Mathematical Excalibur

Volume 5, Number 1 January 2000 – February 2000

Olympiad Corner

8th Taiwan (ROC) Mathematical Olympiad, April 1999:

Time allowed: 4.5 Hours
Each problem is worth 7 points.

Problem 1. Determine all solutions (x, y, z) of positive integers such that

$$(x+1)^{y+1}+1=(x+2)^{z+1}.$$

Problem 2. Let $a_1, a_2, .., a_{1999}$ be a sequence of nonnegative integers such that for any integers i, j, with $i + j \leq 1999$,

$$a_i + a_j \leq a_{i+j} \leq a_i + a_j + 1.$$

Prove that there exists a real number x such that $a_n = [nx]$ for each n = 1, 2, …, 1999, where $[nx]$ denotes the largest integer less than or equal to nx.

Problem 3. There are 1999 people participating in an exhibition. Two of any 50 people do not know each other. Prove that there are at least 41 people, and each of them knows at most 1958 people.

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Acknowledgment: Thanks to Elina Chiu, MATH Dept, HKUST for general assistance.

On-line: http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is March 4, 2000.

For individual subscription for the next five issues for the 00-01 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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The Road to a Solution

Kin Y. Li

Due to family situation, I missed the trip to the 1999 IMO at Romania last summer. Fortunately, our Hong Kong team members were able to send me the problems by email. Of course, once I got the problems, I began to work on them. The first problem is the following.

Determine all finite sets S of at least three points in the plane which satisfy the following condition: for any two distinct points A and B in S, the perpendicular bisector of the line segment AB is an axis of symmetry of S.

This was a nice problem. I spent sometime on it and got a solution. However, later when the team came back and I had a chance to look at the official solution, I found it a little beyond my expectation. Below I will present my solution and the official solution for comparison.

Here is the road I took to get a solution. To start the problem, I looked at the case of three points, say P_1, P_2, P_3, satisfying the condition. Clearly, the three points cannot be collinear (otherwise considering the perpendicular bisector of the segment joining two consecutive points on the line will yield a contradiction). Now by the condition, it follows that P_2 must be on the perpendicular bisector of segment $P_1 P_3$. Hence, $P_1 P_2 = P_2 P_3$. By switching indices, P_3 should be on the perpendicular bisector of $P_2 P_1$ and so $P_2 P_3 = P_3 P_1$. Thus, P_1, P_2, P_3 are the vertices of an equliateral triangle.

Next the case of four points required more observations. Again no three points are collinear. Also, from the condition, none of the point can be inside the triangle having the other three points as vertices. So the four points are the vertices of a convex quadrilateral. Then the sides have equal length as in the case of three points. Considering the perpendicular bisector of any side, by symmetry, the angles at the other two vertices must be the same. Hence all four angles are the same. Therefore, the four points are the vertices of a square.

After the cases of three and four points, it is quite natural to guess such sets are the vertices of regular polygons. The proof of the general case now follows from the reasonings of the two cases we looked at. First, no three points are collinear. Next, the smallest convex set enclosing the points must be a polygonal region with all sides having the same length and all angles the same. So the boundary of the region is a regular polygon. Finally, one last detail is required. In the case of four points, no point is inside the triangle formed by the other three points by inspection. However, for large number of points, inspection is not good enough. To see that none of the points is inside the polygonal region takes a little bit more work.

Again going back to the case of four points, it is natural to look at the situation when one of the point, say P, is inside the triangle formed by the other three points. Considering the perpendicular bisectors of three segments joining P to the other three points, we see that we can always get a contradiction.

