Olympiad Corner

The 42nd International Mathematical Olympiad, Washington DC, USA, 8-9 July 2001

Problem 1. Let \( ABC \) be an acute-angled triangle with circumcentre \( O \). Let \( P \) on \( BC \) be the foot of the altitude from \( A \). Suppose that \( \angle BCA \geq \angle ABC + 30^\circ \). Prove that \( \angle CAB + \angle COP < 90^\circ \).

Problem 2. Prove that
\[
\frac{a}{\sqrt{a^2 + 8bc}} + \frac{b}{\sqrt{b^2 + 8ca}} + \frac{c}{\sqrt{c^2 + 8ab}} \geq 1
\]
for all positive real numbers \( a, b \) and \( c \).

Problem 3. Twenty-one girls and twenty-one boys took part in a mathematical contest.
- Each contestant solved at most six problems.
- For each girl and each boy, at least one problem was solved by both of them.

Prove that there was a problem that was solved by at least three girls and at least three boys.

(continued on page 4)

Pell’s Equation (I)

Kin Y. Li

Let \( d \) be a positive integer that is not a square. The equation \( x^2 - dy^2 = 1 \) with variables \( x, y \) over integers is called Pell’s equation. It was Euler who attributed the equation to John Pell (1611-1685), although Brahmagupta (7th century), Bhaskara (12th century) and Fermat had studied the equation in details earlier.

A solution \((x, y)\) of Pell’s equation is called positive if both \( x \) and \( y \) are positive integers. Hence, positive solutions correspond to the lattice points in the first quadrant that lie on the hyperbola \( x^2 - dy^2 = 1 \). A positive solution \((x_1, y_1)\) is called the least positive solution (or fundamental solution) if it satisfies \( x_1 < x \) and \( y_1 < y \) for every other positive solution \((x, y)\). (As the hyperbola \( x^2 - dy^2 = 1 \) is strictly increasing in the first quadrant, the conditions for being least are the same as requiring \( x_1 + y_1\sqrt{d} < x + y\sqrt{d} \).

Theorem. Pell’s equation \( x^2 - dy^2 = 1 \) has infinitely many positive solutions. If \((x_1, y_1)\) is the least positive solution, then for \( n = 1, 2, 3, \ldots \), define
\[
x_n + y_n\sqrt{d} = (x_1 + y_1\sqrt{d})^n.
\]
The pairs \((x_n, y_n)\) are all the positive solutions of the Pell’s equation. The \( x_n \)'s and \( y_n \)'s are strictly increasing to infinity and satisfy the recurrence relations
\[
x_{n+1} = 2x_n + y_n\sqrt{d}, \quad y_{n+1} = x_n + 2y_n.
\]
We will comment on the proof. The least positive solution is obtained by writing \( \sqrt{d} \) as a simple continued fraction. It turns out
\[
\sqrt{d} = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ldots}}},
\]
where \( a_0 = \lfloor \sqrt{d} \rfloor \) and \( a_1, a_2, \ldots \) is a periodic positive integer sequence. The continued fraction will be denoted by \( \langle a_0, a_1, a_2, \ldots \rangle \). The \( k \)-th convergent of \( \langle a_0, a_1, a_2, \ldots \rangle \) is the number \( \frac{p_k}{q_k} \) with \( p_k, q_k \) relatively prime. Let \( a_1, a_2, \ldots, a_m \) be the period for \( \sqrt{d} \). The least positive solution of Pell’s equation turns out to be
\[
(x_1, y_1) = \left\{ \begin{array}{ll}
(p_{m-1}, q_{m-1}) & \text{if } m \text{ is even} \\
(p_{2m-1}, q_{2m-1}) & \text{if } m \text{ is odd}
\end{array} \right.
\]
For example, \( \sqrt{5} = \langle 1, 2, 1, 2, 1, 2, \ldots \rangle \) and so \( m = 2 \), then \( \langle 1, 1 \rangle = \frac{2}{1} \). We check \( 2^2 - 3 \cdot 1^2 = 1 \) and clearly, \( (2, 1) \) is the least positive solution of \( x^2 - 3y^2 = 1 \). Next, \( \sqrt{2} = \langle 1, 2, 2, \ldots \rangle \) and so \( m = 1 \), then \( \langle 1, 2 \rangle = \frac{3}{2} \). We check \( 3^2 - 2 \cdot 2^2 = 1 \) and again clearly \( (3, 2) \) is the least positive solution of \( x^2 - 2y^2 = 1 \).

