Mathematical Excalibur

Volume 6, Number 4 *October 2001 – December 2001*

Olympiad Corner

The 18th Balkan Mathematical Olympiad, Belgrade, Yugoslavia, 5 May 2001

Problem 1. Let n be a positive integer. Show that if a and b are integers greater then 1 such that $2^n - 1 = ab$, then the number $ab - (a - b) - 1$ is of the form $k \cdot 2^{2m}$, where k is odd and m is a positive integer.

Problem 2. Prove that if a convex pentagon satisfies the following conditions:

(1) all interior angles are congruent; and

(2) the lengths of all sides are rational numbers,

then it is a regular pentagon.

Problem 3. Let a, b, c be positive real numbers such that $a + b + c \geq abc$. Prove that

$$a^2 + b^2 + c^2 \geq \sqrt{3}abc.$$

Problem 4. A cube of dimensions $3\times3\times3$ is divided into 27 congruent unit cubical cells.

(continued on page 4)

Editors: 張百康 (CHEUNG Pak-Hong), Munsang College, HK
高子眉 (KO Tsz-Mei)
梁達榮 (LEUNG Tat-Wing), Applied Math. Dept., HKPU
李健賢 (LI Kin-Yin), Math. Dept., HKUST
吳鏡波 (NG Keng-Po Roger), ITC HKPU

Artist: 楊秀英 (YEUNG Sau-Ying Camille), MFA, CU

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line: http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***15 January 2001***.

For individual subscription for the next five issues for the 01-02 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

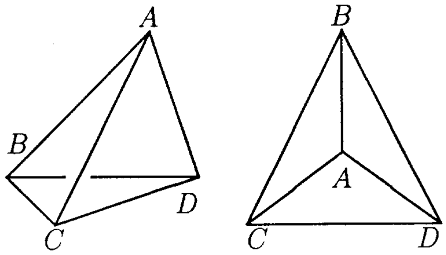
Dr. Kin-Yin LI
Department of Mathematics
The Hong Kong University of Science and Technology
Clear Water Bay, Kowloon, Hong Kong

Fax: (852) 2358 1643
Email: makyli@ust.hk

六個頂點的多面體

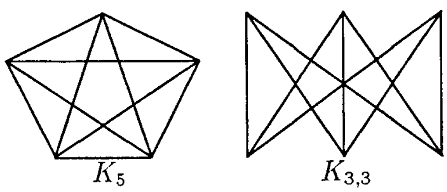
吳主居 **(Richard Travis NG)**

圖一左顯示一個四面體，用邊作輪廓，如將底面擴張，然後把其他的邊壓下去，可得到一個平面圖，各邊祇在頂點處相交，如圖一右所示。



（插圖一）

非平面圖不能是多面體的輪廓，最基本的非平面圖有兩個。第一個有五個頂點，兩兩相連，稱為 K_5 ，見圖二左。第二個有六個頂點，分為兩組，各有三個，同組的互不相連，不同組的則兩兩相連，稱為 $K_{3,3}$，見圖二右。



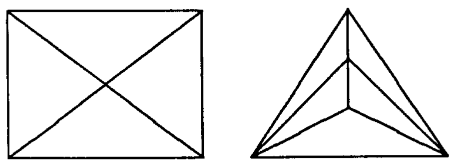
（插圖二）

一個頂點所在邊上的數量，稱為它的度數，一個代表多面體的平面圖，每個頂點的度數，都不能少於 3，所以任何多面體，都不少於四個頂點。假如它祇有四個頂點，它們的度數必定是 (3，3，3，3)，唯一的可能就是圖一的四面體。

假如一個多面體有五個頂點，看來它們的度數可能會是：

(3，3，3，3，3)，(3，3，3，3，4)，(3，3，3，4，4)，(3，3，4，4，4)，(3，4，4，4，4) 或 (4，4，4，4，4)。

不過很快便會發現，左面那三組是不可能的，因為各頂點度數之和，必定是邊數的雙倍，不可能是奇數。右面第一組是個四邊形為底的金字塔，見圖三左，第二組是個三角形為底的雙金字塔，見圖三右。最後一組是 K_5 ，不是平面圖，不能代表多面體。

