Mathematical Excalibur

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Olympiad Corner

The 31st United States of America Mathematical Olympiad 2002

Problem 1. Let S be a set with 2002 elements, and let N be an integer with $0 \le N \le 2^{2002}$. Prove that it is possible to color every subset of S either black or white so that the following conditions hold:

(a) the union of any two white subsets is white;

(b) the union of any two black subsets is black;

(c) there are exactly N white subsets.

Problem 2. Let ABC be a triangle such that

$$\left(\cot\frac{A}{2}\right)^2+\left(2\cot\frac{B}{2}\right)^2+\left(3\cot\frac{C}{2}\right)^2=\left(\frac{6s}{7r}\right)^2,$$

where s and r denote its semiperimeter and its inradius, respectively. Prove that triangle ABC is similar to a triangle T whose side lengths are all positive integers with no common divisor and determine these integers.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line: http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***Sep 20, 2002***.

For individual subscription for the next five issues for the 01-02 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Problem Solving I

Kin-Yin LI

George Polya's famous book *How to Solve It* is a book we highly recommend every student who is interested in problem solving to read. In solving a difficult problem, Polya teaches us to ask the following questions. What is the condition to be satisfied? Have you seen a similar problem? Can you restate the problem in another way or in a related way? Where is the difficulty? If you cannot solve it, can you solve a part of the problem if the condition is relaxed. Can you solve special cases? Is there any pattern you can see from the special cases? Can you guess the answer? What clues can you get from the answer or the special cases? Below we will provide some examples to guide the student in analyzing problems.

Example 1. (*Polya, How to Solve It, pp. 23-25*) Given ΔABC with AB the longest side. Construct a square having two vertices on side AB and one vertex on each of sides BC and CA using a compass and a straightedge (i.e. a ruler without markings).

Analysis. (*Where is the difficulty?*) The difficulty lies in requiring all four vertices on the sides of the triangle. If we relax *four* to *three*, the problem becomes much easier. On CA, take a point P close to A. Draw the perpendicular from P to AB and let the foot be Q. With Q as center and PQ as radius, draw a circle and let it intersect AB at R. Draw the perpendicular line to AB through R and let S be the point on the line which is PQ units from R and on the same side of AB as P. Then $PQRS$ is a square with P on CA and Q, R on AB.

(*What happens if you move the point P on side CA?*) You get a square similar to $PQRS$. (*What happens in the special case P = A?*) You get a point. (*What happens to S if you move P from A toward C?*) As P moves along AC, the triangles APQ will be similar to each other. Then the triangles APS will also be similar to each other and S will trace a line segment from A. This line AS intersects BC at a point S', which is the fourth vertex we need. From S', we can find the three other vertices dropping perpendicular lines and rotating points.

Example 2. (*1995 Russian Math Olympiad*) There are $n > 1$ seats at a merry-go-around. A boy takes n rides. Between each ride, he moves clockwise a certain number (less than n) of places to a new horse. Each time he moves a different number of places. Find all n for which the boy ends up riding each horse.

Analysis. (*Can you solve special cases?*) The cases n = 2, 4, 6 work, but the cases n = 3, 5 do not work. (*Can you guess the answer?*) The answer should be n is even. (*What clues can you get from the special cases?*) From experimenting with cases, we see that if $n > 1$ is odd, then the last ride seems to always repeat the first horse. (*Why?*) From the first to the last ride, the boy moved $1 + 2 + \cdots + (n-1) = n(n-1)/2$ places. If $n > 1$ is odd, this is a multiple of n and so we repeat the first horse.

