Mathematical Excalibur

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Olympiad Corner

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Problem 1. Let ABC be an acute-angled triangle with circumcentre O. Let P on BC be the foot of the altitude from A. Suppose that $\angle BCA \geq \angle ABC + 30^\circ$. Prove that $\angle CAB + \angle COP < 90^\circ$.

Problem 2. Prove that

$$\frac{a}{\sqrt{a^2+8bc}}+\frac{b}{\sqrt{b^2+8ca}}+\frac{c}{\sqrt{c^2+8ab}} \geq 1$$

for all positive real numbers a, b and c.

Problem 3. Twenty-one girls and twenty-one boys took part in a mathematical contest.

- Each contestant solved at most six problems.
- For each girl and each boy, at least one problem was solved by both of them.

Prove that there was a problem that was solved by at least three girls and at least three boys.

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The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***November 10, 2001***.

For individual subscription for the next five issues for the 01-02 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Pell's Equation (I)

Kin Y. Li

Let d be a positive integer that is not a square. The equation $x^2 - dy^2 = 1$ with variables x, y over integers is called ***Pell's equation***. It was Euler who attributed the equation to John Pell (1611-1685), although Brahmagupta (7th century), Bhaskara (12th century) and Fermat had studied the equation in details earlier.

A solution (x, y) of Pell's equation is called *positive* if both x and y are positive integers. Hence, positive solutions correspond to the lattice points in the first quadrant that lie on the hyperbola $x^2 - dy^2 = 1$. A positive solution (x_1, y_1) is called the *least positive solution* (or *fundamental solution*) if it satisfies $x_1 < x$ and $y_1 < y$ for every other positive solution (x, y). (As the hyperbola $x^2 - dy^2 = 1$ is strictly increasing in the first quadrant, the conditions for being least are the same as requiring $x_1 + y_1\sqrt{d} < x + y\sqrt{d}$.)

Theorem. *Pell's equation $x^2 - dy^2 = 1$ has infinitely many positive solutions. If (x_1, y_1) is the least positive solution, then for $n = 1, 2, 3, \ldots,$ define*

$$x_n + y_n\sqrt{d} = (x_1 + y_1\sqrt{d})^n.$$

The pairs (x_n, y_n) are all the positive solutions of the Pell's equation. The x_n's and y_n's are strictly increasing to infinity and satisfy the recurrence relations $x_{n+2} = 2x_1x_{n+1} - x_n$ and $y_{n+2} = 2x_1y_{n+1} - y_n$.

We will comment on the proof. The least positive solution is obtained by writing $\sqrt{d}$ as a simple continued fraction. It turns out

$$\sqrt{d} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}},$$

where $a_0 = [\sqrt{d}]$ and $a_1, a_2, \ldots$ is a periodic positive integer sequence. The continued fraction will be denoted by $\langle a_0, a_1, a_2, \ldots \rangle$. The *k-th convergent* of $\langle a_0, a_1, a_2, \ldots \rangle$ is the number $\frac{p_k}{q_k} = \langle a_0, a_1, a_2, \ldots, a_k \rangle$ with p_k, q_k relatively prime. Let $a_1, a_2, \ldots, a_m$ be the period for $\sqrt{d}$. The least positive solution of Pell's equation turns out to be

$$(x_1, y_1) = \begin{cases} (p_{m-1}, q_{m-1}) & \text{if } m \text{ is even} \\ (p_{2m-1}, q_{2m-1}) & \text{if } m \text{ is odd} \end{cases}.$$

For example, $\sqrt{3} = \langle 1, 1, 2, 1, 2, \ldots \rangle$ and so $m = 2$, then $\langle 1, 1 \rangle = \frac{2}{1}$. We check $2^2 - 3 \cdot 1^2 = 1$ and clearly, $(2, 1)$ is the least positive solution of $x^2 - 3y^2 = 1$. Next, $\sqrt{2} = \langle 1, 2, 2, \ldots \rangle$ and so $m = 1$, then $\langle 1, 2 \rangle = \frac{3}{2}$. We check $3^2 - 2 \cdot 2^2 = 1$ and again clearly $(3, 2)$ is the least positive solution of $x^2 - 2y^2 = 1$.