Next, if there is a positive solution \((x, y)\) such that \( x_n + y_n\sqrt{d} < x + y\sqrt{d} < x_{n+1} + y_{n+1}\sqrt{d} \), then consider \( u + v\sqrt{d} = (x + y\sqrt{d})(x_n + y_n\sqrt{d}) \). We will get \( u + v\sqrt{d} = (x + y\sqrt{d})(x_n + y_n\sqrt{d}) \) so that \( u^2 - dv^2 = (u - v\sqrt{d})(u + v\sqrt{d}) = 1 \), contradicting \((x_1, y_1)\) being the least positive solution.

To obtain the recurrence relations, note that
\[(x_1 + y_1 \sqrt{d})^2 = x_1^2 + dy_1^2 + 2x_1 y_1 \sqrt{d} = 2x_1^2 - 1 + 2x_1 y_1 \sqrt{d} = 2x_1 (x_1 + y_1 \sqrt{d}) - 1.\]

So
\[x_{n+2} + y_{n+2} \sqrt{d} = (x_1 + y_1 \sqrt{d})^n (x_1 + y_1 \sqrt{d})^n - (x_1 + y_1 \sqrt{d})^n = 2x_1 x_{n+1} - x_n + (2x_1 y_{n+1} - y_n) \sqrt{d}.\]

The related equation \(x^2 - dy^2 = -1\) may not have a solution, for example, \(x^2 - 3y^2 = -1\) cannot hold as \(x^2 - 3y^2 = x^2 + y^2 \not\equiv -1 \mod(4)\).

However, if \(d\) is a prime and \(d = 1 \mod(4)\), then a theorem of Lagrange asserts that it will have a solution. In general, if \(x^2 - dy^2 = -1\) has a least positive solution \((x_1, y_1)\), then all its positive solutions are pairs \((x, y)\), where \(x + y \sqrt{d} = (x_1 + y_1 \sqrt{d})^{2k-1}\) for some positive integer \(n\).

In passing, we remark that some \(k\)-th convergent numbers are special. If the length \(m\) of the period for \(\sqrt{d}\) is even, then \(x^2 - dy^2 = 1\) has \((x_n, y_n) = (p_{nm-1}, q_{nm-1})\) as all its positive solutions, but \(x^2 - dy^2 = -1\) has no integer solution. If \(m\) is odd, then \(x^2 - dy^2 = 1\) has \((p_{jm-1}, q_{jm-1})\) with \(j\) even as all its positive solutions and \(x^2 - dy^2 = -1\) has \((p_{jm-1}, q_{jm-1})\) with \(j\) odd as all its positive solutions.

**Example 1.** Prove that there are infinitely many triples of consecutive integers each of which is a sum of two squares. **Solution.** The first such triple is \(8 = 2^2 + 2^2\), \(9 = 3^2 + 0^2\), \(10 = 3^2 + 1^2\), which suggests we consider triples \(x^2 - 1, x^2, x^2 + 1\). Since \(x^2 - 2y^2 = 1\) has infinitely many positive solutions \((x, y)\), we see that \(x^2 - 1 = y^2 + y^2, x^2 = x^2 + 0^2\) and \(x^2 + 1\) satisfy the requirement and there are infinitely many such triples.