(Is there any pattern you can see from the special cases when n is even?) Name the horses 1, 2, …, n in the clockwise direction. For n = 2, we can ride horses 1, 2 in that order and the move sequence is 1. For n = 4, we can ride horses 1, 2, 4, 3 in that order and the move sequence is 1, 2, 3. For n = 6, we can ride horses 1, 2, 6, 3, 5, 4 and the

move sequence is 1, 4, 3, 2, 5. Then for the general even cases n, we can ride horses 1, 2, n, 3, $n-1$, …, $(n/2)+1$ in that order with move sequence 1, $n-2$, 3, $n-4$, …, 2, $n-1$. The numbers in the move sequence are all distinct as it is the result of merging odd numbers 1, 3, …, $n-1$ with even numbers $n-2$, $n-4$, …, 2.

Example 3. (*1982 Putnam Exam*) Let $K(x, y, z)$ be the area of a triangle with sides x, y, z. For any two triangles with sides a, b, c and a', b', c' respectively, show that

$$\sqrt{K(a,b,c)}+\sqrt{K(a',b',c')} \le \sqrt{K(a+a',b+b',c+c')}$$

and determine the case of equality.

Analysis. (*Can you restate the problem in another way?*) As the problem is about the area and sides of a triangle, we bring out Heron's formula, which asserts the area of a triangle with sides x, y, z is given by

$$K(x,y,z)=\sqrt{s(s-x)(s-y)(s-z)},$$

where s is half the perimeter, i.e. $s=\frac{1}{2}(x+y+z)$. Using this formula, the problem becomes showing

$$\sqrt[4]{stuv}+\sqrt[4]{s't'u'v'} \le \sqrt[4]{(s+s')(t+t')(u+u')(v+v')},$$

where $s=\frac{1}{2}(a+b+c)$, $t=s-a$, $u=s-b$, $v=s-c$ and similarly for s',t',u',v'.

(*Have you seen a similar problem or can you relax the condition?*) For those who saw the forward-backward induction proof of the AM-GM inequality before, this is similar to the proof of the case $n=4$ from the case $n=2$. For the others, having groups of four variables are difficult to work with. We may consider the more manageable case $n=2$. If we replace 4 by 2, we get a simpler inequality

$$\sqrt{xy}+\sqrt{x'y'} \le \sqrt{(x+x')(y+y')}.$$

This is easier. Squaring both sides, canceling common terms, then factoring, this turns out to be just $\left(\sqrt{xy'}-\sqrt{x'y}\right)^2 \ge 0$. Equality holds if and only if $x:x'=y:y'$. Applying this simpler inequality twice, we easily get the required inequality

$$\begin{aligned}&\sqrt[4]{stuv}+\sqrt[4]{s't'u'v'}\\ &\le \sqrt{(\sqrt{st}+\sqrt{s't'})(\sqrt{uv}+\sqrt{u'v'})}\\ &\le \sqrt{\sqrt{(s+s')(t+t')}\sqrt{(u+u')(v+v')}}.\end{aligned}$$

Tracing the equality case back to the simpler inequality, we see equality holds if and only if $a:b:c=a':b':c'$, i.e. the triangles are similar.

Example 4. Is there a way to pack 250 $1\times1\times4$ bricks into a $10\times10\times10$ box?

Analysis. (*Where is the difficulty?*) 10 is large for a 3 dimensional cube. We can relax the problem a bit by considering a two dimensional analogous problem with smaller numbers, say 1×2 cards pack into a 8×8 board. This is clearly possible. (*What if we relax the board to be a square, say by taking out two squares from the board?*) This may become impossible. For example, if the 8×8 board is a checkerboard and we take out two black squares, then since every 1×2 card covers exactly one white and one black square, any possible covering must require the board to have equal number of white and black squares.

(*What clue can you get from the special cases?*) Coloring a board can help to solve the problem. (*Can we restate the problem in a related way?*) Is it possible to color the cubes of the $10\times10\times10$ box with four colors in such a way that in every four consecutive cubes each color occurs exactly once, where consecutive cubes are cubes sharing a common face? Yes, we can put color 1 in a corner cube, then extend the coloring to the whole box by putting colors 1, 2, 3, 4 periodically in each of the three perpendicular directions parallel to the edges of the box. However, a counting shows that for the $10\times10\times10$ box, there are 251 color 1 cubes, 251 color 2 cubes, 249 color 3 cubes and 249 color 4 cubes. So the required packing is impossible.