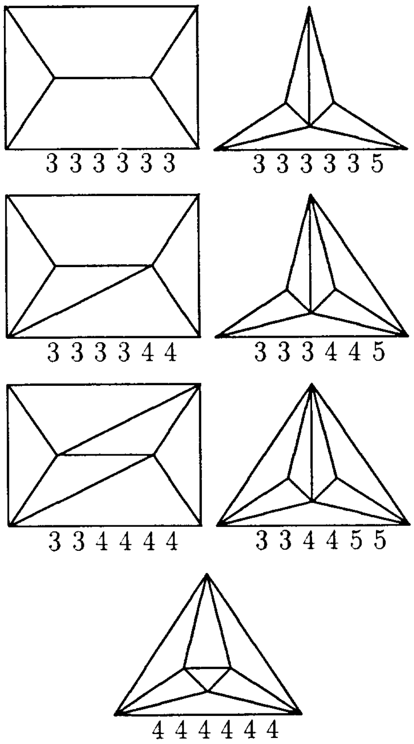


（插圖三）

六個頂點的多面體，有多少個呢？每個頂點的度數，都是 3，4 或 5，有下列可能:

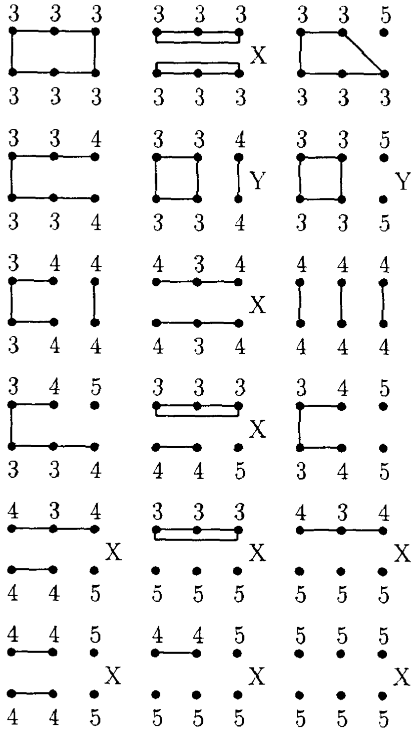
(3，3，3，3，3，3)，(3，3，3，3，3，5)，
(3，3，3，3，4，4)，(3，3，3，3，5，5)，
(3，3，3，4，4，5)，(3，3，4，4，4，4)，
(3，3，3，5，5，5)，(3，3，4，4，5，5)，
(3，4，4，4，4，5)，(4，4，4，4，4，4)，
(3，3，5，5，5，5)，(3，4，4，5，5，5)，
(4，4，4，4，5，5)，(3，5，5，5，5，5)，
(4，4，5，5，5，5)或(5，5，5，5，5，5)。

這十六組其中四組，有兩種表達方式，所以共有二十種情況，我們發現有七個不同的多面體，見圖四。



（插圖四）

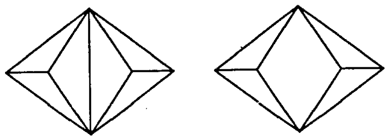
最後證明，就祇有這七種。我們先試劃這些圖，因為頂點太多兩兩相連，祇劃出缺了的邊比較容易，亦立即發現 (3，3，5，5，5，5) 和 (3，5，5，5，5，5) 這兩組是不能成立的，其餘十八種在圖五列出。



（插圖五）

沒做記號那七組，代表我們那七種多面體，用 X 做記號的，都含有 $K_{3,3}$ 在內，所以不可能是平面圖。用 Y 做記號的，雖然它們都是平面圖，但不能代表多面體。

先看圖六左的 (3，3，3，3，5，5)，它僅能代表兩個共邊的四面體，不是一個多面體。再看圖六右 (3，3，3，3，4，4) 的第二種情況，兩個度數為 4 的頂點互不相連，兩個四邊形的面，有兩個不相鄰的公共頂點，這也是不可能的。



（插圖六）

Remarks by Professor Andy Liu
(University of Alberta, Canada)

Polyhedra with Six Vertices is the work of Richard Travis Ng, currently a Grade 12 student at Archbishop MacDonald High School in Edmonton, Canada. The result is equivalent to that in John McClellan's *The Hexahedra Problem* (Recreational Mathematics Magazine, 4, 1961, 34-40), which counts the number of polyhedra with six faces. The problem is also featured in Martin Gardner's "New Mathematical Dviersions" (Mathematical Association of America, 1995, 224-225 and 233). However, the proof in this article is much simpler.

The 2001 Hong Kong IMO team with Professor Andrew Wiles at Washington, DC taken on July 13, 2001. From left to right, *Leung Wai Ying, Yu Hok Pun, Ko Man Ho, Professor Wiles, Cheng Kei Tsi, Chan Kin Hang, Chao Khek Lun*.