**Example 2.** Find all triangles whose sides are consecutive integers and areas are also integers. **Solution.** Let the sides be \(z - 1, z, z + 1\). Then the semiperimeter \(s = \frac{3z}{2}\) and the area is \(A = \frac{\sqrt{3} (z^2 - 4)}{4}\). If \(A\) is an integer, then \(z\) cannot be odd, say \(z = 2x\), and \(z^2 - 4 = 3 \omega^2\). So \(4x^2 - 4 = 3 \omega^2\), which implies \(\omega\) is even, say \(\omega = 2y\). Then \(x^2 - 3y^2 = 1\), which has \((x_1, y_1) = (2, 1)\) as the least positive solution. So all positive solutions are \((x_n, y_n)\), where \(x_n + y_n \sqrt{3} = (2 + \sqrt{3})^n\). Now \(x_n - y_n \sqrt{3} = (2 - \sqrt{3})^n\). Hence,
\[x_n = \frac{(2 + \sqrt{3})^n + (2 - \sqrt{3})^n}{2}\]
and
\[y_n = \frac{(2 + \sqrt{3})^n - (2 - \sqrt{3})^n}{2 \sqrt{3}}.\]

The sides of the triangles are \(2x_n - 1, 2x_n, 2x_n + 1\) and the areas \(A = 3x_n y_n\).

**Example 3.** Find all positive integers \(k, m\) such that \(k^2 + m = 1 + 2 + \cdots + k = (k+1) + (k+2) + \cdots + m\). **Solution.** Adding \(1 + 2 + \cdots + k\) to both sides, we get \(2k(k+1) = m(m+1)\), which can be rewritten as \((2m+1)^2 - 2(2k+1)^2 = -1\). Now the equation \(x^2 - 2y^2 = -1\) has \((1, 1)\) as its least positive solution. So its positive solutions are pairs \(x_n + y_n \sqrt{2} = (1 + \sqrt{2})^{2k-1}\). Then
\[x_n = \frac{(1 + \sqrt{2})^{2k-1} + (1 - \sqrt{2})^{2k-1}}{2}\]
and
\[y_n = \frac{(1 + \sqrt{2})^{2k-1} - (1 - \sqrt{2})^{2k-1}}{2 \sqrt{2}}.\]

Since \(x^2 - 2y^2 = -1\) implies \(x\) is odd, \(x\) is of the form \(2m+1\). Then \(y^2 = 2m^2 + m + 1\) implies \(y\) is odd, so \(y\) is of the form \(2k+1\). Then \((k, m) = \left(\frac{y_n - 1}{2}, \frac{x_n - 1}{2}\right)\) with \(n = 2, 3, 4, \ldots\) are all the solutions. **Example 4.** Prove that there are infinitely many positive integers \(n\) such that \(n^2 + 1\) divides \(n!\). **Solution.** The equation \(x^2 - 5y^2 = -1\) has \((2, 1)\) as the least positive solution. So it has infinitely many positive solutions. Consider those solutions with \(y > 5\). Then \(5 < y < 2y \leq x\) as \(4y^2 \leq 5y^2 - 1 = x^2\). So \(2(x^2 + 1) = 5 \cdot y \cdot 2y\) divides \(x!\), which is more than we want. **Example 5.** For the sequence \(a_n = \left[\sqrt{n^2 + (n+1)^2}\right]\), prove that there are infinitely many \(n\)’s such that \(a_n - a_{n+1} > 1\) and \(a_{n+1} - a_n = 1\). **Solution.** First consider the case \(n^2 + (n+1)^2 = y^2\), which can be rewritten as \((2n+1)^2 - 2y^2 = -1\). As in example 3 above, \(x^2 - 2y^2 = -1\) has infinitely many positive solutions and each \(x\) is odd, say \(x = 2n+1\) for some \(n\). For these \(n\)’s, \(a_n = y \) and \(a_{n+1} = \left[\sqrt{(n+1)^2 + n^2}\right] = \left[\sqrt{y^2 - 4n}\right]\). The equation \(y^2 = n^2 + (n+1)^2\) implies \(n > 2\) and \(a_n - a_{n+1} \leq \sqrt{y^2 - 4n} < y - 1 = a_n - 1\). So \(a_n - a_{n+1} > 1\) for these \(n\)’s.