Example 5. (*1985 Moscow Math Olympiad*) For every integer $n\ge3$, show that $2^n=7x^2+y^2$ for some odd positive integers x and y.

Analysis. (*cf. Arthur Engel, Problem-Solving Strategies, pp. 126-127*) (*Can you solve special cases?*) For $n = 3, 4, \ldots, 10$, we have the table:

n	3	4	5	6	7	8	9	10
$x=x_n$	1	1	1	3	1	5	7	3
$y=y_n$	1	3	5	1	11	9	13	31

(*Is there any pattern you can see from the special cases?*) In cases $n = 3, 5, 8$, it seems that x_{n+1} is the average of x_n and y_n. For cases $n = 4, 6, 7, 9, 10$, the average of x_n and y_n is even and it seems that $|x_n-y_n|=2x_{n+1}$. (*Can you guess the answer?*) The answer should be

$$x_{n+1}=\begin{cases}\frac{1}{2}(x_n+y_n) & \text{if } \frac{1}{2}(x_n+y_n) \text{ is odd}\\ \frac{1}{2}|x_n-y_n| & \text{if } \frac{1}{2}(x_n+y_n) \text{ is even}\end{cases}$$

and

$$\begin{aligned}y_{n+1}&=\sqrt{2^{n+1}-7x_{n+1}^2}\\ &=\sqrt{2(7x_n^2+y_n^2)-7x_{n+1}^2}\\ &=\begin{cases}\frac{1}{2}|7x_n-y_n| & \text{if } \frac{1}{2}(x_n+y_n) \text{ is odd}\\ \frac{1}{2}(7x_n+y_n) & \text{if } \frac{1}{2}(x_n+y_n) \text{ is even}\end{cases}.\end{aligned}$$

(*Is this correct?*) The case $n = 3$ is correct. If $2^n=7x_n^2+y_n^2$, then the choice of y_{n+1} will give $2^{n+1}=7x_{n+1}^2+y_{n+1}^2$. (*Must x_{n+1} and y_{n+1} be odd positive integers?*) Yes, this can be checked by writing x_n and y_n in the form $4k\pm1$.

IMO 2002

IMO 2002 will be held in Glasgow, United Kingdom from July 19 to July 30 this summer. Based on the selection test performances, the following students have been chosen to represent Hong Kong:

CHAO Khek Lun (*St. Paul's College*)
CHAU Suk Ling (*Queen Elizabeth School*)
CHENG Kei Tsi (*La Salle College*)
IP Chi Ho (*St. Joseph College*)
LEUNG Wai Ying (*Queen Elizabeth School*)
YU Hok Pun (*SKH Bishop Baker Secondary Sch*)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. Solutions should be preceded by the solver's name, home (or **email**) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is ***September 20, 2002***.

Problem 151. Every integer greater than 2 can be written as a sum of distinct positive integers. Let $A(n)$ be the maximum number of terms in such a sum for n. Find $A(n)$. (*Source: 1993 German Math Olympiad*)

Problem 152. Let $ABCD$ be a cyclic quadrilateral with E as the intersection of lines AD and BC. Let M be the intersection of line BD with the line through E parallel to AC. From M, draw a tangent line to the circumcircle of $ABCD$ touching the circle at T. Prove that $MT = ME$. (*Source: 1957 Nanjing Math Competition*)

Problem 153. Let R denote the real numbers. Find all functions f: $R \to R$ such that the equality $f(f(x) + y) = f(x^2 - y) + 4f(x)y$ holds for all pairs of real numbers x, y. (*source: 1997 Czech-Slovak Match*)

Problem 154. For nonnegative numbers a, d and positive numbers b, c satisfying $b + c \geq a + d$, what is the minimum value of $\frac{b}{c+d} + \frac{c}{a+b}$?

