Olympiad Corner

XII Asia Pacific Math Olympiad, March 2000:

Time allowed: 4 Hours

Problem 1. Compute the sum
\[ S = \sum_{i=0}^{101} \frac{x_i^3}{1 - 3x_i + 3x_i^2} \quad \text{for} \quad x_i = \frac{i}{101}. \]

Problem 2. Given the following triangular arrangement of circles:

Each of the numbers 1, 2, ..., 9 is to be written into one of these circles, so that each circle contains exactly one of these numbers and
(i) the sums of the four numbers on each side of the triangle are equal;
(continued on page 4)

Coordinate Geometry

Kin Y. Li

When we do a geometry problem, we should first look at the given facts and the conclusion. If all these involve intersection points, midpoints, feet of perpendiculars, parallel lines, then there is a good chance we can solve the problem by coordinate geometry. However, if they involve two or more circles, angle bisectors and areas of triangles, then sometimes it is still possible to solve the problem by choosing a good place to put the origin and the x-axis. Below we will give some examples. It is important to stay away from messy computations!

Example 1. (1995 IMO) Let A, B, C and D be four distinct points on a line, in that order. The circles with diameters AC and BD intersect at the points X and Y. The line XY meets BC at the point Z. Let P be a point on the line XY different from Z. The line CP intersects the circle with diameter AC at the points C and M, and the line BP intersects the circle with diameter BD at the points B and N. Prove that the lines AM, DN, and XY are concurrent.

Solution. (Due to Mok Tze Tao, 1995 Hong Kong Team Member) Set the origin at Z and the x-axis on line AD. Let the coordinates of the circumcenters of triangles AMC and BNDF be (x_1, 0) and (x_2, 0), and the circumradii be r_1 and r_2, respectively. Then the coordinates of A and C are (x_1-r_1, 0) and (x_1+r_1, 0), respectively. Let the coordinates of P be (0, y_0). Since AM \perp CP and the slope of CP is \( -y_0/(x_1+r_1) \), the equation of AM works out to be \((x_1+r_1)x-y_0y=x_1^2-y_0^2\). Let Q be the intersection of AM with XY, then Q has coordinates \((0, (r_1^2-x_1^2)/y_0)\).

Similarly, let Q’ be the intersection of DN with XY, then Q’ has coordinates \((0, (r_2^2-x_2^2)/y_0)\). Since \(r_1^2-x_1^2=ZX^2\) \(=r_2^2-x_2^2\), so \(Q=Q’\).

Example 2. (1998 APMO) Let ABC be a triangle and D the foot of the altitude from A. Let E and F be on a line passing through D such that AE is perpendicular to BE, AF is perpendicular to CF, and E and F are different from D. Let M and N be the midpoints of the line segments BC and EF, respectively. Prove that AN is perpendicular to NM.

(Remarks. We can set the origin at D and the x-axis on line BC. Then computing the coordinates of E and F will be a bit messy. A better choice is to set the line through D, E, F horizontal.)
Solution. (Due to Cheung Pok Man, 1998 Hong Kong Team Member) Set the origin at $A$ and the $x$-axis parallel to line $EF$. Let the coordinates of $E$, $F$ be $(d, b)$, $(e, b)$, respectively. The case $b=0$ leads to $D=E$, which is not allowed. So we may assume $b \neq 0$. Since $BE \perp AE$ and the slope of $AE$ is $b/e$, so the equation of line $BE$ works out to be $ex+by=e^2+b^2$. Similarly, the equations of lines $CF$ and $BC$ are $fx+by=f^2+b^2$ and $dx+by=d^2+b^2$, respectively. Solving the equations for $BE$ and $BC$, we find $B$ has coordinates $(d+e, b-(de+b^2)/2b)$. Similarly, $C$ has coordinates $(e+f, b-(df+b)^2/(2b))$ and $N$ has coordinates $(e+f/2, b)$. So the slope of $AN$ is $2b/(e+f)$ and the slope of $MN$ is $-e/f/(2b)$. Therefore, $AN \perp MN$.

Example 3. (2000 IMO) Two circles $\Gamma_1$ and $\Gamma_2$ intersect at $M$ and $N$. Let $\ell$ be the common tangent to $\Gamma_1$ and $\Gamma_2$ so that $M$ is closer to $\ell$ than $N$. Let $\ell$ touch $\Gamma_1$ at $A$ and $\Gamma_2$ at $B$. Let the line through $M$ parallel to $\ell$. Let $\ell$ meet the circle $\Gamma_1$ again at $C$ and the circle $\Gamma_2$ again at $D$. Lines $CA$ and $DB$ meet at $E$; lines $AN$ and $CD$ meet at $P$; lines $BN$ and $CD$ meet at $Q$. Show that $EP=EQ$.