Next, if there is a positive solution (x, y) such that $x_n + y_n\sqrt{d} < x + y\sqrt{d} < x_{n+1} + y_{n+1}\sqrt{d}$, then consider $u + v\sqrt{d} = (x + y\sqrt{d})/(x_n + y_n\sqrt{d})$. We will get $u + v\sqrt{d} < x_1 + y_1\sqrt{d}$ and $u - v\sqrt{d} = (x - y\sqrt{d})/(x_n - y_n\sqrt{d})$ so that $u^2 - dv^2 = (u - v\sqrt{d})(u + v\sqrt{d}) = 1$, con-tradicting (x_1, y_1) being the least positive solution.

To obtain the recurrence relations, note that

$$(x_1+y_1\sqrt{d})^2 = x_1^2+dy_1^2+2x_1y_1\sqrt{d}$$
$$= 2x_1^2-1+2x_1y_1\sqrt{d}$$
$$= 2x_1(x_1+y_1\sqrt{d})-1.$$

So

$$x_{n+2}+y_{n+2}\sqrt{d}$$
$$=(x_1+y_1\sqrt{d})^2(x_1+y_1\sqrt{d})^n$$
$$=2x_1(x_1+y_1\sqrt{d})^{n+1}-(x_1+y_1\sqrt{d})^n$$
$$=2x_1x_{n+1}-x_n+(2x_1y_{n+1}-y_n)\sqrt{d}.$$

The related equation $x^2-dy^2=-1$ may not have a solution, for example, $x^2-3y^2=-1$ cannot hold as $x^2-3y^2\equiv x^2+y^2\neq -1$ (mod 4). However, if d is a prime and $d\equiv 1$ (mod 4), then a theorem of Lagrange asserts that it will have a solution. In general, if $x^2-dy^2=-1$ has a least positive solution (x_1, y_1), then all its positive solutions are pairs (x, y), where $x+y\sqrt{d} = (x_1+y_1\sqrt{d})^{2n-1}$ for some positive integer n.

In passing, we remark that some k-th convergent numbers are special. If the length m of the period for $\sqrt{d}$ is even, then $x^2-dy^2=1$ has $(x_n, y_n) = (p_{nm-1}, q_{nm-1})$ as all its positive solutions, but $x^2-dy^2=-1$ has no integer solution. If m is odd, then $x^2-dy^2=1$ has (p_{jm-1}, y_{jm-1}) with j even as all its positive solutions and $x^2-dy^2=-1$ has (p_{jm-1}, q_{jm-1}) with j odd as all its positive solutions.

Example 1. Prove that there are infinitely many triples of consecutive integers each of which is a sum of two squares.

Solution. The first such triple is $8=2^2+2^2$, $9=3^2+0^2$, $10=3^2+1^2$, which suggests we consider triples x^2-1, x^2, x^2+1. Since $x^2-2y^2=1$ has infinitely many positive solutions (x, y), we see that $x^2-1=y^2+y^2$, $x^2=x^2+0^2$ and x^2+1 satisfy the requirement and there are infinitely many such triples.

Example 2. Find all triangles whose sides are consecutive integers and areas are also integers.

Solution. Let the sides be $z-1, z, z+1$. Then the semiperimeter $s=\frac{3z}{2}$ and the area is $A=\frac{z\sqrt{3(z^2-4)}}{4}$. If A is an integer, then z cannot be odd, say $z=2x$, and $z^2-4=3\omega^2$. So $4x^2-4=3\omega^2$, which implies ω is even, say $\omega=2y$. Then $x^2-3y^2=1$, which has $(x_1, y_1) =$ (2, 1) as the least positive solution. So all positive solutions are (x_n, y_n), where $x_n+y_n\sqrt{3}=(2+\sqrt{3})^n$. Now $x_n-y_n\sqrt{3}=(2-\sqrt{3})^n$. Hence,

$$x_n=\frac{(2+\sqrt{3})^n+(2-\sqrt{3})^n}{2}$$

and

$$y_n=\frac{(2+\sqrt{3})^n-(2-\sqrt{3})^n}{2\sqrt{3}}.$$

The sides of the triangles are $2x_n-1$, $2x_n, 2x_n+1$ and the areas are $A=3x_ny_n$.