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. Solutions should be preceded by the solver's name, home (or **email**) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, Hong Kong University of Science & Technology, Clear Water Bay, Kowloon*. The deadline for submitting solutions is ***15 January 2001.***

Problem 136. For a triangle *ABC*, if sin *A*, sin *B*, sin *C* are rational, prove that cos *A*, cos *B*, cos *C* must also be rational.
If cos *A*, cos *B*, cos *C* are rational, must at least one of sin *A*, sin *B*, sin *C* be rational?

Problem 137. Prove that for every positive integer *n*,

$$(\sqrt{3}+\sqrt{2})^{1/n}+(\sqrt{3}-\sqrt{2})^{1/n}$$

is irrational.

Problem 138. (*Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain*) If $a+b$ and $a-b$ are relatively prime integers, find the greatest common divisor (or the highest common factor) of $2a+(1+2a)(a^2-b^2)$ and $2a(a^2+2a-b^2)(a^2-b^2)$.

Problem 139. Let a line intersect a pair of concentric circles at points *A*, *B*, *C*, *D* in that order. Let *E* be on the outer circle and *F* be on the inner circle such that chords *AE* and *BF* are parallel. Let *G* and *H* be points on chords *BF* and *AE* that are the feet of perpendiculars from *C* to *BF* and from *D* to *AE*, respectively. Prove that *EH* = *FG*. (*Source: 1958 Shanghai City Math Competition*)

Problem 140. A convex pentagon has five equal sides. Prove that the interior of the five circles with the five sides as diameters do not cover the interior of the pentagon.

****

Solutions

****

Problem 131. Find the greatest common divisor (or highest common factor) of the numbers n^n-n for $n = 3, 5, 7, \ldots$.

Solution. **CHAN Wai Hong** (STFA Leung Kau Kui College, Form 6), **CHUNG Tat Chi** (Queen Elizabeth School, Form 5), **Jack LAU Wai Shun** (Tsuen Wan Public Ho Chuen Yiu Memorial College, Form 6), **LEE Tsun Man Clement** (St. Paul's College, Form 3), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 6), **Boris YIM Shing Yik** (Wah Yan College, Kowloon) and **YUEN Ka Wai** (Carmel Divine Grace Foundation Secondary School, Form 6).

Since the smallest number is $3^3-3=24$, the greatest common divisor is at most 24. For $n=2k+1$,

$$n^n-n=n\left((n^2)^k-1\right)$$

$$=(n-1)n(n+1)(n^{2k-2}+\cdots+1).$$

Now one of $n-1,\ n,\ n+1$ is divisible by 3. Also, $(n-1)(n+1)=4k(k+1)$ is divisible by 8. So n^n-n is divisible by 24. Therefore, the greatest common divisor is 24.

Other commended solvers: **CHAO Khek Lun Harold** (St. Paul's College, Form 7), **CHAU Suk Ling** (Queen Elizabeth School, Form 7), **CHIU Yik Yin** (St. Joseph's Anglo-Chinese School, Form 6), **CHU Tsz Ying** (St. Joseph's Anglo-Chinese School), **KWOK Sze Ming** (Queen Elizabeth School, Form 6), **LAW Siu Lun** (CCC Ming Kei College, Form 7), **Antonio LEI Iat Fong** and **Alvin LEE Kar Wai** (Colchester Royal Grammar School, England), **LEUNG Wai Ying** (Queen Elizabeth School, Form 7), **Campion LOONG** (STFA Leung Kau Kui College, Form 6), **NG Ka Chun** (Queen Elizabeth School, Form 7), **SIU Ho Chung** (Queen's College, Form 3), **TANG Sheung Kon** (STFA Leung Kau Kui College, Form 7), **TSOI Hung Ming** (SKH Lam Woo Memorial Secondary School, Form 7), **WONG Chun Ho** (STFA Leung Kau Kui College, Form 7), **Tak Wai Alan WONG** (University of Toronto, Canada), **WONG Tsz Wai** (Hong Kong Chinese Women's Club College, Form 6), **WONG Wing Hong** (La Salle College, Form 4) and **YUEN Chi Hung** (SKH Chan Young Secondary School, Form 4).