Also, for these \(n\)’s, \(a_{n+1} = \left[\sqrt{(n+1)^2 + (n+2)^2}\right] = \left[\sqrt{y^2 + 4n + 4}\right]\). As \(n < y < 2n + 1\), we easily get \(y + 1 < \sqrt{y^2 + 4n + 4} < y + 2\). So \(a_{n+1} - a_n = (y + 1) - y = 1\). **Example 6.** (American Math Monthly E2606, proposed by R.S. Luthar) Show that there are infinitely many integers \(n\) such that \(2n + 1\) and \(3n + 1\) are perfect squares, and that such \(n\) must be multiples of 40. **Solution.** Consider \(2n + 1 = u^2\) and \(3n + 1 = v^2\). On one hand, \(u^2 + v^2 = 2\) (mod 5) implies \(u^2, v^2 = 1\) (mod 5), which means \(n\) is a multiple of 5. On the other hand, we have \(3u^2 - 2v^2 = 1\). Setting \(u = x + 2y\) and \(v = x + 3y\), the equation becomes \(x^2 - 6y^2 = 1\). It has infinitely many positive solutions. Since \(3u^2 - 2v^2 = 1, u\) is odd, say \(u = 2k + 1\). Then \(n = 2k^2 + 2k\) is even. Since \(3n + 1 = v^2\), so \(v\) is odd, say \(v = 4m + 1\). Then \(3n = 16m^2 - 8m\), which implies \(n\) is also a multiple of 8.
Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration.

Solutions should be preceded by the solver’s name, home (or email) address and school affiliation. Please send submissions to Dr. Kin Y. Li, Department of Mathematics, Hong Kong University of Science & Technology, Clear Water Bay, Kowloon.

The deadline for submitting solutions is November 10, 2001.

Problem 131. Find the greatest common divisor (or highest common factor) of the numbers \( n^3 - n \) for \( n = 3, 5, 7, \ldots \).

Problem 132. Points \( D, E, F \) are chosen on sides \( AB, BC, CA \) of \( \triangle ABC \), respectively, so that \( DE = BE \) and \( FE = CE \). Prove that the center of the circumcircle of \( \triangle ADF \) lies on the angle bisector of \( \angle DEF \). (Source: 1989 USSR Math Olympiad)

Problem 133. (a) Are there real numbers \( a \) and \( b \) such that \( a + b \) is rational and \( a^n + b^n \) is irrational for every integer \( n \geq 2 \)? (b) Are there real numbers \( a \) and \( b \) such that \( a + b \) is rational and \( a^n + b^n \) is rational for every integer \( n \geq 2 \)? (Source: 1989 USSR Math Olympiad)

Problem 134. Ivan and Peter alternatively write down 0 or 1 from left to right until each of them has written 2001 digits. Peter is a winner if the number, interpreted as in base 2, is not the sum of two perfect squares. Prove that Peter has a winning strategy. (Source: 2001 Bulgarian Winter Math Competition)

Problem 135. Show that for \( n \geq 2 \), if \( a_1, a_2, \ldots, a_n > 0 \), then

\[
(a_1^2 + 1)(a_2^2 + 1) \cdots (a_n^2 + 1) \geq (a_1^2 a_2 + 1)(a_2^2 a_3 + 1) \cdots (a_n^2 a_1 + 1).
\]

(Source: 7th Czech-Slovak-Polish Match)

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Solutions

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Problem 126. Prove that every integer can be expressed in the form \( x^2 + y^2 - 5z^2 \), where \( x, y, z \) are integers.

Solution. CHAN Kin Hang (CUHK, Math Major, Year 1), CHENG Kei Tsi Daniel (La Salle College, Form 7), CHENG Man Chuen (CUHK, Math Major, Year 1), CHUNG Tat Chi (Queen Elizabeth School, Form 5), FOK Chi Kwong (Yuen Long Merchants Association Secondary School, Form 5), IP Ivan (St. Joseph’s College, Form 6), KOO Koopa (Boston College, Sophomore), LAM Shek Ming Sherman (La Salle College, Form 6), LAU Wai Shun (Tuen Wan Public Ho Chuen Yiu Memorial College, Form 6), LEE Kevin (La Salle College, Form 6), LEUNG Wai Ying (Queen Elizabeth School, Form 7), MAN Chi Wai (HKSYCIA Wong Tai Shan Memorial College), NG Ka Chun (Queen Elizabeth School, Form 7), SIU Tsz Hang (STFA Leung Kau Kui College, Form 6), YEUNG Kai Sing (La Salle College, Form 5) and YUNG Po Lam (CUHK, Math Major, Year 2).

