Mathematical Excalibur

Volume 5, Number 4 September 2000 – November 2000

Olympiad Corner

The 41st International Mathematical Olympiad, July 2000:

Time allowed: 4 hours 30 minutes
Each problem is worth 7 points.

Problem 1. Two circles Γ_1 and Γ_2 intersect at M and N. Let ℓ be the common tangent to Γ_1 and Γ_2 so that M is closer to ℓ than N is. Let ℓ touches Γ_1 at A and Γ_2 at B. Let the line through M parallel to ℓ meets the circle Γ_1 again at C and the circle Γ_2 at D. Lines CA and DB meet at E; lines AN and CD meet at P; lines BN and CD meet at Q. Show that $EP = EQ$.

Problem 2. Let a, b, c be positive real numbers such that $abc = 1$. Prove that
$(a-1+1/b)(b-1+1/c)(c-1+1/a) \le 1$

Problem 3. Let $n \ge 2$ be a positive integer. Initially, there are n fleas on a horizontal line, not all at the same point. For a positive real number λ, define a move as follows:
Choose any two fleas, at points A and B, with A to the left of B; let the flea at A jump to the point C on the line to the line to the right of B with $BC/AB = \lambda$.

(continued on page 4)

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On-line: http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is December 10, 2000.

For individual subscription for the next five issues for the 01-02 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Jensen's Inequality

Kin Y. Li

In comparing two similar expressions, often they involve a common function. To see which expression is greater, the shape of the graph of the function on an interval is every important. A function f is said to be *convex* on an interval I if for any two points (x_1, $f(x_1)$) and (x_2, $f(x_2)$) on the graph, the segment joining these two points lie on or above the graph of the function over [x_1, x_2]. That is,

$$f((1-t)x_1+tx_2) \le (1-t)\ f(x_1)+tf(x_2)$$

for every t in [0, 1]. If f is continuous on I, then it is equivalent to have

$$f\left(\frac{x_1+x_2}{2}\right) \le \frac{f(x_1)+f(x_2)}{2}$$

for every x_1, x_2 in I. If furthermore f is differentiable, then it is equivalent to have a nondecreasing derivative. Also, f is *strictly convex on I* if f is convex on I and equality holds in the inequalities above only when $x_1 = x_2$. We say a function g is *concave on an interval I* if the function $-g$ is convex on I. Similarly, g is strictly concave on I if $-g$ is strictly convex on I.
The following are examples of strictly convex functions on intervals:

$$x^p \text{ on } [0, \infty) \text{ for } p > 1,$$

$$x^p \text{ on } (0, \infty) \text{ for } p < 0,$$

$$a^x \text{ on } (-\infty, \infty) \text{ for } a > 1,$$

$$\tan x \text{ on } [0, \tfrac{\pi}{2}).$$

The following are examples of strictly concave functions on intervals:

$$x^p \text{ on } [0, \infty) \text{ for } 0 < p < 1,$$

$$\log_a x \text{ on } (0, \infty) \text{ for } a > 1,$$

$$\cos x \text{ on } [-\pi/2, \pi/2],$$

$$\sin x \text{ on } [0, \pi].$$

The most important inequalities con-cerning these functions are the following.

Jensen's Inequality. *If f is convex on an interval I and* $x_1, x_2, ..., x_n$ *are in I, then*

$$f\left(\frac{x_1+x_2+\cdots+x_n}{n}\right) \le \frac{f(x_1)+f(x_2)+\cdots+f(x_n)}{n}.$$

For strictly convex functions, equality holds if and only if $x_1 = x_2 = \cdots = x_n$.

Generalized Jensen's Inequality. *Let f be continuous and convex on an interval I. If* $x_1, ..., x_n$ *are in I and* $0 < t_1, t_2, ..., t_n < 1$ *with* $t_1+t_2+\cdots+t_n = 1$, *then*

$$f(t_1x_1+t_2x_2+\cdots+t_nx_n) \le t_1f(x_1)+t_2f(x_2)+\cdots+t_nf(x_n)$$

(with the same equality condition for strictly convex functions).
Jensen's inequality is proved by doing a forward induction to get the cases $n = 2^k$, then a backward induction to get case $n-1$ from case n by taking x_n to be the arithmetic mean of $x_1, x_2, \ldots, x_{n-1}$. For the generalized Jensen's inequality, the case all t_i's are rational is proved by taking common denominator and the other cases are obtained by using continuity of the function and the density of rational numbers.
There are similar inequalities for concave and strictly concave functions by reversing the inequality signs.