Example 3. Find all positive integers k, m such that $k<m$ and

$$1+2+\cdots+k=(k+1)+(k+2)+\cdots+m.$$

Solution. Adding $1+2+\cdots+k$ to both sides, we get $2k(k+1)=m(m+1)$, which can be rewritten as $(2m+1)^2-2(2k+1)^2=-1$. Now the equation $x^2-2y^2=-1$ has $(1,1)$ as its least positive solution. So its positive solutions are pairs $x_n+y_n\sqrt{2}=(1+\sqrt{2})^{2n-1}$. Then

$$x_n=\frac{(1+\sqrt{2})^{2n-1}+(1-\sqrt{2})^{2n-1}}{2}$$

and

$$y_n=\frac{(1+\sqrt{2})^{2n-1}-(1-\sqrt{2})^{2n-1}}{2\sqrt{2}}.$$

Since $x^2-2y^2=-1$ implies x is odd, so x is of the form $2m+1$. Then $y^2=2m^2+m+1$ implies y is odd, so y is of the form $2k+1$. Then $(k,m)=\left(\frac{y_n-1}{2}, \frac{x_n-1}{2}\right)$ with n = 2, 3, 4, … are all the solutions.

Example 4. Prove that there are infinitely many positive integers n such that n^2+1 divides $n!$.

Solution. The equation $x^2-5y^2=-1$ has $(2,1)$ as the least positive solution. So it has infinitely many positive solutions. Consider those solutions with $y>5$. Then $5<y<2y\le x$ as $4y^2\le 5y^2-1=x^2$. So $2(x^2+1)=5\cdot y\cdot 2y$ divides $x!$, which is more than we want.

Example 5. For the sequence $a_n = \left[\sqrt{n^2+(n+1)^2}\right]$, prove that there are infinitely many n's such that $a_n-a_{n+1}>1$ and $a_{n+1}-a_n=1$.

Solution. First consider the case $n^2+(n+1)^2=y^2$, which can be rewritten as $(2n+1)^2-2y^2=-1$. As in example 3 above, $x^2-2y^2=-1$ has infinitely many positive solutions and each x is odd, say $x=2n+1$ for some n. For these n's, $a_n=y$ and $a_{n-1} = \left[\sqrt{(n-1)^2+n^2}\right] = \left[\sqrt{y^2-4n}\right]$. The equation $y^2=n^2+(n+1)^2$ implies $n>2$ and $a_{n-1}\le\sqrt{y^2-4n}<y-1=a_n-1$. So $a_n-a_{n-1}>1$ for these n's. Also, for these n's, $a_{n+1} = \left[\sqrt{(n+1)^2+(n+2)^2}\right]=\left[\sqrt{y^2+4n+4}\right]$. As $n<y<2n+1$, we easily get $y+1<\sqrt{y^2+4n+4}<y+2$. So $a_{n+1}-a_n = (y+1)-y=1$.

Example 6. (American Math Monthly E2606, proposed by R.S. Luthar) Show that there are infinitely many integers n such that $2n+1$ and $3n+1$ are perfect squares, and that such n must be multiples of 40.

Solution. Consider $2n+1=u^2$ and $3n+1=v^2$. On one hand, $u^2+v^2\equiv 2$ (mod 5) implies u^2, $v^2\equiv 1$ (mod 5), which means n is a multiple of 5.

On the other hand, we have $3u^2-2v^2 = 1$. Setting $u=x+2y$ and $v=x+3y$, the equation becomes $x^2-6y^2=1$. It has infinitely many positive solutions. Since $3u^2-2v^2=1$, u is odd, say $u=2k+1$. Then $n=2k^2+2k$ is even. Since $3n+1=v^2$, so v is odd, say $v=4m\pm 1$. Then $3n=16m^2\pm 8m$, which implies n is also a multiple of 8.

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Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. Solutions should be preceded by the solver's name, home (or **email**) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, Hong Kong University of Science & Technology, Clear Water Bay, Kowloon*. The deadline for submitting solutions is ***November 10, 2001***.

Problem 131. Find the greatest common divisor (or highest common factor) of the numbers $n^n - n$ for $n = 3$, 5, 7, … .

Problem 132. Points D, E, F are chosen on sides AB, BC, CA of ΔABC, respectively, so that $DE = BE$ and $FE = CE$. Prove that the center of the circumcircle of ΔADF lies on the angle bisector of $\angle DEF$. (*Source: 1989 USSR Math Olympiad*)

Problem 133. (a) Are there real numbers a and b such that $a+b$ is rational and $a^n + b^n$ is irrational for every integer $n \ge 2$? (b) Are there real numbers a and b such that $a+b$ is irrational and $a^n + b^n$ is rational for every integer $n \ge 2$? (*Source: 1989 USSR Math Olympiad*)