For \( n \) odd, say \( n = 2k + 1 \), we have \( (2k)^2 + (k + 1)^2 - 5k^2 = 2k + 1 = n \). For \( n \) even, say \( n = 2k \), we have \( (2k - 1)^2 + (k - 2)^2 - 5(k - 1)^2 = 2k = n \).

Problem 127. For positive real numbers \( a, b, c \) with \( a + b + c = abc \), show that

\[
\frac{1}{\sqrt{1+a^2}} + \frac{1}{\sqrt{1+b^2}} + \frac{1}{\sqrt{1+c^2}} \leq \frac{3}{2},
\]

determine when equality occurs. (Source: 1998 South Korean Math Olympiad)

Solution. CHAN Kin Hang (CUHK, Math Major, Year 1), CHENG Kei Tsi Daniel (La Salle College, Form 7), CHENG Man Chuen (CUHK, Math Major, Year 1), CHUNG Tat Chi (Queen Elizabeth School, Form 5), IP Ivan (St. Joseph’s College, Form 6), KOO Koopa (Boston College, Sophomore), LEE Kevin (La Salle College, Form 6), NG Ka Chun (Queen Elizabeth School, Form 7).

Let \( A = \tan^{-1} a, B = \tan^{-1} b, C = \tan^{-1} c \).

Since \( a, b, c > 0 \), we have \( 0 < A, B, C < \frac{\pi}{2} \).

Now \( a + b + c = abc \) is the same as \( \tan A + \tan B + \tan C = \tan A \tan B \tan C \). Then

\[
\tan C = \frac{\tan A + \tan B}{1 - \tan A \tan B},
\]

which implies \( A + B + C = \pi \). In terms of \( A, B, C \), the inequality to be proved is \( \cos A + \cos B + \cos C \leq 3 \), which follows by applying Jensen’s inequality to \( f(x) = \cos x \) on \( (0, \frac{\pi}{2}) \).

Other commended solvers: CHENG Man Chuen (CUHK, Math Major, Year 1), IP Ivan (St. Joseph’s College, Form 6), LAM Shek Ming Sherman (La Salle College, Form 6), LEUNG Wai Ying (Queen Elizabeth School, Form 7), MAN Chi Wai (HKSYCIA Wong Tai Shan Memorial College), TSUI Ka Ho (Hoi Ping Chamber of Commerce Secondary School, Form 7), WONG Wing Hong (La Salle College, Form 4) and YEUNG Kai Sing (La Salle College, Form 5).

Problem 128. Let \( M \) be a point on segment \( AB \). Let \( AMCD, BEHM \) be squares on the same side of \( AB \). Let the circumcircles of these squares intersect at \( M \) and \( N \). Show that \( B, N, C \) are collinear and \( H \) is the orthocenter of \( \triangle ABC \). (Source: 1979 Henan Province Math Competition)

Other commended solvers: CHAN Kin Hang (CUHK, Math Major, Year 1), CHENG Kei Tsi Daniel (La Salle College, Form 7), CHENG Man Chuen (CUHK, Math Major, Year 1), CHUNG Tat Chi (Queen Elizabeth School, Form 5), IP Ivan (St. Joseph’s College, Form 6), KOO Koopa (Boston College, Sophomore), LAM Shek Ming Sherman (La Salle College, Form 6), KWOK Sze Ming (Queen Elizabeth School, Form 6), LAM Shek Ming Sherman (La Salle College, Form 6), LEE Kevin (La Salle College, Form 6), NG Ka Chun (Queen Elizabeth School, Form 7), SIU Tsz Hang (STFA Leung Kau Kui College, Form 6), WONG Wing Hong (La Salle College, Form 4) and YEUNG Kai Sing (La Salle College, Form 5).