Putting all these observations together, here is the solution I got:

Clearly, no three points of such a set is collinear (otherwise considering the perpendicular bisector of the two furthest points of S on that line, we will get a contradiction). Let H be the convex hull of such a set, which is the smallest convex set containing S. Since S is finite, the boundary of H is a polygon with the vertices P_1, $P_2, ..., P_n$ belonging to S. Let $P_i = P_j$ if $i \equiv j \pmod{n}$. For i = 1, 2, …, n, the condition on the set implies P_i is on the perpendicular bisector of $P_{i-1}\ P_{i+1}$. So $P_{i-1}\ P_i = P_i\ \ P_{i+1}$. Considering the perpendicular bisector of $P_{i-1}\ P_{i+2}$, we see that $\angle P_{i-1}\ \ P_i\ P_{i+1} = \angle P_i P_{i+1} P_{i+2}$. So the boundary of *H* is a regular polygon.

Next, there cannot be any point *P* of *S* inside the regular polygon. (To see this, assume such a *P* exists. Place it at the origin and the furthest point *Q* of *S* from *P* on the positive real axis. Since the origin *P* is in the interior of the convex polygon, not all the vertices can lie on or to the right of the *y*-axis. So there exists a vertex P_j to the left of the *y*-axis. Since the perpendicular bisector of *PQ* is an axis of symmetry, the mirror image of P_j will be a point in *S* further than *Q* from *P*, a contradiction.) So *S* is the set of vertices of some regular polygon. Conversely, such a set clearly has the required property.

Next we look at the official solution, which is shorter and goes as follows: Suppose *S* = { X_1, …, X_n } is such a set. Consider the *barycenter* of *S*, which is the point *G* such that

$$\overrightarrow{OG} = \frac{\overrightarrow{OX_1} + \cdots + \overrightarrow{OX_n}}{n}.$$

Note the barycenter does not depend on the origin. To see this, suppose we get a point G' using another origin O', i.e. $\overrightarrow{O'G'}$ is the average of $\overrightarrow{O'X_i}$ for $i = 1, \ldots, n$. Subtracting the two averages, we get $\overrightarrow{OG} - \overrightarrow{O'G'} = \overrightarrow{OO'}$. Adding $\overrightarrow{O'G'}$ to both sides, $\overrightarrow{OG} = \overrightarrow{OG'}$, so $G = G'$.

By the condition on *S*, after reflection with respect to the perpendicular bisector of every segment $X_i X_j$, the points of S are permuted only. So *G* is unchanged, which implies *G* is on every such perpendicular bisector. Hence, *G* is equidistant from all X_i's. Therefore, the X_i's are concyclic. For three consecutive points of *S*, say X_i, X_j, X_k, on the circle, considering the perpendicular bisector of segment $X_i X_k$, we have $X_i X_j = X_j X_k$. It follows that the points of *S* are the vertices of a regular polygon and the converse is clear.

Cavalieri's Principle

Kin Y. Li

Have you ever wondered why the volume of a sphere of radius *r* is given by the formula $\frac{4}{3}\pi r^3$? The r^3 factor can be easily accepted because volume is a three dimensional measurement. The π factor is probably because the sphere is round. Why then is there $\frac{4}{3}$ in the formula?

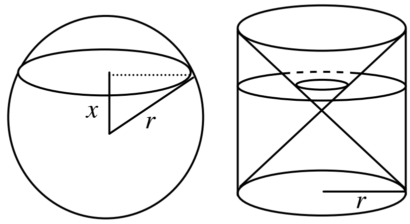
In school, most people told you it came from calculus. Then, how did people get the formula before calculus was invented? In particular, how did the early Egyptian or Greek geometers get it thousands of years ago?

Those who studied the history of mathematics will be able to tell us more of the discovery. Below we will look at one way of getting the formula, which may not be historically the first way, but it has another interesting application as we will see. First, let us introduce

Cavalieri's Principle: *Two objects having the same height and the same cross sectional area at each level must have the same volume.*

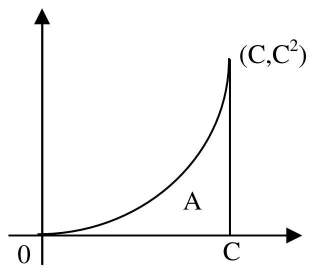
To understand this, imagine the two objects are very large, like pyramids that are built by piling bricks one level on top of another. By definition, the volume of the objects are the numbers of 1×1×1 bricks used to build the objects. If at *each level* of the construction, the number of bricks used (which equals the cross sectional area numerically) is the same for the two objects, then the volume (which equals the total number of bricks used) would be the same for both objects.