(*Source: 1988 All Soviet Math Olympiad*)

Problem 155. We are given 1997 distinct positive integers, any 10 of which have the same least common multiple. Find the maximum possible number of pairwise relatively prime numbers among them. (*Source: 1997 Hungarian Math Olympiad*)

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Solutions

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Problem 146. Is it possible to partition a square into a number of congruent right triangles each containing a 30° angle? (*Source: 1994 Russian Math Olympiad, 3rd Round*)

Solution. **CHAO Khek Lun Harold** (St. Paul's College, Form 7), **CHEUNG Chung Yeung** (STFA Leung Kau Kui College, Form 4), **Antonio LEI** (Colchester Royal Grammar School, UK, Year 12), **LEUNG Wai Ying** (Queen Elizabeth School, Form 7), **POON Ming Fung** (STFA Leung Kau Kui College, Form 4), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 6), **WONG Wing Hong** (La Salle College, Form 4) and **Richard YEUNG Wing Fung** (STFA Leung Kau Kui College, Form 4).

Without loss of generality, let the sides of the triangles be 2, 1, $\sqrt{3}$. Assume n such triangles can partition a square. Since the sides of the square are formed by sides of these triangles, so the sides of the square are of the form $a + b\sqrt{3}$, where a, b are nonnegative integers. Considering the area of the square, we get $(a + b\sqrt{3})^2 = \frac{n\sqrt{3}}{2}$, which is the same as $2(a^2 + 3b^2) = (n - 4ab)\sqrt{3}$. Since a, b are integers and $\sqrt{3}$ is irrational, we must have $a^2 + 3b^2 = 0$ and $n - 4ab = 0$. The first equation implies $a = b = 0$, which forces the sides of the square to be 0, a contradiction.

Other commended solver: **WONG Chun Ho** (STFA Leung Kau Kui College, Form 7).

Problem 147. Factor $x^8 + 4x^2 + 4$ into two nonconstant polynomials with integer coefficients.

Solution. **CHENG Ka Wai** (STFA Leung Kau Kui College, Form 4), **CHEUNG CHUNG YEUNG** (STFA Leung Kau Kui College, Form 4), **FUNG Yi** (La Salle College, Form 4), **LEUNG Wai Ying** (Queen Elizabeth School, Form 7), **POON Ming Fung** (STFA Leung Kau Kui College, Form 4), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 6) and **TANG Sze Ming** (STFA Leung Kau Kui College, Form 4).

$$\begin{aligned} & x^8 + 4x^2 + 4 \\ &= (x^8 + 4x^6 + 8x^4 + 8x^2 + 4) \\ &\quad - (4x^6 + 8x^4 + 4x^2) \\ &= (x^4 + 2x^2 + 2)^2 - (2x^3 + 2x)^2 \\ &= (x^4 + 2x^3 + 2x^2 + 2x + 2) \\ &\quad \times (x^4 - 2x^3 + 2x^2 + 2). \end{aligned}$$

Other commended solvers: **CHAO Khek Lun Harold** (St. Paul's College, Form 7), **HUI Chun Yin John** (Hong Kong Chinese Women's Club College, Form 6), **LAW Siu Lun Jack** (CCC Ming Kei College, Form 7), **WONG Chun Ho** (STFA Leung Kau Kui College, Form 7), **Tak Wai Alan WONG** (University of Toronto, Canada), **WONG Wing Hong** (La Salle College, Form 4) & **YEUNG Kai Tsz Max** (Ju Ching Chu Secondary School, Form 5).