(Remarks. Here if we set the $x$-axis on the line through the centers of the circles, then the equation of the line $AB$ will be complicated. So it is better to have line $AB$ on the $x$-axis.)

Solution. Set the origin at $A$ and the $x$-axis on line $AB$. Let $B, M$ have coordinates $(b, 0), (s, t)$, respectively. Let the centers $O_1, O_2$ of $\Gamma_1, \Gamma_2$ be at $(0, b), (b, r_2)$, respectively. Then $C, D$ have coordinates $(-s, t), (2b-s, t)$, respectively. Since $AB, CD$ are parallel, $CD=2b=2AB$ implies $A, B$ are midpoints of $CE, DE$, respectively. So $E$ is at $(s, -t)$. We see $EM \perp CD$.

To get $EP=EQ$, it is now left to show $M$ is the midpoint of segment $PO$. Since $O_1 O_2 \perp MN$ and the slope of $O_1 O_2$ is $-(r_2-r_)/b$, the equation of line $MN$ is $bx+(r_2-r_1)y=bs+(r_2-r_1)t$. (This line should pass through the midpoint of segment $AB$.) Since $O_1 M=r_2$ and $O_2 M=r_1$, we get

$$(b-s)^2+(r_2-t)^2=r_1^2$$

and

$$s^2+(r_1-t)^2=r_2^2.$$  

Subtracting these equations, we get

$$b^2/2=bs+(r_2-r_1)t$$

which implies $(b/2, 0)$ on line $MN$. Since $PQ, AB$ are parallel and line $MN$ intersects $AB$ at its midpoint, then $M$ must be the midpoint of segment $PQ$. Together with $EM \perp PQ$, we get $EP=EQ$.

Example 4. (2000 APMO) Let $ABC$ be a triangle. Let $M$ and $N$ be the points in which the median and the angle bisector, respectively, at $A$ meet the side $BC$. Let $Q$ and $P$ be the points in which the perpendicular at $N$ to $NA$ meets $MA$ and $BA$, respectively, and $O$ the point in which the perpendicular at $P$ to $BA$ meets $AN$ produced. Prove that $OQ$ is perpendicular to $BC$.

(Remarks. Here the equation of the angle bisector is a bit tricky to obtain unless it is the $x$-axis. In that case, the two sides of the angle are symmetric with respect to the $x$-axis.)

Solution. (Due to Wong Chun Wai, 2000 Hong Kong Team Member) Set the origin at $N$ and the $x$-axis on line $NO$. Let the equation of line $AB$ be $y=ax+b$, then the equation of lines $AC$ and $PO$ are $y=-ax-b$ and $y=-(1/ax)+b$, respectively. Let the equation of $BC$ be $y=cx$. Then $B$ has coordinates $(b/(c-a), bc/(c-a))$, $C$ has coordinates $(-b/(c+a), -bc/(c+a))$, $M$ has coordinates $(ab/(c^2-a^2), abc/(c^2-a^2))$, $A$ has coordinates $(-b/a, 0)$, $O$ has coordinates $(ab, 0)$ and $Q$ has coordinates $(0, abc)$. Then $BC$ has slope $c$ and $QO$ has slope $-1/c$. Therefore, $QO \perp BC$.

Example 5. (1998 IMO) In the convex quadrilateral $ABCD$, the diagonals $AC$ and $BD$ are perpendicular and the opposite sides $AB$ and $DC$ are not parallel. Suppose that the point $P$, where the perpendicular bisectors of $AB$ and $DC$ meet, is inside $ABCD$. Prove that $ABCD$ is a cyclic quadrilateral if and only if the triangles $ABP$ and $CDP$ have equal areas.

(Remarks. The area of a triangle can be computed by taking the half length of the cross product. A natural candidate for the origin is $P$ and having the diagonals parallel to the axes will be helpful.)

Solution. (Due to Leung Wing Chung, 1998 Hong Kong Team Member) Set the origin at $P$ and the $x$-axis parallel to line $AC$. Then the equations of lines $AC$ and $BD$ are $y=p$ and $x=q$, respectively. Let $AP=BP=r$ and $CP=DP=s$. Then the coordinates of $A$, $B$, $C$, $D$ are

$$(q, r^2-p^2), (q, q^2-p^2), (s^2-q^2-p, p),$$

(respectively. Using the determinant formula for finding the area of a triangle, we see that the areas of triangles $ABP$ and $CDP$ are equal if and only if

$$q^2-r^2=q^2-q^2-pq = -q^2-r^2-q^2-pq.$$  

Since $f(x) = -q^2+x^2$ is strictly decreasing when $x \geq |p|$ and $|q|$, equality of areas hold if and only if $x=r$, which is equivalent to $A, B, C, D$ concyclic (since $P$ being on the perpendicular bisectors of $AB, CD$ is the only possible place for the center).