Example 1. For a triangle ABC, show that $\sin A + \sin B + \sin C \le \frac{3\sqrt{3}}{2}$ and determine when equality holds.

Solution. Since $f(x) = \sin x$ is strictly concave on [0, π], so

$$\begin{aligned} &\sin A + \sin B + \sin C \\ &= f(A)+f(B)+f(C) \\ &\le 3f\left(\frac{A+B+C}{3}\right) \\ &= 3\sin\left(\frac{A+B+C}{3}\right) \\ &= \frac{3\sqrt{3}}{2}. \end{aligned}$$

Equality holds if and only if $A = B = C = \pi/3$, i.e. ΔABC is equilateral.

Example 2. If $a, b, c > 0$ and

$$a + b + c = 1,$$

then find the minimum of

$$\left(a+\frac{1}{a}\right)^{10}+\left(b+\frac{1}{b}\right)^{10}+\left(c+\frac{1}{c}\right)^{10}.$$

Solution. Note $0 < a, b, c < 1$. Let $f(x) = \left(x+\frac{1}{x}\right)^{10}$ on $I = (0, 1)$, then f is strictly convex on I because its second derivative

$$90\left(x+\frac{1}{x}\right)^8\left(1-\frac{1}{x^2}\right)^2+10\left(x+\frac{1}{x}\right)^9\left(\frac{2}{x^3}\right)$$

is positive on I. By Jensen's inequality,

$$\frac{10^{10}}{3^9}=3f\left(\frac{a+b+c}{3}\right)$$
$$\leq f(a)+f(b)+f(c)$$
$$=\left(a+\frac{1}{a}\right)^{10}+\left(b+\frac{1}{b}\right)^{10}+\left(c+\frac{1}{c}\right)^{10}.$$

So the minimum is $10^{10}/3^9$, attained when $a = b = c = 1/3$.

Example 3. Prove that AM-GM in-equality, which states that if a_1, $a_2, \ldots, a_n \geq 0$, then

$$\frac{a_1+a_2+\cdots+a_n}{n} \geq \sqrt[n]{a_1a_2\cdots a_n}.$$

Solution. If one of the a_i's is 0, then the right side is 0 and the inequality is clear. If $a_1, a_2, \ldots, a_n > 0$, then since $f(x) = \log x$ is strictly concave on $(0, \infty)$, by Jensen's inequality,

$$\log\left(\frac{a_1+a_2+\cdots+a_n}{n}\right)$$
$$\geq \frac{\log a_1+\log a_2+\cdots+\log a_n}{n}$$
$$=\log\left(\sqrt[n]{a_1a_2\cdots a_n}\right).$$

Exponentiating both sides, we get the AM-GM inequality.

Remarks. If we use the generalized Jensen's inequality instead, we can get the weighted AM-GM inequality. It states that if $a_1, \ldots, a_n > 0$ and $0 < t_1, \ldots, t_n < 1$ satisfying $t_1 + \cdots + t_n = 1$, then $t_1 a_1 + \cdots + t_n a_n \geq a_1^{t_1}\cdots a_n^{t_n}$ with equality if and only if all a_i's are equal.

Example 4. Prove the power mean inequality, which states that for a_1, $a_2, \ldots, a_n > 0$ and $s < t$, if

$$S_r=\left(\frac{a_1^r+a_2^r+\cdots+a_n^r}{n}\right)^{1/r},$$

then $S_s \leq S_t$. Equality holds if and only if $a_1 = a_2 = \cdots = a_n$.

Remarks. S_1 is the arithmetic mean (AM) and S_{-1} is the harmonic mean (HM) and S_2 is the root-mean-square (RMS) of a_1, $a_2, \cdots, a_n$. Taking limits, it can be shown that $S_{+\infty}$ is the maximum (MAX), S_0 is the geometric mean (GM) and $S_{-\infty}$ is the minimum (MIN) of $a_1, a_2, \cdots, a_n$.