Problem 134. Ivan and Peter alternatively write down 0 or 1 from left to right until each of them has written 2001 digits. Peter is a winner if the number, interpreted as in base 2, is not the sum of two perfect squares. Prove that Peter has a winning strategy. (*Source: 2001 Bulgarian Winter Math Competition*)

Problem 135. Show that for $n \ge 2$, if $a_1, a_2, ..., a_n > 0$, then

$$(a_1^3+1)(a_2^3+1)\cdots(a_n^3+1) \ge$$

$$(a_1^2a_2+1)(a_2^2a_3+1)\cdots(a_n^2a_1+1).$$

(*Source: 7th Czech-Slovak-Polish Match*)

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Solutions

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Problem 126. Prove that every integer can be expressed in the form $x^2 + y^2 - 5z^2$, where x, y, z are integers.

Solution. **CHAN Kin Hang** (CUHK, Math Major, Year 1), **CHENG Kei Tsi Daniel** (La Salle College, Form 7), **CHENG Man Chuen** (CUHK, Math Major, Year 1), **CHUNG Tat Chi** (Queen Elizabeth School, Form 5), **FOK Chi Kwong** (Yuen Long Merchants Association Secondary School, Form 5), **IP Ivan** (St. Joseph's College, Form 6), **KOO Koopa** (Boston College, Sophomore), **LAM Shek Ming Sherman** (La Salle College, Form 6), **LAU Wai Shun** (Tsuen Wan Public Ho Chuen Yiu Memorial College, Form 6), **LEE Kevin** (La Salle College, Form 6), **LEUNG Wai Ying** (Queen Elizabeth School, Form 7), **MAN Chi Wai** (HKSYC IA Wong Tai Shan Memorial College), **NG Ka Chun** (Queen Elizabeth School, Form 7), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 6), **YEUNG Kai Sing** (La Salle College, Form 5) and **YUNG Po Lam** (CUHK, Math Major, Year 2).

For n odd, say $n = 2k + 1$, we have $(2k)^2 + (k+1)^2 - 5k^2 = 2k + 1 = n$. For n even, say $n = 2k$, we have $(2k-1)^2 + (k-2)^2 - 5(k-1)^2 = 2k = n$.

Problem 127. For positive real numbers a, b, c with $a + b + c = abc$, show that

$$\frac{1}{\sqrt{1+a^2}} + \frac{1}{\sqrt{1+b^2}} + \frac{1}{\sqrt{1+c^2}} \le \frac{3}{2},$$

and determine when equality occurs. (*Source: 1998 South Korean Math Olympiad*)

Solution. **CHAN Kin Hang** (CUHK, Math Major, Year 1), **CHENG Kei Tsi Daniel** (La Salle College, Form 7), **KOO Koopa** (Boston College, Sophomore), **LEE Kevin** (La Salle College, Form 6) and **NG Ka Chun** (Queen Elizabeth School, Form 7).

Let $A = \tan^{-1} a$, $B = \tan^{-1} b$, $C = \tan^{-1} c$. Since $a, b, c > 0$, we have $0 < A, B, C < \frac{\pi}{2}$. Now $a + b + c = abc$ is the same as $\tan A + \tan B + \tan C = \tan A \tan B \tan C$. Then

$$\tan C = \frac{-(\tan A + \tan B)}{1 - \tan A \tan B} = \tan(\pi - A - B)$$

which implies $A + B + C = \pi$. In terms of A, B, C the inequality to be proved is $\cos A + \cos B + \cos C \le \frac{3}{2}$, which follows by applying Jensen's inequality to $f(x) = \cos x$ on $(0, \frac{\pi}{2})$.

Other commended solvers: **CHENG Man Chuen** (CUHK, Math Major, Year 1), **IP Ivan** (St. Joseph's College, Form 6), **LAM Shek Ming Sherman** (La Salle College, Form 6), **LEUNG Wai Ying** (Queen Elizabeth School, Form 7), **MAN Chi Wai** (HKSYC&IA Wong Tai Shan Memorial College), **TSUI Ka Ho** (Hoi Ping Chamber of Commerce Secondary School, Form 7), **WONG Wing Hong** (La Salle College, Form 4) and **YEUNG Kai Sing** (La Salle College, Form 5).

Problem 128. Let M be a point on segment AB. Let $AMCD$, $BEHM$ be squares on the same side of AB. Let the circumcircles of these squares intersect at M and N. Show that B, N, C are collinear and H is the orthocenter of ΔABC. (*Source: 1979 Henan Province Math Competition*)

Solution. **LEUNG Wai Ying** (Queen Elizabeth School, Form 7), **MAN Chi Wai** (HKSYC&IA Wong Tai Shan Memorial College) and **YUNG Po Lam** (CUHK, Math Major, Year 2).