After seeing these examples, we would like to remind the readers that there are pure geometric proofs to each of the problems. For examples (1) and (3), there are proofs that only take a few lines. We encourage the readers to discover these simple proofs.

Although in the opinions of many people, a pure geometric proof is better and more beautiful than a coordinate geometric proof, we should point out that sometimes the coordinate geometric proofs may be preferred when there are many cases. For example (2), the different possible orderings of the points $D, E, F$ on the line can all happen as some pictures will show. The coordinate geometric proofs above cover all cases.
Problem Corner

We welcome readers to submit solutions to the problems posed below for publication consideration. Solutions should be preceded by the solver’s name, home address and school affiliation. Please send submissions to Dr. Kin Y. Li, Department of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon. The deadline for submitting solutions is October 10, 2000.

Problem 106. Find all positive integer ordered pairs \((a,b)\) such that 
\[ \gcd(a,b)+\text{lcm}(a,b)=a+b+6, \]
where \(\gcd\) stands for greatest common divisor (or highest common factor) and \(\text{lcm}\) stands for least common multiple.

Problem 107. For \(a, b, c > 0\), if \(abc=1\), then show that 
\[ \frac{b+c}{\sqrt{a}} + \frac{c+a}{\sqrt{b}} + \frac{a+b}{\sqrt{c}} \geq \sqrt{a} + \sqrt{b} + \sqrt{c} + 3. \]

Problem 108. Circles \(C_1\) and \(C_2\) with centers \(O_1\) and \(O_2\) (respectively) meet at points \(A, B\). The radii \(O_1B\) and \(O_2B\) intersect \(C_1\) and \(C_2\) at \(F\) and \(E\). The line parallel to \(EF\) through \(B\) meets \(C_1\) and \(C_2\) at \(M\) and \(N\), respectively. Prove that \(MN=AE+AF\). (Source: 17th Iranian Mathematical Olympiad)

Problem 109. Show that there exists an increasing sequence \(a_1, a_2, a_3, \ldots\) of positive integers such that for every nonnegative integer \(k\), the sequence \(k+a_1, k+a_2, k+a_3, \ldots\) contains only finitely many prime numbers. (Source: 1997 Math Olympiad of Czech and Slovak Republics)

Problem 110. In a park, 10000 trees have been placed in a square lattice. Determine the maximum number of trees that can be cut down so that from any stump, you cannot see any other stump. (Assume the trees have negligible radius compared to the distance between adjacent trees.) (Source: 1997 German Mathematical Olympiad)

Comments. You may think of the trees being placed at \((x,y)\), where \(x, y = 0, 1, 2, \ldots, 99\).

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Solutions
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Problem 101. A triple of numbers \((a_1, a_2, a_3) = (3, 4, 12)\) is given. We now perform the following operation: choose two numbers \(a_i\) and \(a_j\) \((i \neq j)\), and exchange them by \(0.6a_i-0.8a_j\) and \(0.8a_i+0.6a_j\). Is it possible to obtain after several steps the unorderd triple \((2, 8, 10)\)? (Source: 1999 National Math Competition in Croatia)

Solution. FAN Wai Tong (St. Mark’s School, Form 7), KO Man Ho (Wah Yan College, Kowloon, Form 6) and LAW Hiu Fai (Wah Yan College, Kowloon, Form 6).

Since \((0.6a_i-0.8a_j)^2 + (0.8a_i+0.6a_j)^2 \neq a_i^2 + a_j^2\), the sum of the squares of the triple of numbers before and after an operation stays the same. Since \(3^2 + 4^2 + 12^2 \neq 2^2 + 8^2 + 10^2\), so \((2, 8, 10)\) cannot be obtained.

Problem 102. Let \(a\) be a positive real number and \((x_1, x_2, \ldots)\) be a sequence of real numbers such that \(x_1=a\) and 
\[ x_{n+1} \geq (n+2)x_n - \frac{1}{2} \frac{1}{n+1} k_x, \text{ for all } n \geq 1. \]

Show that there exists a positive integer \(n\) such that \(x_n > 1999!\). (Source: 1999 Romanian Third Selection Examination)

Solution. FAN Wai Tong (St. Mark’s School, Form 7).