Solution. In the cases $0 < s < t$ or $s < 0 < t$, we can apply Jensen's inequality to $f(x) = x^{t/s}$. In the case $s < t < 0$, we let $b_i = 1/a_i$ and apply the case $0 < -t < -s$. The other cases can be obtained by taking limit of the cases proved.

Example 5. Show that for $x, y, z > 0$,

$$x^5+y^5+z^5$$
$$\leq x^5\sqrt{\frac{x^2}{yz}}+y^5\sqrt{\frac{y^2}{zx}}+z^5\sqrt{\frac{z^2}{xy}}.$$

Solution. Let $a = \sqrt{x}$, $b = \sqrt{y}$, $c = \sqrt{z}$, then the inequality becomes

$$a^{10}+b^{10}+c^{10} \leq \frac{a^{13}+b^{13}+c^{13}}{abc}.$$

By the power mean inequality,

$$a^{13}+b^{13}+c^{13}=3S_{13}^{13}$$
$$=3S_{13}^{10}\,S_{13}^3 \geq 3S_{10}^{10}\,S_0^3$$
$$=\left(a^{10}+b^{10}+c^{10}\right)abc.$$

Example 6. Prove Hölder's inequality, which states that if $p, q > 1$ satisfy $\frac{1}{p}+\frac{1}{q} = 1$ and $a_1, \ldots, a_n, b_1, \ldots, b_n$ are real (or complex) numbers, then

$$\sum_{i=1}^n\left|a_ib_i\right| \leq \left(\sum_{i=1}^n\left|a_i\right|^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^n\left|b_i\right|^q\right)^{\frac{1}{q}}.$$

(The case $p = q = 2$ is the Cauchy-Schwarz inequality.)

Solution. Let

$$A=\left|a_1\right|^p+\cdots+\left|a_n\right|^p.$$
$$B=\left|b_1\right|^p+\cdots+\left|b_n\right|^q.$$

If A or B is 0, then either all a_i's or all b_i's are 0, which will make both sides of the inequality 0.

So we need only consider the case $A \neq 0$ and $B \neq 0$. Let $t_1 = 1/p$ and $t_2 = 1/q$, then $0 < t_1, t_2 < 1$ and $t_1 + t_2 = 1$. Let $x_i = \left|a_i\right|^p/A$ and $y_i = \left|b_i\right|^q/B$, then

$$x_1+\cdots+x_n=1, \qquad y_1+\cdots+y_n=1.$$

Since $f(x) = e^x$ is strictly convex on $(-\infty, \infty)$, by the generalized Jensen's inequality,

$$x_i^{1/p}y_i^{1/q} = f(t_1\ln x_i+t_2\ln y_i)$$
$$\leq t_1 f(\ln x_i)+t_2 f(\ln y_i)=\frac{x_i}{p}+\frac{y_i}{q}.$$

Adding these for $i = 1, \ldots, n$, we get

$$\sum_{i=1}^n\frac{|a_i||b_i|}{A^{1/p}B^{1/q}}=\sum_{i=1}^n x_i^{1/p}y_i^{1/q}$$
$$\leq \frac{1}{p}\sum_{i=1}^n x_i+\frac{1}{q}\sum_{i=1}^n y_i=1.$$

Therefore,

$$\sum_{i=1}^n|a_i||b_i| \leq A^{1/p}B^{1/q}$$
$$=\left(\sum_{i=1}^n|a_i|^p\right)^{1/p}\left(\sum_{i=1}^n|b_i|^q\right)^{1/q}.$$

Example 7. If $a, b, c, d > 0$ and

$$c^2+d^2=(a^2+b^2)^3,$$

then show that

$$\frac{a^3}{c}+\frac{b^3}{d} \geq 1.$$

Solution 1. Let

$$x_1=\sqrt{a^3/c}, \quad x_2=\sqrt{b^3/d},$$
$$y_1=\sqrt{ac}, \quad y_2=\sqrt{bd}.$$

By the Cauchy-Schwarz inequality,

$$\left(\frac{a^3}{c}+\frac{b^3}{d}\right)(ac+bd)$$
$$=\left(x_1^2+x_2^2\right)\left(y_1^2+y_2^2\right)$$
$$\geq (x_1y_1+x_2y_2)^2$$
$$=(a^2+b^2)^2$$
$$=\sqrt{(a^2+b^2)(c^2+d^2)}$$
$$\geq ac+bd.$$

Cancelling $ac + bd$ on both sides, we get the desired inequality.

