Mathematical Excalibur

Volume 5, Number 3 May 2000 – Septl 2000

Olympiad Corner

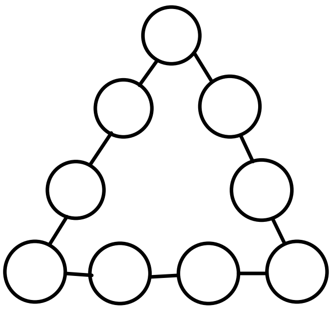
XII Asia Pacific Math Olympiad, March 2000:

Time allowed: 4 Hours

Problem 1. Compute the sum

$$S=\sum_{i=0}^{101}\frac{x_i^3}{1-3x_i+3x_i^2} \text{ for } x_i=\frac{i}{101}.$$

Problem 2. Given the following triangular arrangement of circles:



Each of the numbers 1,2,...,9 is to be written into one of these circles, so that each circle contains exactly one of these numbers and

(i) the sums of the four numbers on each side of the triangle are equal;

(continued on page 4)

Acknowledgment: Thanks to Janet Wong, MATH Dept, HKUST for general assistance.

On-line: http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is October 10, 2000.

For individual subscription for the next five issues for the 00-01 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

Dr. Kin-Yin Li
Department of Mathematics
Hong Kong University of Science and Technology
Clear Water Bay, Kowloon, Hong Kong

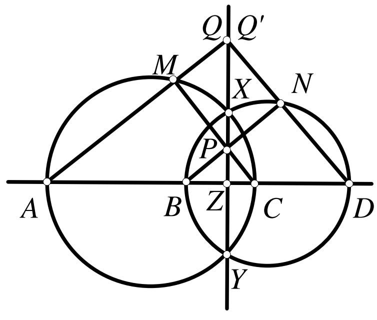
Fax: 2358-1643
Email: makyli@ust.hk

Coordinate Geometry

Kin Y. Li

When we do a geometry problem, we should first look at the given facts and the conclusion. If all these involve intersection points, midpoints, feet of perpendiculars, parallel lines, then there is a good chance we can solve the problem by coordinate geometry. However, if they involve two or more circles, angle bisectors and areas of triangles, then sometimes it is still possible to solve the problem by choosing a good place to put the origin and the x-axis. Below we will give some examples. *It is important to stay away from messy computations !*

Example 1. (1995 IMO) Let *A, B, C* and *D* be four distinct points on a line, in that order. The circles with diameters *AC* and *BD* intersect at the points *X* and *Y*. The line *XY* meets *BC* at the point *Z*. Let *P* be a point on the line *XY* different from *Z*. The line *CP* intersects the circle with diameter *AC* at the points *C* and *M*, and the line *BP* intersects the circle with diameter *BD* at the points *B* and *N*. Prove that the lines *AM, DN*, and *XY* are concurrent.

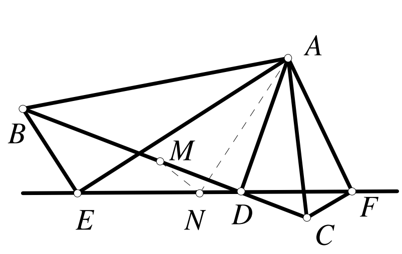


(*Remarks*. Quite obvious we should set the origin at *Z*. Although the figure is not symmetric with respect to line *XY*, there are pairs such as *M, N* and *A, D* and *B, C* that are symmetric in *roles*! So we work on the left half of the figure, the computations will be similar for the right half.)

Solution. (*Due to Mok Tze Tao, 1995 Hong Kong Team Member*) Set the origin at *Z* and the *x*-axis on line *AD*. Let the coordinates of the circumcenters of triangles *AMC* and *BND* be $(x_1, 0)$ and $(x_2, 0)$, and the circumradii be r_1 and r_2, respectively. Then the coordinates of A and *C* are $(x_1-r_1, 0)$ and $(x_1+r_1, 0)$, respectively. Let the coordinates of *P* be $(0, y_0)$. Since $AM \perp CP$ and the slope of *CP* is $-y_0/(x_1+r_1)$, the equation of *AM* works out to be $(x_1+r_1)x-y_0y=x_1^2-r_1^2$. Let *Q* be the intersection of *AM* with *XY*, then *Q* has coordinates $(0,(r_1^2-x_1^2)/y_0)$.