Problem 148. Find all distinct prime numbers p, q, r, s such that their sum is also prime and both $p^2 + qs$, $p^2 + qr$ are perfect square numbers. (*Source: 1994 Russian Math Olympiad, 4th Round*)

Solution. **CHAO Khek Lun Harold** (St. Paul's College, Form 7), **CHEUNG CHUNG YEUNG** (STFA Leung Kau Kui College, Form 4), **LAW Siu Lun Jack** (CCC Ming Kei College, Form 7), **Antonio LEI** (Colchester Royal Grammar School, UK, Year 12), **LEUNG Wai Ying** (Queen Elizabeth School, Form 7), **POON Ming Fung** (STFA Leung Kau Kui College, Form 4), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 6), **TANG Chun Pong Ricky** (La Salle College, Form 4), **WONG Chun Ho** (STFA Leung Kau Kui College, Form 7), **WONG Wing Hong** (La Salle College, Form 4), **Richard YEUNG Wing Fung** (STFA Leung Kau Kui College, Form 4) and **YUEN Ka Wai** (Carmel Divine Grace Foundation Secondary School, Form 6).

Since the sum of the primes p, q, r, s is a prime greater than 2, one of p, q, r, s is 2. Suppose $p \neq 2$. Then one of q, r, s is 2 so that one of $p^2 + qs$, $p^2 + qr$ is of the form $(2m+1)^2 + 2(2n + 1) = 4(m^2 + m + n) + 3$, which cannot be a perfect square as perfect squares are of the form $(2k)^2 = 4k^2$ or $(2k+1)^2 = 4(k^2 + k) + 1$. So $p = 2$.

Suppose $2^2 + qs = a^2$, then q, s odd implies a odd and $qs = (a + 2)(a - 2)$. Since q, s are prime, the smaller factor $a - 2 = 1$, q or s. In the first case, $a = 3$ and $qs = 5$, which is impossible. In the remaining two cases, either $q = a - 2$, $s = a + 2 = q + 4$ or $s = a - 2$, $q = a + 2 = s + 4$. Next $2^2 + qr = b^2$ will similarly implies q, r differe by 4. As q, r, s are distinct primes, one of r, s is $q - 4$ and the other is $q = 4$. Note that $q - 4$, q, $q + 4$ have different remainders when they are divided by 3. One of them is 3 and it must be $q - 4$. Thus there are two solutions $(p, q, r, s) = (2, 7, 3, 11)$ or $(2, 7, 11, 3)$. It is easy to check both solutions satisfy all conditions.

Other commended solvers: **WONG Wai Yi** (True Light Girl's College, Form 4)

Problem 149. In a 2000 × 2000 table, every square is filled with $a + 1$ or $a - 1$. It is known that the sum of these numbers is nonnegative. Prove that there are 1000 columns and 1000 rows such that the sum of the numbers in these intersection squares is at least 1000. (*Source: 1994 Russian Math Olympiad, 5th Round*)

Solution 1. **LEUNG Wai Ying** (Queen Elizabeth School, Form 7).

Since the numbers have a nonnegative sum, there is a column with a nonnegative sum. Hence there are at least one thousand squares in that column filled with +1. Thus, without loss of generality we may assume the squares in rows 1 to 1000 of column 1 are filled with +1. Evaluate the sums of the numbers in the squares of rows 1 to 1000 for each of the remaining columns. Pick the 999 columns with the largest sums in these evaluations. If these 999 columns have a nonnegative total sum S, then we are done (simply take rows 1 to 1000 and the first column with these 999 columns). Otherwise, $S < 0$ and at least one of the 999 columns has a negative sum. Since the sum of the first 100 squares in each column must be even, the sum of the first 100 squares in that column is at most –2. Then the total sum of all squares in rows 1 to 1000 is at most $1000 + S + (-2)1000 < -1000$.

Since the sum of the whole table is nonnegative, the sum of all squares in rows 1001 to 2000 would then be greater than 1000. Then choose the squares in these rows and the 1000 columns with the greatest sums. If these squares have a sum at least 1000, then we are done. Otherwise, assume the sum is less than 1000, then at least one of these 1000 columns will have a nonpositive sum. Thus, the remaining 1000 columns will each have a nonpositive sum. This will lead to the sum of all squares in rows 1001 to 2000 be less than 1000 + (0)1000 = 1000, a contradiction.