Solution 2. Let

$$x=(a^3/c)^{2/3}, \quad y=(b^3/d)^{2/3}.$$

By the $p = 3$, $q = 3/2$ case of Hölder's inequality,

$$a^2+b^2$$
$$=(c^{2/3})x+(d^{2/3})y$$
$$\leq (c^2+d^2)^{1/3}(x^{3/2}+y^{3/2})^{2/3}$$

Cancelling $a^2+b^2=(c^2+d^2)^{1/3}$ on both sides, we get $1 \leq x^{3/2}+y^{3/2} = (a^3/c)+(b^3/d)$.

Problem Corner

We welcome readers to submit solutions to the problems posed below for publication consideration. Solutions should be preceeded by the solver's name, home address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics*, *Hong Kong University of Science & Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is *December 10, 2000.*

Problem 111. Is it possible to place 100 solid balls in space so that no two of them have a common interior point, and each of them touches at least one-third of the others? *(Source: 1997 Czech-Slovak Match)*

Problem 112. Find all positive integers (x, n) such that x^n+2^n+1 is divisor of $x^{n+1}+2^{n+1}+1$. (*Source: 1998 Romanian Math Olympiad*)

Problem 113. Let $a, b, c > 0$ and $abc \le 1$. Prove that

$$\frac{a}{c}+\frac{b}{a}+\frac{c}{b}\ge a+b+c.$$

(Hint: Consider the case $abc = 1$ first.)

Problem 114. (*Proposed by Mohammed Aassila, Universite Louis Pasteur, Strasbourg, France*) An infinite chessboard is given, with n black squares and the remainder white. Let the collection of black squares be denoted by G_0. At each moment $t = 1, 2, 3, \ldots$, a simultaneous change of colour takes place throughout the board according to the following rule: every square gets the colour that dominates in the three square configuration consisting of the square itself, the square above and the square to the right. New collections of black squares G_1, G_2, G_3, … are so formed. Prove that G_n is empty.

Problem 115. (*Proposed by Mohammed Aassila, Universite Louis Pasteur, Strasbourg, France*) Find the locus of the points P in the plane of an equilateral triangle ABC for which the triangle formed with lengths PA, PB and PC has constant area.

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Solutions

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Problem 106. Find all positive integer ordered pairs (a, b) such that

$$\gcd(a,b)+\operatorname{lcm}(a,b)=a+b+6,$$

where gcd stands for greatest common divisor (or highest common factor) and lcm stands for least common multiple.

Solution. **CHAN An Jack** and **LAW Siu Lun Jack** (Mei Kei College, Form 6), **CHAN Chin Fei** (STFA Leung Kau Kui College), **CHAO Khek Lun Harold** (St. Paul's College, Form 6), **CHAU Suk Ling** (Queen Elizabeth School, Form 6), **CHENG Man Chuen** (Tsuen Wan Government Secondary School, Form 7), **FUNG Wing Kiu Ricky** (La Salle College), **HUNG Chung Hei** (Pui Ching Middle School, Form 5), **KO Man Ho** (Wah Yan College, Kowloon, Form 7), **LAM Shek Ming Sherman** (La Salle College, Form 5), **LAW Ka Ho** (HKU, Year 1), **LEE Kevin** (La Salle College), **LEUNG Wai Ying** (Queen Elizabeth School, Form 6), **MAK Hoi Kwan Calvin** (La Salle College), **OR Kin** (SKH Bishop Mok Sau Tseng Secondary School), **POON Wing Sze Jessica** (STFA Leung Kau Kui College, Form 7), **TANG Sheung Kon** (STFA Leung Kau Kui College, Form 6), **TONG Chin Fung** (SKH Lam Woo Memorial Secondary School, Form 6), **WONG Wing Hong** (La Salle College, Form 3) and **YEUNG Kai Shing** (La Salle College, Form 4).