To get the volume of a sphere, let us apply Cavalieri's principle to a solid sphere *S* of radius *r* and an object *T* made out from a solid right circular cylinder with height 2*r* and base radius *r* removing a pair of right circular cones with height *r* and base radius *r* having the center of the cylinder as the apex of each cone.



Both *S* and *T* have the same height 2*r*. Now consider the cross sectional area of each at a level *x* units from the equatorial plane of *S* and *T*. The cross section for *S* is a circular disk of radius $\sqrt{r^2 - x^2}$ by Pythagoras' theorem, which has area $\pi(r^2 - x^2)$. The cross section for *T* is an annular ring of outer radius *r* and inner radius *x*, which has the same area $\pi r^2 - \pi x^2$. By Cavalieri's principle, *S* and *T* have the same volume. Since the volume of *T* is $\pi r^2(2r) - 2\times\frac{1}{3}\pi r^2 r = \frac{4}{3}\pi r^3$, so the volume of *S* is the same.

Cavalieri's principle is not only useful in getting the volume of special solids, but it can also be used to get the area of special regions in a plane! Consider the region *A* bounded by the graph of $y = x^2$, the *x*-axis and the line $x = c$ in the first quadrant.



The area of this region is less than the area of the triangle with vertices at (0, 0), (*c*, 0), (*c*, c^2), which is $\frac{1}{2}c^3$. If you ask a little kid to guess the answer, you may get $\frac{1}{3}c^3$ since he knows $\frac{1}{3} < \frac{1}{2}$. For those who know calculus, the answer is easily seen to be correct. How can one explain this without calculus?

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Problem Corner

We welcome readers to submit solutions to the problems posed below for publication consideration. Solutions should be preceeded by the solver's name, home address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is *March 4, 2000.*

Problem 96. If every point in a plane is colored red or blue, show that there exists a rectangle all of its vertices are of the same color.

Problem 97. A group of boys and girls went to a restaurant where only big pizzas cut into 12 pieces were served. Every boy could eat 6 or 7 pieces and every girl 2 or 3 pieces. It turned out that 4 pizzas were not enough and that 5 pizzas were too many. How many boys and how many girls were there? (*Source: 1999 National Math Olympiad in Slovenia*)

Problem 98. Let ABC be a triangle with $BC > CA > AB$. Select points D on BC and E on the extension of AB such that $BD = BE = AC$. The circumcircle of BED intersects AC at point P and BP meets the circumcircle of ABC at point Q. Show that $AQ + CQ = BP$. (*Source: 1998-99 Iranian Math Olympiad*)

Problem 99. At Port Aventura there are 16 secret agents. Each agent is watching one or more other agents, but no two agents are both watching each other. Moreover, any 10 agents can be ordered so that the first is watching the second, the second is watching the third, etc., and the last is watching the first. Show that any 11 agents can also be so ordered. (*Source: 1996 Spanish Math Olympiad*)

Problem 100. The arithmetic mean of a number of pairwise distinct prime numbers equals 27. Determine the biggest prime that can occur among them. (*Source: 1999 Czech and Slovak Math Olympiad*)

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Solutions

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Problem 91. Solve the system of equations:

$$\sqrt{3x}\left(1+\frac{1}{x+y}\right)=2$$

$$\sqrt{7y}\left(1-\frac{1}{x+y}\right)=4\sqrt{2}\,.$$

(*This is the corrected version of problem 86.*)