Similarly, let *Q'* be the intersection of *DN* with *XY*, then *Q'* has coordinates $(0,(r_2^2-x_2^2)/y_0)$. Since $r_1^2-x_1^2=ZX^2=r_2^2-x_2^2$, so $Q=Q'$.

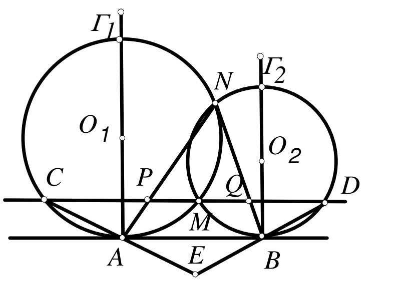
Example 2. (1998 APMO) Let *ABC* be a triangle and *D* the foot of the altitude from *A*. Let *E* and *F* be on a line passing through *D* such that *AE* is perpendicular to *BE, AF* is perpendicular to *CF*, and *E* and *F* are different from *D*. Let *M* and *N* be the midpoints of the line segments *BC* and *EF*, respectively. Prove that *AN* is perpendicular to *NM*.



(*Remarks*. We can set the origin at *D* and the *x*-axis on line *BC*. Then computing the coordinates of *E* and *F* will be a bit messy. A better choice is to set the line through *D,E.F* horizontal.)

Solution. (*Due to Cheung Pok Man, 1998 Hong Kong Team Member*) Set the origin at A and the x-axis parallel to line EF. Let the coordinates of D, E, F be $(d, b), (e, b), (f, b)$, respectively. The case $b=0$ leads to $D=E$, which is not allowed. So we may assume $b\neq 0$. Since $BE \perp AE$ and the slope of AE is b/e, so the equation of line BE works out to be $ex+by=e^2+b^2$. Similarly, the equations of lines CF and BC are $fx+by=f^2+b^2$ and $dx+by=d^2+b^2$, respectively. Solving the equations for BE and BC, we find B has coordinates $(d+e, b-(de/b))$. Similarly, C has coordinates $(d+f, b-(df/b))$. Then M has coordinates $(d+(e+f)/2, b-(de+df)/(2b))$ and N has coordinates $((e+f)/2, b)$. So the slope of AN is $2b/(e+f)$ and the slope of MN is $-(e+f)/(2b)$. Therefore, $AN \perp MN$.

Example 3. (2000 IMO) Two circles Γ_1 and Γ_2 intersect at M and N. Let ℓ be the common tangent to Γ_1 and Γ_2 so that M is closer to ℓ than N is. Let ℓ touch Γ_1 at A and Γ_2 at B. Let the line through M parallel to ℓ meet the circle Γ_1 again at C and the circle Γ_2 again at D. Lines CA and DB meet at E; lines AN and CD meet at P; lines BN and CD meet at Q. Show that $EP=EQ$.



(*Remarks.* Here if we set the x-axis on the line through the centers of the circles, then the equation of the line AB will be complicated. So it is better to have line AB on the x-axis.)

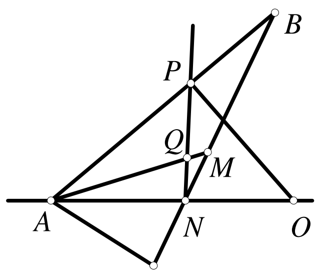
Solution. Set the origin at A and the x-axis on line AB. Let B, M have coordinates $(b,0)$, (s,t), respectively. Let the centers O_1, O_2 of Γ_1, Γ_2 be at $(0, r_1)$, (b, r_2), respectively. Then C, D have coordinates $(-s, t)$, $(2b-s,t)$, respectively. Since AB, CD are parallel, $CD=2b=2AB$ implies A, B are midpoints of CE, DE, respectively. So E is at $(s, -t)$. We see $EM \perp CD$.