We will prove by induction that \(x_m \geq 3x_{m-1}\) for every positive integer \(j\). The case \(j=1\) is true by the given inequality. Assume the cases \(j = 1, \ldots, n-1\) are true. Then \(x_{n} \geq 3x_{n-1} \geq 9x_{n-2} \geq \ldots \geq \frac{x_2}{x_1} \geq (n+2)\frac{x_n}{x_{n-1}} - \sum_{k=1}^{n-1} \frac{k}{n+1} x_k \geq (n+2) - \sum_{k=1}^{n-1} \frac{n-1}{3^{n-k}} \geq (n+2) - \sum_{k=1}^{n-1} \frac{n-1}{3^{n-k}}
\[
\geq \frac{n+5}{2} \geq 3.
\]

So the case \(j = n\) is also true.

Since \(a > 0\), we can take \(n > 1 + \log_3 (1999!a)\).

Then \(x_n \geq 3^{n-1} x_1 = 3^{n-1} a > 1999!\).

Problem 103. Two circles intersect in points \(A\) and \(B\). A line \(l\) that contains the point \(A\) intersects again the circles in the points \(C, D\), respectively. Let \(M, N\) be the midpoints of the arcs \(BC\) and \(BD\), which do not contain the point \(A\), and let \(K\) be the midpoint of the segment \(CD\). Show that \(\angle MKN = 90^\circ\). (Source: 1999 Romanian Fourth Selection Examination)

Solution. FAN Wai Tong (St. Mark’s School, Form 7).

Let \(M’\) and \(N’\) be the midpoints of chords \(BC\) and \(BD\) respectively. From the midpoint theorem, we see that \(BM’KN’\) is a parallelogram. Now 
\[ \angle KN’N = \angle KN’B + 90^\circ = \angle KM’B + 90^\circ = \angle KM’M. \]

Let \(\alpha = \angle NDB = \angle NAB\). Then 
\[ \angle KN’N = \frac{MB}{NN’} \tan \alpha = \frac{1}{2} \frac{BC}{BD} \tan \alpha. \]

Now 
\[ \angle MCB = \angle MCB = \frac{1}{2} \angle CAB = \frac{1}{2} (180^\circ - \angle DAB) = 90^\circ - \angle NAB = 90^\circ - \alpha. \]
Then \( KN' / KN' = MM' / M'K \). So triangles \( MM'K, KN'N \) are similar. Then \( \triangle M'K = \angle N'K' \) and

\[
\angle MKN = \angle M'K'N' - \angle M'KN = \angle N'K'N = \angle K'DM - (\angle N'KN + \angle N'K'N) = 90^\circ.
\]

Other commended solvers: WONG Chun Wai (Choi Hung Estate Catholic Secondary School, Form 7).

Problem 104. Find all positive integers \( n \) such that \( 2^n-1 \) is a multiple of 3 and \((2^n-1)/3\) is a divisor of \( 4m^2 + 1 \) for some integer \( m \). (Source: 1999 Korean Mathematical Olympiad)

Solution. (Official Solution)

(Some checks should suggest \( n \) is a power of 2.) Now \( 2^n-1 \) is a multiple of 3 if and only if \( (1)^n \equiv 2^n \equiv (\text{mod } 3) \), that is \( n \) is even. Suppose for some even \( n \), \((2^n-1)/3\) is a divisor of \( 4m^2 + 1 \) for some \( m \). Assume \( n \) has an odd prime divisor \( d \). Now \( 2^{2d} \equiv 1 \) (mod 4) implies one of its prime divisor \( p \) is of the form \( 4k+3 \). Then \( p \) divides \( 2^{2d} - 1 \), which divides \( 4m^2 + 1 \). Then \( p \) and \( 2m \) are relatively prime and so

\[
1 \equiv (2m)^n \equiv (4m^2)^{2d} \equiv 1 \pmod{p},
\]

a contradiction. So \( n \) cannot have any odd prime divisor. Hence \( n=2 \) for some positive integer \( j \).