Solution. (CHENG Kei Tsi, LEE Kar Wai, TANG Yat Fai) (La Salle College, Form 5), **CHEUNG Yui Ho Yves** (University of Toronto), **HON Chin Wing** (Pui Ching Middle School, Form 5) **KU Hong Tung** (Carmel Divine Grace Foundation Secondary School, Form 6), **LAU Chung Ming Vincent** (STFA Leung Kau Kui College, Form 5), **LAW Siu Lun Jack** (Ming Kei College, Form 5), **Kevin LEE** (La Salle College, Form 4), **LEUNG Wai Ying** (Queen Elizabeth School, Form 5), **MAK Hoi Kwan Calvin** (Form 4), **NG Chok Ming Lewis** (STFA Leung Kau Kui College, Form 6), **NG Ka Wing Gary** (STFA Leung Kau Kui College, Form 7), **NGAN Chung Wai Hubert** (St. Paul's Co-educational College, Form 7), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 4), **TANG King Fun** (Valtorta College, Form 5), **WONG Chi Man** (Valtorta College, Form 5) and **WONG Chun Ho Terry** (STFA Leung Kau Kui College, Form 5). (All solutions received were essentially the same.) Clearly, if (x, y) is a solution, then $x, y > 0$ and

$$1+\frac{1}{x+y}=\frac{2}{\sqrt{3x}}$$

$$1-\frac{1}{x+y}=\frac{4\sqrt{2}}{\sqrt{7y}}\,.$$

Taking the difference of the squares of both equations, we get

$$\frac{4}{x+y}=\frac{4}{3x}-\frac{32}{7y}\,.$$

Simplifying this, we get $0 = 7y^2 - 38xy - 24x^2 = (7y+4x)(y-6x)$. Since $x, y > 0$, $y = 6x$. Substituting this into the first given equation, we get $\sqrt{3x}\left(1+\frac{1}{7x}\right)=2$, which simplifies to $7\sqrt{3x}-14\sqrt{x}+\sqrt{3}=0$. By the quadratic formula, $\sqrt{x}=(7\pm2\sqrt{7})/(7\sqrt{3})$. Then $x=(11\pm4\sqrt{7})/21$ and $y = 6x = (22\pm8\sqrt{7})/7$. Direct checking shows these are solutions.

Comments: An alternative way to get the answers is to substitute $u = \sqrt{x}$, $v = \sqrt{y}$, $z = u + iv$, then the given equations become the real and imaginary parts of the complex equation $z+\frac{1}{z}=c$, where $c = \frac{2}{\sqrt{3}}+i\frac{4\sqrt{2}}{\sqrt{7}}$. Multiplying by z, we can apply the quadratic formula to get $u + iv$, then squaring u, v, we can get x, y.

Problem 92. Let $a_1, a_2, \ldots, a_n$ $(n > 3)$ be real numbers such that $a_1+a_2+\cdots+a_n \ge n$ and $a_1^2+a_2^2+\cdots+a_n^2 \ge n^2$. Prove that $\max(a_1, a_2, \ldots, a_n) \ge 2$. (*Source: 1999 USA Math Olympiad*).

Solution. FAN Wai Tong Louis (St. Mark's School, Form 7).

Suppose $\max(a_1, a_2, \ldots, a_n) < 2$. By relabeling the indices, we may assume $2 > a_1 \ge a_2 \ge \cdots \ge a_n$. Let j be the largest index such that $a_j \ge 0$. For $i > j$, let $b_i = -a_i > 0$. Then

$$2j-n>(a_1+\cdots+a_j)-n\ge b_{j+1}+\cdots+b_n.$$

So $(2j-n)^2 > b_{j+1}^2+\cdots+b_n^2$. Then

$$4j+(2j-n)^2 > a_1^2+\cdots+a_n^2 \ge n^2,$$

which implies $j > n - 1$. Therefore, $j = n$ and all $a_i \ge 0$. This yields $4n > a_1^2+\cdots+a_n^2 \ge n^2$, which gives the contradiction that $3 \ge n$.

Other recommended solvers: **LEUNG Wai Ying** (Queen Elizabeth School, Form 5), **NG Ka Wing Gary** (STFA Leung Kau Kui College, Form 7), **NGAN Chung Wai Hubert** (St. Paul's Co-educational College, Form 7) and **WONG Wing Hong** (La Salle College, Form 2).