To get $EP=EQ$, it is now left to show M is the midpoint of segment PQ. Since $O_1O_2 \perp MN$ and the slope of O_1O_2 is $(r_2-r_1)/b$, the equation of line MN is $bx+(r_2-r_1)y=bs+(r_2-r_1)t$. (This line should pass through the midpoint of segment AB.) Since $O_2M=r_2$ and $O_1M=r_1$, we get

$$(b-s)^2+(r_2-t)^2=r_2^2 \quad \text{and}$$
$$s^2+(r_1-t)^2=r_1^2.$$

Subtracting these equations, we get $b^2/2=bs+(r_2-r_1)t$, which implies $(b/2, 0)$ is on line MN. Since PQ, AB are parallel and line MN intersects AB at its midpoint, then M must be the midpoint of segment PQ. Together with $EM \perp PQ$, we get $EP=EQ$.

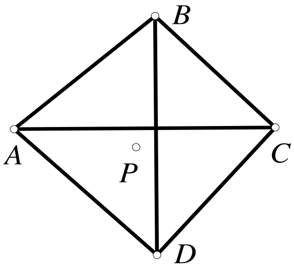
Example 4. (2000 APMO) Let ABC be a triangle. Let M and N be the points in which the median and the angle bisector, respectively, at A meet the side BC. Let Q and P be the points in which the perpendicular at N to NA meets MA and BA, respectively, and O the point in which the perpendicular at P to BA meets AN produced. Prove that QO is perpendicular to BC.



(*Remarks.* Here the equation of the angle bisector is a bit tricky to obtain unless it is the x-axis. In that case, the two sides of the angle is symmetric with respect to the x-axis.)

Solution. *(Due to Wong Chun Wai, 2000 Hong Kong Team Member)* Set the origin at N and the x-axis on line NO. Let the equation of line AB be $y=ax+b$, then the equation of lines AC and PO are $y=-ax-b$ and $y=(-1/a)x+b$, respectively. Let the equation of BC be $y=cx$. Then B has coordinates $(b/(c-a), bc/(c-a))$, C has coordinates $(-b/(c+a), -bc(c+a))$, M has coordinates $(ab/(c^2-a^2), abc/(c^2-a^2))$, A has coordinates $(-b/a, 0)$, O has coordinates $(ab, 0)$ and Q has coordinates $(0, ab/c)$. Then BC has slope c and QO has slope $-1/c$. Therefore, $QO \perp BC$.

Example 5. (1998 IMO) In the convex quadrilateral $ABCD$, the diagonals AC and BD are perpendicular and the opposite sides AB and DC are not parallel. Suppose that the point P, where the perpendicular bisectors of AB and DC meet, is inside $ABCD$. Prove that $ABCD$ is a cyclic quadrilateral if and only if the triangles ABP and CDP have equal areas.



(*Remarks*. The area of a triangle can be computed by taking the half length of the cross product. A natural candidate for the origin is P and having the diagonals parallel to the axes will be helpful.)

Solution. (*Due to Leung Wing Chung, 1998 Hong Kong Team Member*) Set the origin at P and the x-axis parallel to line AC. Then the equations of lines AC and BD are $y=p$ and $x=q$, respectively. Let $AP=BP=r$ and $CP=DP=s$. Then the coordinates of A, B, C, D are $(-\sqrt{r^2-p^2}, p)$, $(q, \sqrt{r^2-q^2})$, $(\sqrt{s^2-p^2}, p)$, $(q, -\sqrt{s^2-q^2})$, respectively. Using the determinant formula for finding the area of a triangle, we see that the areas of triangles ABP and CDP are equal if and only if

$$-\sqrt{r^2-p^2}\sqrt{r^2-q^2}-pq=-\sqrt{s^2-p^2}\sqrt{s^2-q^2}-pq.$$

Since $f(x) = -\sqrt{x^2-p^2}\ \sqrt{x^2-q^2}-pq$ is strictly decreasing when $x \geq |p|$ *and* $|q|$, equality of areas hold if and only if $r=s$, which is equivalent to A, B, C, D concyclic (since P being on the perpendicular bisectors of AB, CD is the only possible place for the center).

After seeing these examples, we would like to remind the readers that there are pure geometric proofs to each of the problems. For examples (1) and (3), there are proofs that only take a few lines. We encourage the readers to discover these simple proofs.

Although in the opinions of many people, a pure geometric proof is better and more beautiful than a coordinate geometric proof, we should point out that sometimes the coordinate geometric proofs may be preferred when there are many cases. For example (2), the different possible orderings of the points D, E, F on the line can all happen as some pictures will show. The coordinate geometric proofs above cover all cases.

