a solid sphere of radius \( r \) and an object \( T \) made out from a solid right circular cylinder with height \( 2r \) and base radius \( r \) removing a pair of right circular cones with height \( r \) and base radius \( r \) having the center of the cylinder as the apex of each cone.

Both \( S \) and \( T \) have the same height \( 2r \). Now consider the cross sectional area of each at a level \( x \) units from the equatorial plane of \( S \) and \( T \). The cross section for \( S \) is a circular disk of radius \( \sqrt{r^2 - x^2} \) by Pythagoras’ theorem, which has area \( \pi(r^2 - x^2) \). The cross section for \( T \) is an annular ring of outer radius \( r \) and inner radius \( x \), which has the same area \( \pi r^2 - \pi x^2 \). By Cavalieri’s principle, \( S \) and \( T \) have the same volume. Since the volume of \( T \) is \( \pi r^2(2r) - 2\times\frac{1}{3}\pi r^2 r = \frac{4}{3}\pi r^3 \), so the volume of \( S \) is the same.

Cavalieri’s principle is not only useful in getting the volume of special solids, but it can also be used to get the area of special regions in a plane! Consider the region \( A \) bounded by the graph of \( y = x^2 \), the \( x \)-axis and the line \( x = c \) in the first quadrant.

The area of this region is less than the area of the triangle with vertices at \((0,0)\), \((c,0)\), \((c, c^2)\), which is \( \frac{1}{2}c^3 \). If you ask a little kid to guess the answer, you may get \( \frac{1}{3}c^3 \) since he knows \( \frac{1}{3} \leq \frac{1}{2} \). For those who know calculus, the answer is easily seen to be correct. How can one explain this without calculus?

(continued on page 4)
Problem Corner

We welcome readers to submit solutions to the problems posed below for publication consideration. Solutions should be preceded by the solver’s name, home address and school affiliation. Please send submissions to Dr. Kin Y. Li, Department of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon. The deadline for submitting solutions is March 4, 2000.

Problem 96. If every point in a plane is colored red or blue, show that there exists a rectangle all of its vertices are of the same color.

Problem 97. A group of boys and girls went to a restaurant where only big pizzas cut into 12 pieces were served. Every boy could eat 6 or 7 pieces and every girl 2 or 3 pieces. It turned out that 4 pizzas were not enough and that 5 pizzas were too many. How many boys and how many girls were there? (Source: 1999 National Math Olympiad in Slovenia)

Problem 98. Let ABC be a triangle with BC > CA > AB. Select points D on BC and E on the extension of AB such that BD = BE = AC. The circumcircle of BED intersects AC at point P and BP meets the circumcircle of ABC at point Q. Show that AQ + CQ = BP. (Source: 1998-99 Iranian Math Olympiad)

Problem 99. At Port Aventura there are 16 secret agents. Each agent is watching one or more other agents, but no two agents are both watching each other. Moreover, any 10 agents can be ordered so that the first is watching the second, the second is watching the third, etc., and the last is watching the first. Show that any 11 agents can also be so ordered. (Source: 1996 Spanish Math Olympiad)

Problem 100. The arithmetic mean of a number of pairwise distinct prime numbers equals 27. Determine the biggest prime that can occur among them. (Source: 1999 Czech and Slovak Math Olympiad)

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Solutions

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Problem 91. Solve the system of equations:

\[ \sqrt{3x}\left(1+ \frac{1}{x+y}\right) = 2 \]
\[ \sqrt{7y}\left(1- \frac{1}{x+y}\right) = 4\sqrt{2}. \]

(This is the corrected version of problem 86.)