Problem 93. Two circles of radii R and r are tangent to line L at points A and B respectively and intersect each other at C and D. Prove that the radius of the circumcircle of triangle ABC does not depend on the length of segment AB. (*Source: 1995 Russian Math Olympiad*).

Solution. CHAO Khek Lun (St. Paul's College, Form 5).

Let O, O' be the centers of the circles of radius R, r, respectively. Let $\alpha=\angle CAB = \angle AOC/2$ and $\beta=\angle CBA=\angle BO'C/2$. Then $AC = 2R\sin\alpha$ and $BC = 2r\sin\beta$. The distance from C to AB is $AC\sin\alpha = BC\sin\beta$, which implies $\sin\alpha / \sin\beta = \sqrt{r/R}$. The circumradius of triangle

ABC is

$$\frac{AC}{2\sin\beta}=\frac{R\sin\alpha}{\sin\beta}=\sqrt{Rr}\ ,$$

which does not depend on the length of AB.

Other recommended solvers: **CHAN Chi Fung** (Carmel Divine Grace Foundation Secondary School, Form 6), **FAN Wai Tong Louis** (St. Mark's School, Form 7), **LEUNG Wai Ying** (Queen Elizabeth School, Form 5), **NG Ka Chun Bartholomew** (Queen Elizabeth School), **NGAN Chung Wai Hubert** (St. Paul's Co-educational College, Form 7) and **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 4).

Problem 94. Determine all pairs (m, n) of positive integers for which 2^m+3^n is a square.

Solution. NGAN Chung Wai Hubert (St. Paul's Co-educational College, Form 7) and **YEUNG Kai Sing** (La Salle College, Form 3).

Let $2^m+3^n=a^2$. Then a is odd and $a^2=2^m+3^n\equiv(-1)^m$ (mod 3). Since squares are 0 or 1 (mod 3), m is even. Next $(-1)^n\equiv 2^m+3^n=a^2\equiv 1$ (mod 4) implies n is even, say $n = 2k$, $k\geq 1$. Then $2^m=(a+3^k)(a-3^k)$. So $a+3^k=2^r$, $a-3^k=2^s$ for integers $r>s\geq 0$ with $r + s = m$. Then $2\cdot 3^k=2^r-2^s$ implies $s = 1$, so $2^{r-1}-1=3^k$. Now $r+1=m$ implies r is odd. So

$$\left(2^{(r-1)/2}+1\right)\left(2^{(r-1)/2}-1\right)=3^k.$$

Since the difference of the factors is 2, not both are divisible by 3. Then the factor $2^{(r-1)/2}-1=1$. Therefore, $r=3$, $k=1$, $(m,n)=(4,2)$, which is easily checked to be a solution.

Other recommended solvers: **CHAO Khek Lun** (St. Paul's College, Form 5), **CHENG Kei Tsi** (La Salle College, Form 5), **FAN Wai Tong Louis** (St. Mark's School, Form 7), **KU Hong Tung** (Carmel Divine Grace Foundation Secondary School, Form 6), **LAW Siu Lun Jack** (Ming Kei College, Form 5), **LEUNG Wai Ying** (Queen Elizabeth School, Form 5), **NG Ka Chun Batholomew** (Queen Elizabeth School), **NG Ka Wing Gary** (STFA Leung Kau Kui College, Form 7), **NG Ting Chi** (TWGH Chang Ming Thien College, Form 7) and **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 4).

Problem 95. Pieces are placed on an $n\times n$ board. Each piece "attacks" all squares that belong to its row, column, and the northwest-southeast diagonal which contains it. Determine the least number of pieces which are necessary to attack all the squares of the board. (*Source: 1995 Iberoamerican Olympiad*).

Solution. LEUNG Wai Ying (Queen Elizabeth School, Form 5).