Solution 2. **CHAO Khek Lun Harold** (St. Paul's College, Form 7).

We first prove that for a $n\times n$ square filled with +1 and -1 and the sum is at least m, where m, n are of the same parity and $m < n$, there exists a $(n-1)\times(n-1)$ square the numbers there have a sum at least $m + 1$. If the sum of the numbers in the $n\times n$ square is greater than m, we may convert some of the +1 squares to -1 to make the sum equal m. Let the sum of the numbers in rows 1 to n be $r_1,\ldots,r_n$. Since $r_1+\cdots+r_n = m < n$, there is a $r_j \le 0$. For each square in row j, add up the numbers in the row and column on which the square lies. Let them be $a_1,\ldots,a_n$. Now $a_1+\cdots+a_n = m+(n-1)r_j \le m < n$. Since a_i is the sum of the numbers in $2n-1$ squares, each a_i is odd. So there exists some $a_k \le -1$. Removing row j and column k, the sum of the numbers in the remaining $(n-1)\times(n-1)$ square is $m - a_k \ge m + 1$. Finally convert back the –1 squares to +1 above and the result follows.

For the problem, start with $n = 2000$ and $m = 0$, then apply the result above 1000 times to get the desired statement.

Problem 150. Prove that in a convex n-sided polygon, no more than n diagonals can pairwise intersect. For what n, can there be n pairwise intersecting diagonals? (Here intersection points may be vertices.) (*Source: 1962 Hungarian Math Olympiad*)

Solution. **CHAO Khek Lun Harold** (St. Paul's College, Form 7) and **TANG Sze Ming** (STFA Leung Kau Kui College, Form 4.

For $n = 3$, there is no diagonal and for $n = 4$, there are exactly two intersecting diagonals. So let $n \ge 5$. Note two diagonals intersect if and only if the pairs of vertices of the diagonals share a common vertex or separate each other on the boundary. Thus, without loss of generality, we may assume the polygon is regular. For each diagonal, consider its perpendicular bisector. If n is odd, the perpendicular bisectors are exactly the n lines joining a vertex to the midpoint of its opposite side. If n is even, the perpendicular bisectors are either lines joining opposite vertices or lines joining the midpoints of opposite edges and again there are exactly n such lines. Two diagonals intersect if and only if their perpendicular bisectors do not coincide. So there can be no more than n pairwise intersecting diagonals. For $n \ge 5$, since there are exactly n different perpendicular bisectors, so there are n pairwise intersecting diagonals.

Other commended solvers: **Antonio LEI** (Colchester Royal Grammar School, UK, Year 12), **LEUNG Wai Ying** (Queen Elizabeth School, Form 7) and **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 6).

Olympiad Corner

(continued from page 1)

Problem 3. Prove that any monic polynomial (a polynomial with leading coefficient 1) of degree n with real coefficients is the average of two monic polynomials of degree n with n real roots.

Problem 4. Let R be the set of real numbers. Determine all functions $f: R \to R$ such that

$$f(x^2 - y^2) = xf(x) - yf(y)$$

for all real numbers x and y.

Problem 5. Let a, b be integers greater than 2. Prove that there exists a positive integer k and a finite sequence $n_1, n_2, \ldots, n_k$ of positive integers such that $n_1 = a$, $n_k = b$, and $n_i n_{i+1}$ is divisible by $n_i + n_{i+1}$ for each $i (1 \le i < k)$.

Problem 6. I have an $n\times n$ sheet of stamps, from which I've been asked to tear out blocks of three adjacent stamps in a single row or column. (I can only tear along the perforations separating adjacent stamps, and each block must come out of a sheet in one piece.) Let $b(n)$ be the smallest number of blocks I can tear out and make it impossible to tear out any more blocks. Prove that there are real constants c and d such that

$$\frac{1}{7}n^2 - cn \le b(n) \le \frac{1}{5}n^2 + dn$$

for all $n > 0$.