Since $\angle BNM = \angle BHM = 45^\circ = \angle CDM = \angle CDM$, it follows B, N, C are collinear. Next, $CH \perp AB$. Also, $BH \perp ME$ and $ME \| AC$ imply $BH \perp AC$. So H is the orthocenter of ΔABC.

Other commended solvers: **CHAN Kin Hang** (CUHK, Math Major, Year 1), **CHENG Kei Tsi Daniel** (La Salle College, Form 7), **CHENG Man Chuen** (CUHK, Math Major, Year 1), **CHUNG Tat Chi** (Queen Elizabeth School, Form 5), **IP Ivan** (St. Joseph's College, Form 6), **KWOK Sze Ming** (Queen Elizabeth School, Form 6), **LAM Shek Ming Sherman** (La Salle College, Form 6), **Lee Kevin** (La Salle College, Form 6), **NG Ka Chun** (Queen Elizabeth School, Form 7), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 6), **WONG Wing Hong** (La Salle College, Form 4) and **YEUNG Kai Sing** (La Salle College, Form 5).

Problem 129. If $f(x)$ is a polynomial of degree $2m + 1$ with integral coefficients for which there are $2m + 1$ integers $k_1, k_2, \ldots, k_{2m+1}$ such that $f(k_i) = 1$ for $i = 1, 2, \ldots, 2m + 1$, prove that $f(x)$ is not the product of two nonconstant polynomials with integral coefficients.

Solution. **CHAN Kin Hang** (CUHK, Math Major, Year 1), **CHENG Kei Tsi Daniel** (La Salle College, Form 7), **CHENG Man Chuen** (CUHK, Math Major, Year 1), **IP Ivan** (St. Joseph's College, Form 6), **KOO Koopa** (Boston College, Sophomore), **LAM Shek Ming Sherman** (La Salle College, Form 6), **LEE Kevin** (La Salle College, Form 6), **LEUNG Wai Ying** (Queen Elizabeth School, Form

7), **MAN Chi Wai** (HKSYC&IA Wong Tai Shan Memorial College**),** **YEUNG Kai Sing** (La Salle College, Form 5) and **YUNG Po Lam** (CUHK, Math Major, Year 2).

Suppose f is the product of two non-constant polynomials with integral co-efficients, say $f = PQ$. Since $1 = f(k_i) = P(k_i)\,Q(k_i)$ and $P(k_i)$, $Q(k_i)$ are integers, so either both are 1 or both are –1. As there are $2m + 1$ k_i 's, either $P(k_i) = Q(k_i) = 1$ for at least $m + 1$ k_i 's or $P(k_i) = Q(k_i) = -1$ for at least $m + 1$ k_i 's. Since deg $f = 2m + 1$, one of deg P or deg Q is at most m. This forces P or Q to be a constant polynomial, a contradiction.

Other commended solvers: **NG Cheuk Chi** (Tsuen Wan Public Ho Chuen Yiu Memorial College) and **NG Ka Chun** (Queen Elizabeth School, Form 7).

Problem 130. Prove that for each positive integer n, there exists a circle in the xy-plane which contains exactly n lattice points in its interior, where a *lattice point* is a point with integral coordinates. (*Source: H. Steinhaus, Zadanie 498, Matematyka 10 (1957), p. 58*)

Solution. **CHENG Man Chuen** (CUHK, Math Major, Year 1) and **IP Ivan** (St. Joseph's College, Form 6).

Let $P = \left(\sqrt{2}, \frac{1}{3}\right)$. Suppose lattice points $(x_0, y_0), (x_1, y_1)$ are the same distance from P. Then $\left(x_0 - \sqrt{2}\right)^2 + \left(y_0 - \frac{1}{3}\right)^2 = \left(x_1 - \sqrt{2}\right)^2 + \left(y_1 - \frac{1}{3}\right)^2$. Moving the x terms to the left, the y terms to the right and factoring, we get

$$(x_0 - x_1)\left(x_0 + x_1 - 2\sqrt{2}\right)$$

$$= (y_0 - y_1)\left(y_0 + y_1 - \frac{2}{3}\right).$$

As the right side is rational and $\sqrt{2}$ is irrational, we must have $x_0 = x_1$. Then the left side is 0, which forces $y_1 = y_0$ since $y_1 + y_0$ is integer. So the lattice points are the same.