Assign coordinates to the squares so (x, y) represents the square on the x-th column from the west and y-th row from the south. Suppose k pieces are enough to attack all squares. Then at least $n - k$ columns, say columns $x_1,\ldots,x_{n-k}$, and $n - k$ rows, say $y_1,\ldots,y_{n-k}$, do not contain any of the k pieces. Consider the $2(n - k) - 1$ squares (x_1,y_1), (x_1,y_2), …, (x_1, y_{n-k}), (x_2,y_1), (x_3,y_1), …, (x_{n-k}, y_1). As they are on different diagonals and must be attacked diagonally by the k pieces, we have $k\geq 2(n-k)-1$. Solving for k, we get $k\geq (2n-1)/3$. Now let k be the least integer such that $k\geq(2n-1)/3$. We will show k is the answer. The case $n = 1$ is clear. Next if $n = 3a + 2$ for a nonnegative integer a, then place $k = 2a + 1$ pieces at $(1, n)$, $(2, n - 2)$, $(3, n - 4)$, …, $(a + 1, n - 2a)$, $(a + 2, n - 1)$, $(a + 3, n - 3)$, $(a + 4, n - 5)$, …, $(2a + 1, n - 2a + 1)$. So squares with $x\leq 2a + 1$ or $y\geq n - 2a$ are under attacked horizontally or vertically. The other squares, with $2a + 2\leq x\leq n$ and $1\leq y\leq n - 2a - 1$, have $2a + 3\leq x + y\leq 2n - 2a - 1$. Now the sums $x + y$ of the k pieces range from $n - a + 1 = 2a + 3$ to $n + a + 1 = 2n - 2a - 1$. So the k pieces also attack the other squares diagonally.

Next, if $n = 3a + 3$, then $k = 2a + 2$ and we can use the $2a + 1$ pieces above and add a piece at the southeast corner to attack all squares. Finally, if $n = 3a + 4$, then $k = 2a + 3$ and again use the $2a + 2$ pieces in the last case and add another piece at the southeast corner.

Other recommended solvers: **(LEE Kar Wai Alvin, CHENG Kei Tsi Daniel, LI Chi Pang Bill, TANG Yat Fai Roger)** (La Salle College, Form 5), **NGAN Chung Wai Hubert** (St. Paul's Co-educational College, Form 7).

Olympiad Corner

(continued from page 1)

Problem 4. Let P^* denote all the odd primes less than 10000. Determine all possible primes $p\in P^*$ such that for each subset S of P^*, say $S = \{p_1, p_2, \ldots, p_k\}$, with $k\geq 2$, whenever $p\notin S$, there must be some q in P^*, but not in S, such that $q + 1$ is a divisor of $(p_1+1)\ (p_2+1)\ \ldots\ (p_k+1)$.

Problem 5. The altitudes through the vertices A, B, C of an acute-angled triangle ABC meet the opposite sides at D, E, F, respectively, and $AB > AC$. The line EF meets BC at P, and the line through D parallel to EF meets the lines AC and AB at Q and R, respectively. N is a point on the side BC such that $\angle NQP + \angle NRP < 180^\circ$. Prove that $BN > CN$.

Problem 6. There are 8 different symbols designed on n different T-shirts, where $n\geq 2$. It is known that each shirt contains at least one symbol, and for any two shirts, the symbols on them are not all the same. Suppose that for any k symbols, $1\leq k\leq 7$, the number of shirts containing at least one of the k symbols is even. Find the value on n.

Cavalieri

(continued from page 2)

To get the answer, we will apply Cavalieri's principle. Consider a solid right cylinder with height 1 and base region A. Numerically, the volume of this solid equals the area of the region A. Now rotate the solid so that the $1\times c^2$ rectangular face becomes the base. As we expect the answer to be $\frac{1}{3}c^3$, we compare this rotated solid with a solid right pyramid with height c and square base of side c.

Both solids have height c. At a level x units below the top, the cross section of the rotated solid is a $1\times x^2$ rectangle. The cross section of the right pyramid is a square of side x. So both solids have the same cross sectional areas at all levels. Therefore, the area of A equals numerically to the volume of the pyramid, which is $\frac{1}{3}c^3$.