Let $m = \gcd(a, b)$, then $a = mx$ and $b = my$ with $\gcd(x, y) = 1$. In that case, $\operatorname{lcm}(a, b) = mxy$. So the equation becomes $m + mxy = mx + my + 6$. This is equivalent to $m(x-1)(y-1) = 6$. Taking all possible positive integer factorizations of 6 and requiring $\gcd(x, y) = 1$, we have $(m, x, y) = (1, 2, 7)$, $(1, 7, 2)$, $(1, 3, 4)$, $(1, 4, 3)$, $(3, 2, 3)$ and $(3, 3, 2)$. Then $(a, b) = (2, 7)$, $(7, 2)$, $(3, 4)$, $(4, 3)$, $(6, 9)$ and $(9, 6)$. Each of these is easily checked to be a solution.

Other recommended solvers: **CHAN Kin Hang Andy** (Bishop Hall Jubilee School, Form 7) and **CHENG Kei Tsi Daniel** (La Salle College, Form 6).

Problem 107. For $a, b, c > 0$, if $abc = 1$, then show that

$$\frac{b+c}{\sqrt{a}}+\frac{c+a}{\sqrt{b}}+\frac{a+b}{\sqrt{c}}\ge\sqrt{a}+\sqrt{b}+\sqrt{c}+3.$$

Solution 1. **CHAN Hiu Fai Philip** (STFA Leung Kau Kui College, Form 7), **LAW Ka Ho** (HKU, Year 1) and **TSUI Ka Ho Willie** (Hoi Ping Chamber of Commerce Secondary School, Form 7).

By the AM-GM inequality and the fact $abc = 1$, we get

$$\frac{b+c}{\sqrt{a}}+\frac{c+a}{\sqrt{b}}+\frac{a+b}{\sqrt{c}}\ge 2\left(\sqrt{\frac{bc}{a}}+\sqrt{\frac{ca}{b}}+\sqrt{\frac{ab}{c}}\right)$$

$$=\left(\sqrt{\frac{ca}{b}}+\sqrt{\frac{ab}{c}}\right)+\left(\sqrt{\frac{ab}{c}}+\sqrt{\frac{bc}{a}}\right)+\left(\sqrt{\frac{bc}{a}}+\sqrt{\frac{ca}{b}}\right)\ge 2\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)\ge$$

$$\sqrt{a}+\sqrt{b}+\sqrt{c}+3\sqrt[6]{abc}=\sqrt{a}+\sqrt{b}+\sqrt{c}+3.$$

Solution 2. **CHAN Kin Hang Andy** (Bishop Hall Jubliee School, Form 7), **CHAO Khek Lun Harold** (St. Paul's College, Form 6), **CHAU Suk Ling** (Queen Elizabeth School, Form 6), **CHENG Kei Tsi** (La Salle College, Form 6), **CHENG Man Chuen** (Tsuen Wan Government Secondary School, Form 7), **LAW Ka Ho** (HKU, Year 1) and **LEUNG Wai Ying** (Queen Elizabeth School, Form 6).

Without loss of generality, assume $a\ge b\ge c$. Then $1/\sqrt{a}\le 1/\sqrt{b}\le 1/\sqrt{c}$. By the rearrangement inequality,

$$\frac{b}{\sqrt{a}}+\frac{c}{\sqrt{b}}+\frac{a}{\sqrt{c}}\ge\frac{a}{\sqrt{a}}+\frac{b}{\sqrt{b}}+\frac{c}{\sqrt{c}}=\sqrt{a}+\sqrt{b}+\sqrt{c}$$

Also, by the AM-GM inequality,

$$\frac{c}{\sqrt{a}}+\frac{a}{\sqrt{b}}+\frac{b}{\sqrt{c}}\ge 3.$$

Adding these two inequalities, we get the desired inequality.