Problem Corner

We welcome readers to submit solutions to the problems posed below for publication consideration. Solutions should be preceded by the solver's name, home address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is *October 10, 2000.*

Problem 106. Find all positive integer ordered pairs (a,b) such that

$$gcd(a,b)+lcm(a,b)=a+b+6,$$

where gcd stands for greastest common divisor (or highest common factor) and lcm stands for least common multiple.

Problem 107. For $a, b, c > 0$, if $abc=1$, then show that

$$\frac{b+c}{\sqrt{a}}+\frac{c+a}{\sqrt{b}}+\frac{a+b}{\sqrt{c}} \ge \sqrt{a}+\sqrt{b}+\sqrt{c}+3.$$

Problem 108. Circles C_1 and C_2 with centers O_1 and O_2 (respectively) meet at points A, B. The radii O_1B and O_2B intersect C_1 and C_2 at F and E. The line parallel to EF through B meets C_1 and C_2 at M and N, respectively. Prove that $MN=AE+AF$. (Source: 17th Iranian Mathematical Olympiad)

Problem 109. Show that there exists an increasing sequence a_1, a_2, a_3, ... of positive integers such that for every nonnegative integer k,the sequence $k+a_1$, $k+a_2$, $k+a_3$ ... contains only finitely many prime numbers. (Source: 1997 Math Olympiad of Czech and Slovak Republics)

Problem 110. In a park, 10000 trees have been placed in a square lattice. Determine the maximum number of trees that can be cut down so that from any stump, you cannot see any other stump. (Assume the trees have negligible radius compared to the distance between adjacent trees.) (Source: 1997 German Mathematical Olympiad)

Comments. You may think of the trees being placed at (x,y), where $x, y = 0, 1, 2, ..., 99$.

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Solutions

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Problem 101. A triple of numbers (a_1, a_2, a_3)=(3, 4, 12) is given. We now perform the following operation: choose two numbers a_i and a_j, $(i \ne j)$, and exchange them by $0.6a_i-0.8a_j$ and $0.8a_i+0.6a_j$. Is it possible to obtain after several steps the (unordered) triple (2, 8, 10) ? (*Source:* 1999 National Math Competition in Croatia)

Solution. **FAN Wai Tong** (St. Mark's School, Form 7), **KO Man Ho** (Wah Yan College, Kowloon, Form 6) and **LAW Hiu Fai** (Wah Yan College, Kowloon, Form 6).

Since $(0.6a_i-0.8a_j)^2 + (0.8a_i+0.6a_j)^2 = a_i^2+a_j^2$, the sum of the squares of the triple of numbers before and after an operation stays the same. Since $3^2 + 4^2 + 12^2 \ne 2^2 + 8^2 + 10^2$, so (2,8,10) cannot be obtained.

Problem 102. Let a be a positive real number and $(x_n)_{n\ge 1}$ be a sequence of real numbers such that $x_1=a$ and

$$x_{n+1} \ge (n+2)x_n - \sum_{k=1}^{n-1} kx_k, \text{ for all } n \ge 1.$$

Show that there exists a positive integer n such that $x_n > 1999!$ (*Source: 1999 Romanian Third Selection Examination*)

Solution. **FAN Wai Tong** (St. Mark's School, Form 7).

We will prove by induction that $x_{j+1} \ge 3x_j$ for every positive integer j. The case $j=1$ is true by the given inequality. Assume the cases j =1, ..., $n-1$ are true. Then $x_n \ge 3x_{n-1} \ge 9x_{n-2} \ge ...$ and

$$\frac{x_{n+1}}{x_n} \ge (n+2) - \sum_{k=1}^{n-1} \frac{kx_k}{x_n}$$

$$\ge (n+2) - \sum_{k=1}^{n-1} \frac{n-1}{3^{n-k}}$$

$$\ge (n+2)-(n-1)(\frac{1}{3}+\frac{1}{9}+...)$$

$$=\frac{n+5}{2}$$

$$\ge 3.$$

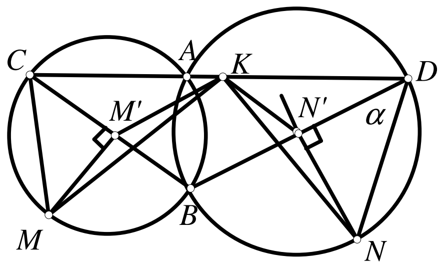
So the case $j = n$ is also true.