Conversely, suppose \( n=2^j \). Let \( F_i = 2^i + 1 \). Using the factorization \( 2^{2d} - 1 = (2^d - 1) \times (2^d + 1) \) repeatedly on the numerator, we get

\[
\frac{2^n - 1}{3} = F_1 F_2 \ldots F_{2j-1}.
\]

Since \( F_i \) divides \( F_{i+2} \) for \( i < j \), the \( F_i \)'s are pairwise relatively prime. By the Chinese remainder theorem, there is a positive integer \( x \) satisfying the simultaneous equations \( x \equiv 0 \pmod{2} \) and \( x \equiv 2^{-x} \pmod{F_i} \) for \( i=1, 2, \ldots, j-1 \). Then \( x \equiv 2m \) for some positive integer \( m \) and \( 4m^2 + 1 = x^2 + 1 = 0 \pmod{F_i} \) for \( i=1, 2, \ldots, j-1 \). So \( 4m^2 + 1 \) is divisible by \( F_i F_{2i} \ldots F_{i+2} = (2^i - 1)/3 \).\]

Problem 105. A rectangular parallelepiped (box) is given, such that its intersection with a plane is a regular hexagon. Prove that the rectangular parallelepiped is a cube. (Source: 1999 National Math Olympiad in Slovenia)

Solution. (Official Solution)

As in the figure, an equilateral triangle \( XYZ \) is formed by extending three alternate sides of the regular hexagon.

The right triangles \( XBJ \) and \( YBJ \) are congruent as they have a common side \( BJ \) and the hypotenuses have equal length. So \( BZ = BY \) and similarly \( BX = BZ \). As the pyramids \( XBYZ \) and \( OBN'Z \) are similar and \( ON = \frac{3}{4} NY \), it follows \( B'Z \)

\[
= \frac{1}{3} BZ.
\]

Thus we have \( BB' = \frac{2}{3} BZ \) and similarly \( AB = \frac{2}{3} BX \) and \( CB = \frac{2}{3} BY \).

Since \( BX = BY = BZ \), we get \( AB = BC = BB' \).

Other commended solvers: FAN Wai Tong (St. Mark’s School, Form 7).

Olympiad Corner

(continued from page 1)

(ii) the sums of the squares of the four numbers on each side of the triangle are equal.

Find all ways in which this can be done.

Problem 3. Let \( ABC \) be a triangle. Let \( M \) and \( N \) be the points in which the median and the angle bisector, respectively, at \( A \) meet the side \( BC \). Let \( Q \) and \( P \) be the points in which the perpendicular at \( N \) to \( NA \) meets \( MA \) and \( BA \), respectively, and \( O \) the point in which the perpendicular at \( P \) to \( BA \) meets \( AN \) produced. Proved that \( QO \) is perpendicular to \( BC \).

Problem 4. Let \( n, k \) be given positive integers with \( n > k \). Prove that

\[
\frac{1}{n+1} \leq k^{n-k} \leq \frac{n}{k(n-k)!} \leq k^{n-k}.
\]

Problem 5. Given a permutation \( (a_0, a_1, ..., a_n) \) of the sequence \( 0, 1, ..., n \). A transposition of \( a_i \) with \( a_j \) is called legal if \( i > 0 \), \( a_i = 0 \) and \( a_{i-1} + 1 = a_j \). The permutation \( (a_0, a_1, ..., a_n) \) is called regular if after a number of legal transpositions it becomes \( (1, 2, ..., n, 0) \). For which numbers \( n \) is the permutation \( (1, n, n-1, ..., 3, 2, 0) \) regular?

2000 APMO and IMO

In April this year, Hong Kong IMO trainees participated in the XII Asia Pacific Mathematical Olympiad. The winners were

Gold Award

Fan Wai Tong (Form 7, St Mark’s School)

Silver Award

Wong Chun Wai (Form 7, Choi Hung Estate Catholic Secondary School)
Chao Khek Lun (Form 5, St. Paul’s College)

Bronze Award

Law Ka Ho (Form 7, Queen Elizabeth School)
Ng Ka Chun (Form 5, Queen Elizabeth School)
Yu Hok Pun (Form 4, SKH Bishop Baker Secondary School)
Chan Kin Hang (Form 6, Bishop Hall Jubilee School)

Honorable Mention

Ng Ka Wing (Form 7, STFA Leung Kau Kui College)
Chau Suk Ling (Form 5, Queen Elizabeth School)
Choy Ting Pong (Form 7, Ming Kei College)

Based on the APMO and previous test results, the following trainees were selected to be the Hong Kong team members to the 2000 International Mathematical Olympiad, which was held in July in South Korea.

Wong Chun Wai (Form 7, Choi Hung Estate Catholic Secondary School)
Ng Ka Wing (Form 7, STFA Leung Kau Kui College)
Law Ka Ho (Form 7, Queen Elizabeth School)
Chan Kin Hang (Form 6, Bishop Hall Jubilee School)
Yu Hok Pun (Form 4, SKH Bishop Baker Secondary School)
Fan Wai Tong (Form 7, St. Mark’s School)