Now consider the circle with center at P and radius r. As r increases from 0 to infinity, the number of lattice points inside the circle increase from 0 to infinity. As the last paragraph shows, the increase cannot jump by 2 or more. So the statement is true.

Other commended solvers: **CHENG Kei Tsi Daniel** (La Salle College, Form 7), **KOO Koopa** (Boston College, Sophomore), **LEUNG Wai Ying** (Queen Elizabeth School, Form 7), **MAN Chi Wai** (HKSYC&IA Wong Tai Shan Memorial College), **NG Ka Chun** (Queen Elizabeth School, Form 7) and **YEUNG Kai Sing** (La Salle College, Form 4).

Olympiad Corner

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Problem 4. Let n be an odd integer greater than 1, let $k_1, k_2, ..., k_n$ be given integers. For each of the n! permutations $a = (a_1, a_2, ..., a_n)$ of 1, 2, …, n, let

$$S(a) = \sum_{i=1}^{n} k_i a_i .$$

Prove that there are two permutations b and c, $b \neq c$, such that n! is a divisor of $S(b) – S(c)$.

Problem 5. In a triangle ABC, let AP bisect $\angle BAC$, with P on BC, and let BQ bisect $\angle ABC$, with Q on CA. It is known that $\angle BAC = 60^\circ$ and that $AB + BP = AQ + QB$.

What are the possible angles of triangle ABC?

Problem 6. Let a, b, c, d be integers with $a > b > c > d > 0$. Suppose that $ac + bd = (b + d + a – c)(b + d – a + c)$. Prove that $ab + cd$ is not prime.

Pell's Equation (I)

(continued from page 2)

Example 7. Prove that the only positive integral solution of $5^a - 3^b = 2$ is $a = b = 1$.

Solution. Clearly, if a or b is 1, then the other one is 1, too. Suppose (a, b) is a solution with both $a, b > 1$. Considering (mod 4), we have $1 – (–1)^b \equiv 2$ (mod 4), which implies b is odd. Considering (mod 3), we have $(-1)^a \equiv 2$ (mod 3), which implies a is odd.

Setting $x = 3^b + 1$ and $y = 3^{(b-1)/2} 5^{(a-1)/2}$, we get $15y^2 = 3^b 5^a = 3^b(3^b + 2) = (3^b + 1)^2 - 1 = x^2 - 1$. So (x, y) is a positive solution of $x^2 - 15y^2 = 1$. The least positive solution is $(4,1)$. Then $(x, y) = (x_n, y_n)$ for some positive integer n, where $x_n + y_n\sqrt{15} = (4 + \sqrt{15})^n$. After examining the first few y_n 's, we observe that y_{3k} are the only terms that are divisible by 3. However, they also seem to be divisible by 7, hence cannot be of the form $3^c 5^d$.

To confirm this, we use the recurrence relations on y_n . Since $y_1 = 1$, $y_2 = 8$ and $y_{n+2} = 8y_{n+1} - y_n$, taking y_n (mod 3), we get the sequence 1, 2, 0, 1, 2, 0… and taking y_n (mod 7), we get 1, 1, 0, -1, -1, 0, 1, 1, 0, -1, -1, 0, ….

Therefore, no $y = y_n$ is of the form $3^c 5^d$ and $a, b > 1$ cannot be solution to $5^a - 3^b = 2$.

Example 8. Show that the equation $a^2 + b^3 = c^4$ has infinitely many solutions.

Solution. We will use the identity

$$1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2,$$

which is a standard exercise of mathematical induction. From the identity, we get $\left(\frac{(n-1)n}{2}\right)^2 + n^3 = \left(\frac{n(n+1)}{2}\right)^2$ for $n > 1$. All we need to do now is to show there are infinitely many positive integers n such that $n(n + 1)/2 = k^2$ for some positive integers k. Then $(a, b, c) = ((n – 1)n/2, n, k)$ solves the problem.

Now $n(n + 1)/2 = k^2$ can be rewritten as $(2n+1)^2 - 2(2k)^2 = 1$. We know $x^2 - 2y^2 = 1$ has infinitely many positive solutions. For any such (x, y), clearly x is odd, say $x = 2m + 1$. They $y^2 = 2m^2 + 2m$ implies y is even. So any such (x, y) is of the form $(2n+1,\ 2k)$. Therefore, there are infinitely many such n.