Since $a > 0$, we can take

$$n > 1+\log_3 (1999!/a).$$

Then $x_n \ge 3^{n-1}x_1 = 3^{n-1}a > 1999!$.

Problem 103. Two circles intersect in points A and B. A line l that contains the point A intersects again the circles in the points C, D, respectively. Let M, N be the midpoints of the arcs BC and BD, which do not contain the point A, and let K be the midpoint of the segment CD. Show that $\angle MKN=90°$. (*Source:* 1999 Romanian Fourth Selection Examination)



Solution. **FAN Wai Tong** (St. Mark's School, Form 7)

Let M' and N' be the midpoints of chords BC and BD respectively. From the midpoint theorem, we see that $BM'KN'$ is a parallelogram. Now

$$\angle KN'N = \angle KN'B + 90^\circ$$
$$= \angle KM'B + 90^\circ$$
$$= \angle KM'M.$$

Let $\alpha = \angle NDB = \angle NAB$. Then

$$\frac{KN'}{N'N} = \frac{M'B}{N'D \tan\alpha} = \frac{\frac{1}{2}BC}{\frac{1}{2}BD\tan\alpha}.$$

Now

$$\angle MCB = \angle MCB = \frac{1}{2}\angle CAB$$
$$= \frac{1}{2}(180^\circ - \angle DAB)$$
$$= 90^\circ - \angle NAB$$
$$= 90^\circ - \alpha.$$

So

$$\frac{MM'}{M'K}=\frac{CM'\cot\alpha}{BN'}=\frac{\frac{1}{2}BC\cot\alpha}{\frac{1}{2}BD}.$$

Then $KN'/N'N=MM'/M'K$. So triangles $MM'K, KN'N$ are similar. Then $\angle M'KM = \angle N'NK$ and

$$\begin{aligned}\angle MKN &= \angle M'KN' - \angle M'KM - \angle N'KN\\ &= \angle KN'D - (\angle N'NK + \angle N'KN)\\ &= 90^{\circ}\end{aligned}$$

Other commended solvers: **WONG Chun Wai** (Choi Hung Estate Catholic Secondary School, Form 7).

Problem 104. Find all positive integers n such that 2^n-1 is a multiple of 3 and $(2^n-1)/3$ is a divisor of $4m^2+1$ for some integer m. (*Source:* 1999 Korean Mathematical Olympiad)

Solution. (Official Solution)

(Some checkings should suggest n is a power of 2.) Now 2^n-1 is a multiple of 3 if and only if $(-1)^n\equiv 2^n\equiv$(mod 3), that is n is even. Suppose for some even n, $(2^n-1)/3$ is a divisor of $4m^2+1$ for some m. Assume n has an odd prime divisor d. Now $2^d-1\equiv 3$ (mod 4) implies one of its prime divisor p is of the form $4k+3$. Then p divides 2^d-1, which divides 2^n-1, which divides $4m^2+1$. Then p and $2m$ are relatively prime and so

$$1\equiv(2m)^{p-1}=(4m^2)^{2k+1}\equiv-1 \pmod p,$$

a contradiction. So n cannot have any odd prime divisor. Hence $n=2^j$ for some positive integer j.

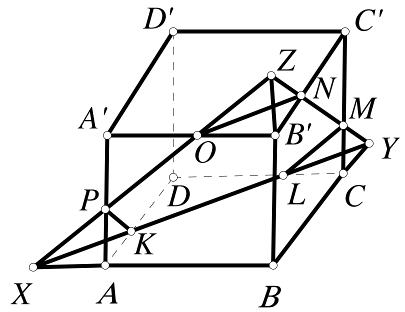
Conversely, suppose $n=2^j$. Let $F_i=2^{2^i}+1$. Using the factorization $2^{2b}-1=(2^b-1)\times(2^b+1)$ repeatedly on the numerator, we get

$$\frac{2^n-1}{3}=F_1F_2\cdots F_{j-1}.$$

Since F_i divides F_j-2 for $i<j$, the F_i's are pairwise relatively prime. By the Chinese remainder theorem, there is a positive integer x satisfying the simultaneous equations $x\equiv 0$ (mod 2) and $x\equiv 2^{2^{i-1}}$ (mod F_i) for i=1, 2, ..., $j-1$. Then $x=2m$ for some positive integer m and $4m^2+1=x^2+1\equiv 0$ (mod F_i) for i=1, 2,…$j-1$. So $4m^2+1$ is divisible by $F_1F_2\ldots F_{j-1}=(2^n-1)/3$.