Problem 132. Points *D*, *E*, *F* are chosen on sides *AB, BC, CA* of ΔABC, respectively, so that *DE* = *BE* and *FE* = *CE*. Prove that the center of the circumcircle of ΔADF lies on the angle bisector of $\angle DEF$. (*Source: 1989 USSR Math Olympiad*)

Solution. **CHAN Wai Hong** (STFA Leung Kau Kui College, Form 6), **CHAO Khek Lun Harold** (St. Paul's College, Form 7), **CHAU Suk Ling** (Queen Elizabeth School, Form 7), **CHIU Yik Yin** (St. Joseph's Anglo-Chinese School, Form 6), **CHU Tsz Ying** (St. Joseph's Anglo-Chinese School), **CHUNG Tat Chi** (Queen Elizabeth School, Form 5), **FOK Chi Kwong** (Yuen Long Merchants Association Secondary School, Form 5), **KWOK Sze Ming** (Queen Elizabeth School, Form 6), **KWONG Tin Yan** (True Light Girls' College, Form 6), **Antonio LEI Iat Fong** and **Alvin LEE Kar Wai** (Colchester Royal Grammar School, England), **LEUNG Wai Ying** (Queen Elizabeth School, Form 7), **SIU Ho Chung** (Queen's College, Form 3), **WONG Tsz Wai** (Hong Kong Chinese Women's Club College, Form 6) and **WONG Wing Hong** (La Salle College, Form 4).

Let *O* be the circumcenter of ΔADF and α, β, γ be the measures of angles *A*, *B*, *C* of ΔABC . Then $\angle DOF = 2\alpha$ and $180^\circ - \angle DEF = \angle BED + \angle CEF = 360^\circ - 2\beta - 2\gamma = 2\alpha = \angle DOF$. So *ODEF* is a cyclic quadrilateral. Since *OD* = *OF*, $\angle DEO = \angle FEO$. So *O* is on the angle bisector of $\angle DEF$.

Other commended solvers: **NG Ka Chun** (Queen Elizabeth School, Form 7), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 6), **TSOI Hung Ming** (SKH Lam Woo Memorial Secondary School, Form 7) and **YUEN Chi Hung** (SKH Chan Young Secondary School, Form 4).

Problem 133. (a) Are there real numbers *a* and *b* such that $a+b$ is rational and a^n+b^n is irrational for every integer $n \ge 2$? (b) Are there real numbers *a* and *b* such that $a+b$ is irrational and a^n+b^n is rational for every integer $n \ge 2$? (*Source: 1989 USSR Math Olympiad*)

Solution. **CHAO Khek Lun Harold** (St. Paul's College, Form 7), **LEUNG Wai Ying** (Queen Elizabeth School, Form 7) and **YUEN Chi Hung** (SKH Chan Young Secondary School, Form 4).

(a) Let $a = \sqrt{2}+1$ and $b = -\sqrt{2}$. Then $a+b=1$ is rational. For an integer $n \ge 2$, from the binomial theorem, since binomial coefficients are positive integers, we get

$$(\sqrt{2}+1)^n = r_n\sqrt{2}+s_n,$$

where r_n, s_n are positive integers. For every positive integer *k*, we have $a^{2k} + b^{2k} = r_{2k}\sqrt{2} + s_{2k} + 2^k$ and $a^{2k+1} + b^{2k+1} = (r_{2k+1}-2^k)\sqrt{2} + s_{2k+1}$. Since

$$r_{2k+1} \ge 2^k + C_2^{2k+1}2^{k-1} > 2^k,$$

$a^n + b^n$ is irrational for $n \ge 2$.

(b) Suppose such *a* and *b* exist. Then neither of them can be zero from cases *n* = 2 and 3. Now

$$(a^2+b^2)^2 = (a^4+b^4)+2a^2b^2$$

implies a^2b^2 is rational, but then

$$(a^2+b^2)(a^3+b^3)$$
$$=(a^5+b^5)+a^2b^2(a+b)$$

will imply $a+b$ is rational, which is a contradiction.

Other commended solvers: **NG Ka Chun** (Queen Elizabeth School, Form 7), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 6) and **TSUI Chun Wa** (Carmel Divine Grace Foundation Secondary School, Form 6).

Problem 134. Ivan and Peter alternatively write down 0 or 1 from left to right until each of them has written 2001 digits. Peter is a winner if the number, interpreted as in base 2, is not the sum of two perfect squares. Prove that Peter has a winning strategy. (*Source: 2001 Bulgarian Winter Math Competition*)

Solution. (*Official Solution)*

Peter may use the following strategy: he plans to write three 1's and 1998 0's, until Ivan begins to write a 1. Once Ivan writes his first 1, then Peter will switch to follow Ivan exactly from that point to the end.

If Peter succeeded to write three 1's and 1998 0's, then Ivan wrote only 0's and the number formed would be 21×4^{1998}. This is not the sum of two perfect squares since 21 is not the sum of two perfect squares.