Generalization: Professor Murray S. Klamkin (University of Alberta, Canada) sent in a solution, which proved a stronger inequality and later generalized it to n variables. He made the sub-stitutions $x_1=\sqrt{a}$, $x_2=\sqrt{b}$, $x_3=\sqrt{c}$ to get rid of square roots and let $S_m = x_1^m+x_2^m+x_3^m$ so that the inequality became

$$\frac{x_2^2+x_3^2}{x_1}+\frac{x_3^2+x_1^2}{x_2}+\frac{x_1^2+x_2^2}{x_3}\ge S_1+3.$$

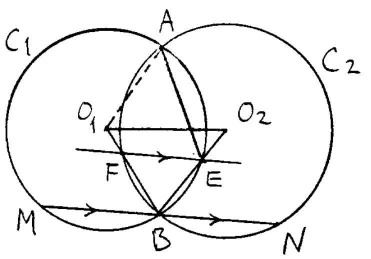
By the AM-GM inequality, $S_m \ge 3\sqrt[3]{x_1^m x_2^m x_3^m} = 3$. Since $S_2/3\ge(S_1/3)^2 \ge S_1/3$ by the power mean inequality, we would get a stronger inequality by replacing $S_1 + 3$ by $2S_2$. Rearranging terms, this stronger inequality could be rewritten as $S_2(S_{-1} - 3) \ge S_1 - S_2$. Now the left side is nonnegative, but the right side is nonpositive. So the stronger inequality is true. If we replace 3 by n

and assume $x_1 \cdots x_n = 1$, then as above, we will get $S_m(S_{1-m} - n) \ge S_1 - S_m$ by the AM-GM and power mean inequalities. Expanding and regrouping terms, we get the stronger inequality in n variables, namely

$$\sum_{i=1}^{n} \frac{S_m - x_i^m}{x_i^{m-1}} \ge (n-1)S_m.$$

Other recommended solvers: **CHAN Chin Fei** (STFA Leung Kau Kui College), **LAM Shek Ming Sherman** (La Salle College, Form 5), **LAW Hiu Fai** (Wah Yan College, Kowloon, Form 7), **LEE Kevin** (La Salle College, Form 5), **MAK Hoi Kwan Calvin** (La Salle College), **OR Kin** (SKH Bishop Mok Sau Tseng Secondary School) and **YEUNG Kai Shing** (La Salle College, Form 4).

Problem 108. Circles C_1 and C_2 with centers O_1 and O_2 (respectively) meet at points A, B. The radii O_1B and O_2B intersect C_1 and C_2 at F and E. The line parallel to EF through B meets C_1 and C_2 at M and N, respectively. Prove that $MN = AE + AF$. (*Source: 17th Iranian Mathematical Olympiad*)



Solution. **YEUNG Kai Shing** (La Salle College, Form 4).

As the case $F = E = B$ would make the problem nonsensible, the radius O_1B of C_1 can only intersect C_2, say at F. Then the radius O_2B of C_2 intersect C_1 at E. Since $\Delta\ EO_1B$ and $\Delta\ FO_2B$ are isosceles, $\angle EO_1F = 180^\circ - 2\angle FBE = \angle EO_2F$. Thus, E, O_2, O_1, F are concyclic. Then $\angle AEB = (360^\circ - \angle AO_1B)/2 = 180^\circ - \angle O_2O_1F = \angle O_2EF = \angle EBM$. So arc$AMB$ = arcMAE. Subtracting minor arcAM from both sides, we get minor arcMB = minor arcAE. So $MB = AE$. Similarly, $NB = AF$. Then $MN = MB + NB = AE + AF$.

Other recommended solvers: **Chan Kin Hang Andy** (Bishop Hall Jubilee School, Form 7), **CHAU Suk Ling** (Queen Elizabeth School, Form 6) and **LEUNG Wai Ying** (Queen Elizabeth School, Form 6).

Problem 109. Show that there exists an increasing sequence a_1, a_2, a_3, … of positive integers such that for every nonnegative integer k, the sequence $k + a_1$, $k + a_2$, $k + a_3$, … contains only finitely many prime numbers. (*Source: 1997 Math Olympiad of Czech and Slovak Republics*)

Solution. **CHAU Suk Ling** (Queen Elizabeth School, Form 6), **CHENG Kei Tsi** (La Salle College, Form 6), **CHENG Man Chuen** (Tsuen Wan Government Secondary School, Form 7), **LAM Shek Ming Sherman** (La Salle College, Form 5), **LAW Hiu Fai** (Wah Yan College, Kowloon, Form 7), **LAW Ka Ho** (HKU, Year 1) and **YEUNG Kai Shing** (La Salle College, Form 4).