Problem 105. A rectangular parallelepiped (box) is given, such that its intersection with a plane is a regular hexagon. Prove that the rectangular parallelepiped is a cube. (*Source:* 1999 National Math Olympiad in Slovenia)

Solution. (Official Solution)



As in the figure, an equilateral triangle XYZ is formed by extending three alternate sides of the regular hexagon.

The right triangles XBZ and YBZ are congruent as they have a common side BZ and the hypotenuses have equal length. So $BX=BY$ and similarly $BX=BZ$. As the pyramids $XBYZ$ and $OB'NZ$ are similar and $ON=\frac{1}{3}XY$, it follows $B'Z=\frac{1}{3}BZ$. Thus we have $BB'=\frac{2}{3}BZ$ and similarly $AB=\frac{2}{3}BX$ and $CB=\frac{2}{3}BY$. Since $BX=BY=BZ$, we get $AB=BC=BB'$.

Other commended solvers: **FAN Wai Tong** (St. Mark's School, Form 7).

Olympiad Corner

(continued from page 1)

(ii) the sums of the squares of the four numbers on each side of the triangle are equal.

Find all ways in which this can be done.

Problem 3. Let ABC be a triangle. Let M and N be the points in which the median and the angle bisector, respectively, at A meet the side BC. Let Q and P be the points in which the perpendicular at N to NA meets MA and BA, respectively, and O the point in which the perpendicular at P to BA meets AN produced. Proved that QO is perpendicular to BC.

Problem 4. Let n, k be given positive integers with n > k. Prove that

$$\frac{1}{n+1}\cdot\frac{n^n}{k^k(n-k)^{n-k}}<\frac{n!}{k!(n-k)!}<\frac{n^n}{k^k(n-k)^{n-k}}.$$

Problem 5. Given a permutation $(a_0, a_1, ..., a_n)$ of the sequence 0,1,..., n. A transposition of a_i with a_j is called *legal* if $i>0$, $a_i=0$ and $a_{i-1}+1=a_j$. The permutation $(a_0, a_1, ..., a_n)$ is called *regular* if after a number of legal transpositions it becomes (1,2, ...,n,0). For which numbers n is the permutation (1, n, $n-1$, ..., 3, 2, 0) regular ?

2000 APMO and IMO

In April this year, Hong Kong IMO trainees participated in the XII Asia Pacific Mathematical Olympiad. The winners were

<u>Gold Award</u>

Fan Wai Tong (Form 7, St Mark's School)

<u>Silver Award</u>

Wong Chun Wai (Form 7, Choi Hung Estate Catholic Secondary School)
Chao Khek Lun (Form 5, St. Paul's College)

<u>Bronze Award</u>

Law Ka Ho (Form 7, Queen Elizabeth School)
Ng Ka Chun (Form 5, Queen Elizabeth School)
Yu Hok Pun (Form 4, SKH Bishop Baker Secondary School)
Chan Kin Hang (Form 6, Bishop Hall Jubilee School)

<u>Honorable Mention</u>

Ng Ka Wing (Form 7, STFA Leung Kau Kui College)
Chau Suk Ling (Form 5, Queen Elizabeth School)
Choy Ting Pong (Form 7, Ming Kei College)

Based on the APMO and previous test results, the following trainees were selected to be the Hong Kong team members to the 2000 International Mathematical Olympiad, which was held in July in South Korea.

Wong Chun Wai (Form 7, Choi Hung Estate Catholic Secondary School)
Ng Ka Wing (Form 7, STFA Leung Kau Kui College)
Law Ka Ho (Form 7, Queen Elizabeth School)
Chan Kin Hang (Form 6, Bishop Hall Jubilee School)
Yu Hok Pun (Form 4, SKH Bishop Baker Secondary School)
Fan Wai Tong (Form 7, St. Mark's School)