If Ivan wrote a 1 at some point, then Peter's strategy would cause the number to have an even number of 0's on the right preceded by two 1's. Hence, the number would be of the form $(4n+3)4^m$. This kind of numbers are also not the sums of two perfect squares, otherwise we have integers *x*, *y* such that

$$x^2+y^2=(4n+3)4^m,$$

which implies *x*, *y* are both even if *m* is a positive integer. Keep on canceling 2 from both *x* and *y*. Then at the end, we will get 4*n* + 3 as a sum of two perfect squares, which is impossible by checking the sum of odd and even perfect squares.

Other commended solvers: **LEUNG Wai Ying** (Queen Elizabeth School, Form 7) and **NG Ka Chun** (Queen Elizabeth School, Form 7).

Problem 135. Show that for $n \ge 2$, if $a_1, a_2, ..., a_n > 0$, then

$$(a_1^3+1)(a_2^3+1)\cdots(a_n^3+1) \ge$$
$$(a_1^2a_2+1)(a_2^2a_3+1)\cdots(a_n^2a_1+1).$$

(*Source: 7th Czech-Slovak-Polish Match*)

Solution 1. **CHIU Yik Yin** (St. Joseph's Anglo-Chinese School, Form 6), **CHU Tsz Ying** (St. Joseph's Anglo-Chinese School), **FOK Chi Kwong** (Yuen Long Merchants Association Secondary School, Form 5) and **WONG Tsz Wai** (Hong Kong Chinese Women's Club College, Form 6).

First we shall prove that

$$\left(a_1^3+1\right)^2\left(a_2^3+1\right) \ge \left(a_1^2a_2+1\right)^3.$$

By expansion, this is the same as

$$a_1^6a_2^3+2a_1^3a_2^3+a_2^3+a_1^6+2a_1^3+1$$
$$\ge a_1^6a_2^3+3a_1^4a_2^2+3a_1^2a_2+1.$$

This follows by regrouping and factoring to get

$$a_1^3(a_1-a_2)^2(a_1+2a_2)$$
$$+\left(a_1-a_2\right)^2(2a_1+a_2)\ge 0$$

or from

$$a_2^3+2a_1^3 \ge 3\left(a_2^3a_1^3a_1^3\right)^{1/3} = 3a_1^2a_2,$$
$$2a_1^3a_2^3+a_1^6 \ge 3\left(a_1^{12}a_2^6\right)^{1/3} = 3a_1^4a_2^2,$$

by the AM-GM inequality. Similarly, we get

$$\left(a_i^3+1\right)^2\left(a_{i+1}^3+1\right) \ge \left(a_i^2a_{i+1}+1\right)^3$$

for i = 2, 3, …, n with $a_{n+1} = a_1$. Multiplying these inequalities and taking cube root, we get the desired inequality.

Solution 2. **Murray KLAMKIN** (University of Alberta, Canada) and **NG Ka Chun** (Queen Elizabeth School, Form 7).

Let $a_{n+1} = a_1$. For *i* = 1, 2, …, *n*, by Hölder's inequality, we have

$$(a_i^3+1)^{2/3}(a_{i+1}^3+1)^{1/3}$$
$$\ge (a_i^3)^{2/3}(a_{i+1}^3)^{1/3}+1.$$

Multiplying these *n* inequalities, we get the desired inequality.

Comments: For the statement and proof of Hölder's inequality, we refer the readers to vol. 5, no. 4, page 2 of *Math Excalibur*.

Other commended solvers: **CHAO Khek Lun Harold** (St. Paul's College, Form 7), **Antonio LEI Iat Fong** and **Alvin LEE Kar Wai** (Colchester Royal Grammar School, England), **LEUNG Wai Ying** (Queen Elizabeth School, Form 7), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 6), **TSOI Hung Ming** (SKH Lam Woo Memorial Secondary School, Form 7), **WONG Chun Ho** (STFA Leung Kau Kui College, Form 7), and **YUEN Chi Hung** (SKH Chan Young Secondary School, Form 4).

Olympiad Corner

(continued from page 1)

One of these cells is empty and the others are filled with unit cubes labeled in an arbitrary manner with numbers 1, 2, …, 26. An admissible move is the moving of a unit cube into an adjacent empty cell. Is there a finite sequence of admissible moves after which the unit cube labeled with *k* and the unit cube labeled with 27 – *k* are interchanged, for each *k* = 1, 2, …, 13? (Two cells are said to be adjacent if they share a common face.)