Solution. (CHENG Kei Tsi, LEE Kar Wai, TANG Yat Fai) (La Salle College, Form 5), CHEUNG Yui Ho Yves (University of Toronto), HON Ching Wing (Pui Ching Middle School, Form 5), KU Hong Tung (Carmel Divine Grace Foundation Secondary College, Form 6), LAU Chung Ming Vincent (STFA Leung Kau Kui College, Form 5), LAW Siu Lun Jack (Ming Kei College, Form 5), KEVIN LEE (La Salle College, Form 4), LEUNG Wai Ying (Queen Elizabeth School, Form 5), MAK Hoi Kwan Calvin (Form 4), NG Chok Ming Lewis (STFA Leung Kau Kui College, Form 6), NG Ka Wing Gary (STFA Leung Kau Kui College, Form 7), NGAN Chung Wai Hubert (St. Paul’s Co-educational College, Form 7), SIU Tsz Hang (STFA Leung Kau Kui College, Form 4), TANG Kun Fu (Valtorta College, Form 5), WONG Chi Man (Valtorta College, Form 5) and WONG Chun Ho Terry (STFA Leung Kau Kui College, Form 5).

(All solutions were essentially the same.) Clearly, if \((x, y)\) is a solution, then \(x, y > 0\) and

\[ 1 + \frac{1}{x+y} = \frac{2}{\sqrt{3x}}, \]
\[ 1 - \frac{1}{x+y} = \frac{4\sqrt{2}}{\sqrt{7y}}. \]

Taking the difference of the squares of both equations, we get

\[ \frac{4}{x+y} = \frac{4}{\sqrt{3x}} - \frac{32}{\sqrt{7y}}. \]

Simplifying this, we get \(0 = 7y^2 - 38xy - 24x^2 = (7y+4x)(y-6x).\) Since \(x, y > 0, y = 6x.\) Substituting this into the first given equation, we get \(\sqrt{3x}\left(1+ \frac{1}{x+y}\right) = 2,\) which simplifies to \(7\sqrt{x} - 14\sqrt{x} + \sqrt{3} = 0.\) By the quadratic formula, \(\sqrt{x} = (7\pm 2\sqrt{7})/7\). Then \(x = (11 \pm 4\sqrt{7})/21\) and \(y = 6x = (22 \pm 8\sqrt{7})/7.\)

Direct checking shows these are solutions.

Comments: An alternative way to get the answers is to substitute \(u = \sqrt{x}, v = \sqrt{y}, z = u + iv,\) then the given equations become the real and imaginary parts of the complex equation \(\frac{1}{z} = c,\) where \(c = \frac{2}{\sqrt{3} + i\frac{4\sqrt{2}}{\sqrt{7}}}.\) Multiplying by \(z,\) we can apply the quadratic formula to get \(u + iv,\) then squaring \(u, v,\) we can get \(x, y.\)

Problem 92. Let \(a_1, a_2, \ldots, a_n (n > 3)\) be real numbers such that \(a_1 + a_2 + \ldots + a_n \geq n\) and \(a_1^2 + a_2^2 + \ldots + a_n^2 \geq n^2.\)

Prove that max \((a_1, a_2, \ldots, a_n) \geq 2.\) (Source: 1999 USA Math Olympiad)

Solution. FAN Wai Tong Louis (St. Mark’s School, Form 7).

Suppose max \((a_1, a_2, \ldots, a_n) < 2.\) By relabeling the indices, we may assume \(2 > a_1 \geq a_2 \geq \ldots \geq a_n.\) Let \(j\) be the largest index such that \(a_j \geq 0.\) For \(i > j,\) let \(b_i = -a_i > 0.\) Then

\[ 2j-n > (a_1 + \cdots + a_j) - n \geq b_{j+1} + \cdots + b_n. \]

So \((2j-n)^2 > b_{j+1}^2 + \cdots + b_n^2.\) Then

\[ 4j + (2j-n)^2 > a_1^2 + \cdots + a_n^2 \geq n^2, \]

which implies \(j > n - 1.\) Therefore, \(j = n\) and all \(a_j \geq 0.\) This yields \(4n > a_1^2 + \cdots + a_n^2 \geq n^2,\) which gives the contradiction that \(3 \geq n.\)

Other recommended solvers: LEUNG Wai Ying (Queen Elizabeth School, Form 5), NG Ka Wing Gary (STFA Leung Kau Kui College, Form 7), NGAN Chung Wai Hubert (St. Paul’s Co-educational College, Form 7) and WONG Wing Hong (La Salle College, Form 2).