Problem 129. If \( f(x) \) is a polynomial of degree \( 2m + 1 \) with integral coefficients for which there are \( 2m + 1 \) integers \( k_1, k_2, \ldots, k_{2m+1} \) such that \( f(k_j) = 1 \) for \( i = 1, 2, \ldots, 2m + 1 \), prove that \( f(x) \) is not the product of two nonconstant polynomials with integral coefficients.

Solution. CHAN Kin Hang (CUHK, Math Major, Year 1), CHENG Kei Tsi Daniel (La Salle College, Form 7), CHENG Man Chuen (CUHK, Math Major, Year 1), IP Ivan (St. Joseph’s College, Form 6), KOO Koopa (Boston College, Sophomore), LAM Shek Ming Sherman (La Salle College, Form 6), LEE Kevin (La Salle College, Form 6), LEUNG Wai Ying (Queen Elizabeth School, Form 7), YEUNG Kai Sing (La Salle College, Form 5).
Problem 130. Prove that for each positive integer $n$, there exists a circle in the $xy$-plane which contains exactly $n$ lattice points in its interior, where a lattice point is a point with integral coordinates. (Source: H. Steinhaus, Zaslanie 498, Matematyka 10 (1957), p. 38)

**Solution.** CHENG Man Chuen (CUHK, Math Major, Year 1) and IP Ivan (St. Joseph’s College, Form 6).

Let $P = \left(\sqrt{2}, \frac{1}{3}\right)$. Suppose lattice points $(x_0, y_0), (x_1, y_1)$ are at the same distance from $P$. Then

\[
\left(x_0 - \sqrt{2}\right)^2 + \left(y_0 - \frac{1}{3}\right)^2 = \left(x_1 - \sqrt{2}\right)^2 + \left(y_1 - \frac{1}{3}\right)^2. \tag{1}
\]

Moving the $x$ terms to the left, the $y$ terms to the right and factoring, we get

\[
(x_0 - x_1)(x_0 + x_1 - 2\sqrt{2}) = (y_0 - y_1)(y_0 + y_1 - \frac{2}{3}).
\]

As the right side is rational and $\sqrt{2}$ is irrational, we must have $x_0 = x_1$. Then the left side is $0$, which forces $y_1 = y_0$ since $y_1 + y_0$ is integer. So all the lattice points are the same.

Now consider the circle with center at $(0, 0)$ and radius $r$. As $r$ increases from $0$ to infinity, the number of lattice points inside the circle increase from $0$ to infinity. As the last paragraph shows, the increase cannot jump by $2$ or more. So the statement is true.

*Other commended solvers:* NG Cheuk Chi (Tsuen Wan Public Ho Chuen Yiu Memorial College) and NG Ka Chun (Queen Elizabeth School, Form 7).

**Example 8.** Show that the equation $a^2 + b^2 = c^4$ has infinitely many solutions. **Solution.** We will use the identity

\[
1^3 + 2^3 + \ldots + n^3 = \left(\frac{n(n+1)^2}{2}\right)^2,
\]

which is a standard exercise of mathematical induction. From this identity, we get

\[
\left(\frac{(n-1)n}{2}\right)^2 + n^3 = \left(\frac{n(n+1)}{2}\right)^2.
\]

For $n > 1$. All we need to do now is to show there are infinitely many positive integers $n$ such that $n(n + 1)/2 = k^2$ for some positive integer $k$. Then $(a, b, c) = ((n - 1)/2, n, k)$ solves the problem.

Now $n(n + 1)/2 = k^2$ can be rewritten as $(2n + 1)^2 - 2(2k)^2 = 1$. We know $x^2 - 2y^2 = 1$ has infinitely many positive solutions. For any such $(x, y)$, clearly $x$ is odd, say $x = 2m + 1$. They $y^2 = 2m^2 + 2m$ implies $y$ is even. So any such $(x, y)$ is of the form $(2m + 1, 2k)$. Therefore, there are infinitely many such $n$. 