Let $a_n = n! + 2$. Then for every non-negative integer k, if $n \ge k + 2$, then $k + a_n$ is divisible by $k + 2$ and is greater than $k + 2$, hence not prime.

Other commended solvers: **CHAN Kin Hang Andy** (Bishop Hall Jubliee School, Form 7), **KO Man Ho** (Wah Yan College, Form 7), **LEE Kevin** (La Salle College, Form 5) and **LEUNG Wai Ying** (Queen Elizabeth School, Form 6).

Problem 110. In a park, 1000 trees have been placed in a square lattice. Determine the maximum number of trees that can be cut down so that from any stump, you cannot see any other stump. (Assume the trees have negligible radius compared to the distance between adjacent trees.) (*Source: 1997 German Mathematical Olympiad*)

Solution. **CHAN Kin Hang Andy** (Bishop Hall Jubliee School, Form 7), **CHAO Khek Lun Harold** (St. Paul's College, Form 6), **Chau Suk Ling** (Queen Elizabeth School, Form 6), **CHENG Kei Tsi** (La Salle College, Form 6), **CHENG Man Chuen** (Tsuen Wan Government Secondary School, Form 7), **FUNG Wing Kiu Ricky** (La Salle College), **LAM Shek Ming Sherman** (La Salle College, Form 5), **LAW Ka Ho** (HKU, Year 1), **LEE Kevin** (La Salle College, Form 5), **LEUNG Wai Ying** (Queen Elizabeth School, Form 6), **LYN Kwong To** and **KO Man Ho** (Wah Yan College, Kowloon, Form 7), **POON Wing Sze Jessica** (STFA Leung Kau Kui College, Form 7) and **YEUNG Kai Shing** (La Salle College, Form 4).

In every 2×2 subsquare, only one tree can be cut. So a maximum of 2500 trees can be cut down. Now let the trees be at (x, y), where $x, y = 0, 1, 2, \ldots, 99$. If we cut down the 2500 trees at (x, y) with both x and y even, then the condition will be satisfied. To see this, consider the stumps at (x_1, y_1) and (x_2, y_2) with x_1, y_1, x_2, y_2 even. The cases $x_1 = x_2$ or $y_1 = y_2$ are clear. Otherwise, write $(y_2 - y_1)/(x_2 - x_1) = m/n$ in lowest term. Then either m or n is odd and so the tree at $(x_1 + m, y_1 + n)$ will be between (x_1, y_1) and (x_2, y_2).

Other recommended solvers: **NG Chok Ming Lewis** (STFA Leung Kau Kui College, Form 7).

Olympiad Corner

(continued from page 1)

Problem 3. ***(cont'd)***
Determine all values of λ such that, for any point M on the line and any initial position of the n fleas, there is a finite sequence of moves that will take all the fleas to positions to the right of M.

Problem 4. A magician has one hundred cards numbered 1 to 100. He puts them into three boxes, a red one, a white one a blue one, so that each contains at least one card.

A member of the audience selects two of the three boxes, chooses one card from each and announces the sum of the numbers on the chosen cards. Given this sum, the magician identifies the box from which no card has been chosen.

How many ways are there to put all the cards into the boxes so that this trick always works? (Two ways are considered different if at least one card is put into a different box.)

Problem 5. Determine whether or not there exists a positive integer n such that n is divisible by exactly 2000 different prime numbers, and $2^n + 1$ is divisible by n.

Problem 6. Let AH_1, BH_2, CH_3, be the altitudes of an acute-angled triangle ABC. The incircle of the triangle ABC touches the sides BC, CA, AB at T_1, T_2, T_3, respectively. Let the lines ℓ_1, ℓ_2, ℓ_3 be the reflections of the lines H_2H_3, H_3H_1, H_1H_2 in the lines T_2T_3, T_3T_1, T_1T_2, respectively.

Prove that ℓ_1, ℓ_2, ℓ_3 determine a triangle whose vertices lie on the incircle of the triangle ABC.