Problem 93. Two circles of radii \(R\) and \(r\) are tangent to line \(L\) at points \(A\) and \(B\) respectively and intersect each other at \(C\) and \(D.\) Prove that the radius of the circumcircle of triangle \(ABC\) does not depend on the length of segment \(AB.\) (Source: 1995 Russian Math Olympiad)

Solution. CHAO Khek Lun (St. Paul’s College, Form 5).

Let \(O, O’\) be the centers of the circles of radii \(R\) and \(r,\) respectively. Let \(\alpha = \angle CAB = \angle AOC/2\) and \(\beta = \angle CBA = \angle BO’C/2.\)

Then \(AC = 2R \sin \alpha = BC \sin \beta,\) which implies \(\sin \alpha / \sin \beta = \sqrt{r/R}.\) The circumradius of triangle...
**ABC is**
\[
\frac{AC}{\sin \alpha} = \frac{R \sin \alpha}{\sin \beta} = \sqrt{R^2 - \beta^2},
\]
which does not depend on the length of $AB$.

**Other recommended solvers:** CHAN Chi Fung (Carmel Divine Grace Foundation Secondary School, Form 6), FAN Wai Tong Louis (St. Mark’s School, Form 7), LEUNG Wai Ying (Queen Elizabeth School, Form 5), NG Ka Chun Bartholomew (Queen Elizabeth School), NGAN Chung Wai Hubert (St. Paul’s Co-educational College, Form 7) and SIU Tsz Hang (STFA Leung Kau Kui College, Form 4).

**Problem 94.** Determine all pairs $(m, n)$ of positive integers for which $2^m + 3^n$ is a square.

**Solution.** NGAN Chung Wai Hubert (St. Paul’s Co-educational College, Form 7) and YEUNG Kai Sing (La Salle College, Form 3).

Let $2^m + 3^n = a^2$. Then $a$ is odd and $a^2 = 2^m + 3^n \equiv (-1)^m \pmod{3}$. Since squares are 0 or 1 (mod 3), $m$ is even. Now $(-1)^m = 2^m + 3^n \equiv a^2 \equiv 1 \pmod{4}$ implies $n$ is even, say $n = 2k$. Then $2^n = (a + k^2)(a - k^2)$. So $a + k^2 = 2^r$, $a - k^2 = 2^s$. Then $2^{n-1} = 2^{r-s} - 2$ implies $n = r + s$. Now $r + 1 = m$ implies $r$ is odd. So
\[
\left(2^{(r-1)/2} + 1\right) \left(2^{(r-1)/2} - 1\right) = 3^k.
\]
Since the difference of the factors is 2, not both are divisible by 3. Then the factor $2^{(r-1)/2} - 1 = 1$. Therefore, $r = 3$, $k = 1$, $(m, n) = (4, 2)$, which is easily checked to be a solution.

**Other recommended solvers:** CHAO Khek Lun (St. Paul’s College, Form 5), CHENG Kei Tsi (La Salle College, Form 5), FAN Wai Tong Louis (St. Mark’s School, Form 7), KU Hong Tung (Carmel Divine Grace Foundation Secondary School, Form 6), LAW Siu Lun Jack (Ming Kei College, Form 5), LEUNG Wai Ying (Queen Elizabeth School, Form 5), NG Ka Chun Bartholomew (Queen Elizabeth School), NG Ka Wing Gary (STFA Leung Kau Kui College, Form 7), NG Ting Chi (TWGH Chang Ming Thien College, Form 7) and SIU Tsz Hang (STFA Leung Kau Kui College, Form 4).

**Problem 95.** Pieces are placed on an $n \times n$ board. Each piece “attacks” all squares that belong to its row, column, and the northwest-southeast diagonal which contains it. Determine the least number of pieces which are necessary to attack all the squares of the board. (Source: 1995 Iberoamerican Olympiad).

**Solution.** LEUNG Wai Ying (Queen Elizabeth School, Form 5).

Assign coordinates to the squares so $(x, y)$ represents the square on the $x$-th column from the west and $y$-th row from the south. Suppose $k$ pieces are enough to attack all squares. Then at least $n - k$ columns, say columns $x_1, ..., x_{n-k}$, and $n - k$ rows, say $y_1, ..., y_{n-k}$ do not contain any of the $k$ pieces. Consider the $2(n-k)$ - 1 squares $(x_1, y_1), (x_1, y_2), ..., (x_1, y_{n-k}), (x_2, y_1), (x_2, y_2), ..., (x_2, y_{n-k}), ..., (x_{n-k}, y_1)$. They are on different diagonals and must be attacked diagonally by the $k$ pieces, we have $k \geq (2(n-k)) - 1$. Solving for $k$, we get $k \geq (2n-1)/3$. Now let $k$ be the least integer such that $k \geq (2n-1)/3$. We will show $k$ is the answer. The case $n = 1$ is clear. Next if $n = 3a + 2$ for a nonnegative integer $a$, then place $k = 2a + 1$ pieces at $(1, n), (2, n-2), (3, n-4), ..., (a+1, n-2a), (a+2, n-1), (a+3, n-3), (a+4, n-5), ..., (2a+1, n-2a+1)$. So squares with $x \leq 2a+1$ or $y \leq 2a-2a$ are under attacked horizontally or vertically. The other squares, with $2a+2 \leq x \leq n$ and $1 \leq y \leq n - 2a - 1$, have $2a + 3 \leq x + y \leq 2n - 2a - 1$. Now the sums $x + y$ of the $k$ pieces range from $n - a + 1 = 2a + 3$ to $n + a + 1 = 2n - 2a - 1$. So the $k$ pieces also attack other squares diagonally.

Next, if $n = 3a + 3$, then $k = 2a + 2$ and we can use the $2a + 1$ pieces above and add a piece at the southeast corner to attack all squares. Finally, if $n = 3a + 4$, then $k = 2a + 3$ and again use the $2a + 2$ pieces in the last case and add another piece at the southeast corner.

**Other recommended solvers:** (LEE Kar Wai Alvin, CHENG Kei Tsi Daniel, LI Chi Pang Bill, TANG Yat Fai Roger) (La Salle College, Form 5), NGAN Chung Wai Hubert (St. Paul’s Co-educational College, Form 7).

**Olympiad Corner** (continued from page 1)

**Problem 4.** Let $P^*$ denote all the odd primes less than 10000. Determine all possible primes $p \in P^*$ such that for each subset $S$ of $P^*$, say $S = \{p_1, p_2, ..., p_k\}$, with $k \geq 2$, whenever $p \not\in S$, there must be some $q \in P^*$, but not in $S$, such that $q + 1$ is a divisor of $(p_1 + 1)(p_2 + 1) \cdots (p_k + 1)$.

**Problem 5.** The altitudes through the vertices $A, B, C$ of an acute-angled triangle $ABC$ meet the opposite sides at $D, E, F$, respectively, and $AB > AC$. The line $EF$ meets $BC$ at $P$, and the line through $D$ parallel to $EF$ meets the lines $AC$ and $AB$ at $R$ and $Q$, respectively. $N$ is a point on the side $BC$ such that $\angle NQR + \angle NRP < 180^\circ$. Prove that $BN > CN$.

**Problem 6.** There are 8 different designs symbolized on 8 different T-shirts, where $n \geq 2$. It is known that each shirt contains at least one symbol, and for any two shirts, the symbols on them are not all the same. Suppose that for any $k$ symbols, $1 \leq k \leq 7$, the number of shirts containing at least one of the $k$ symbols is even. Find the value on $n$.

**Cavalieri** (continued from page 2)

To get the answer, we will apply Cavalieri’s principle. Consider a solid right cylinder with height $1$ and base region $A$. Numerically, the volume of this solid equals the area of the region $A$. Now rotate the solid so that the $1 \times c^2$ rectangular face becomes the base. As we expect the answer to be $\frac{1}{3}c^3$, we compare this rotated solid with a solid right pyramid with height $c$ and square base of side $c$.

Both solids have height $c$. At a level $x$ units below the top, the cross section of the rotated solid is a $1 \times x^2$ rectangle. The cross section of the right pyramid is a square of side $x$. So both solids have the same cross sectional areas at all levels. Therefore, the area of $A$ equals numerically to the volume of the pyramid, which is $\frac{1}{3}c^3$. 