Mathematical Excalibur

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Olympiad Corner

The 43rd International Mathematical Olympiad 2002.

Problem 1. Let n be a positive integer. Let T be the set of points (x, y) in the plane where x and y are non-negative integers and $x + y < n$. Each point of T is colored red or blue. If a point (x, y) is red, then so are all points (x', y') of T with both $x' \leq x$ and $y' \leq y$. Define an X-set to be a set of n blue points having distinct x-coordinates, and a Y-set to be a set of n blue points having distinct y-coordinates. Prove that the number of X-sets is equal to the number of Y-sets.

Problem 2. Let BC be a diameter of the circle Γ with center O. Let A be a point on Γ such that $0° < \angle AOB < 120°$. Let D be the midpoint of the arc AB not containing C. The line through O parallel to DA meets the line AC at J. The perpendicular bisector of OA meets Γ at E and at F. Prove that J is the incentre of the triangle CEF.

Problem 3. Find all pairs of integers such that there exist infinitely many positive integers a for which

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Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line: http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***November 2, 2002***.

For individual subscription for the next five issues for the 01-02 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Mathematical Games (I)

Kin Y. Li

An *invariant* is a quantity that does not change. A *monovariant* is a quantity that keeps on increasing or keeps on decreasing. In some mathematical games, winning often comes from understanding the invariants or the monovariants that are controlling the games.

Example 1. (*1974 Kiev Math Olympiad*) Numbers 1, 2, 3, …, 1974 are written on a board. You are allowed to replace any two of these numbers by one number, which is either the sum or the difference of these numbers. Show that after 1973 times performing this operation, the only number left on the board cannot be 0.

Solution. There are 987 odd numbers on the board in the beginning. Every time the operation is performed, the number of odd numbers left either stay the same (when the numbers taken out are not both odd) or decreases by two (when the numbers taken out are both odd). So the number of odd numbers left on the board after each operation is always odd. Therefore, when one number is left, it must be odd and so it cannot be 0.

Example 2. In an 8×8 board, there are 32 white pieces and 32 black pieces, one piece in each square. If a player can change all the white pieces to black and all the black pieces to white in any row or column in a single move, then is it possible that after finitely many moves, there will be exactly one black piece left on the board?

Solution. No. If there are exactly k black pieces in a row or column before a move is made to that row or column, then after the moves, the number of black pieces in the row or in the column will become $8 - k$, a change of $(8 - k) - k = 8 - 2k$ black pieces on the board. Since $8 - 2k$ is even, the parity of the number of black pieces stay the same before and after the move. Since at the start, there are 32 black pieces, there cannot be 1 black piece left at any time.

Example 3. Four x's and five o's are written around the circle in an arbitrary order. If two consecutive symbols are the same, then insert a new x between them. Otherwise insert a new o between them. Remove the old x's and o's. Keep on repeating this operation. Is it possible to get nine o's?

Solution. If we let $x = 1$ and $o = -1$, then note that consecutive symbols are replaced by their product. If we consider the product P of the nine values before and after each operation, we will see that the new P is the square of the old P. Hence, P will always equal 1 after an operation. So nine o's yielding $P = -1$ can never happen.

Example 4. There are three piles of stones numbering 19, 8 and 9, respectively. You are allowed to choose two piles and transfer one stone from each of these two piles to the third piles. After several of these operations, is it possible that each of the three piles has 12 stones?

Solution. No. Let the number of stones in the three piles be *a, b* and *c,* respectively. Consider (mod 3) of these numbers. In the beginning, they are 1, 2, 0. After one operation, they become 0, 1, 2 no matter which two piles have stones transfer to the third pile. So the remainders are always 0, 1, 2 in some order. Therefore, all piles having 12 stones are impossible.

Example 5. Two boys play the following game with two piles of candies. In the first pile, there are 12 candies and in the second pile, there are 13 candies. Each boy takes turn to make a move consisting of eating two candies from one of the piles or transferring a candy from the first pile to the second. The boy who cannot make a move loses. Show that the boy who played second cannot lose. Can he win?

Solution. Consider S to be the number of candies in the second pile minus the first. Initially, $S = 13 - 12 = 1$. After each move, S increases or decreases by 2. So S (mod 4) has the pattern 1, 3, 1, 3, …. Every time after the boy who played first made a move, S (mod 4) would always be 3. Now a boy loses if and only if there are no candies left in the second pile, then $S = 1 - 0 = 1$. So the boy who played second can always make a move, hence he cannot lose.

Since either the total number of candies decreases or the number of candies in the first pile decreases, so eventually the game must stop, so the boy who played second must win.

Example 6. Each member of a club has at most three enemies in the club. (Here enemies are mutual.) Show that the members can be divided into two groups so that each member in each group has at most one enemy in the group.

Solution. In the beginning, randomly divide the members into two groups. Let S be the sum of the number of the pairs of enemies in each group. If a member has at least two enemies in the same group, then the member has at most one enemy in the other group. Transferring the member to the other group, we will decrease S by at least one. Since S is a nonnegative integer, it cannot be decreased forever. So after finitely many transfers, each member can have at most one enemy in the same group.

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IMO 2002

Kin Y. Li

The International Mathematical Olympiad 2002 was held in Glasgow, United Kingdom from July 19 to 30. There were a total of 479 students from 84 countries and regions participated in the Olympiad.

The Hong Kong team members were

Chao Khek Lun (St. Paul's College)
Chau Suk Ling (Queen Elizabeth School)
Cheng Kei Tsi (La Salle College)
Ip Chi Ho (St. Joseph's College)
Leung Wai Ying (Queen Elizabeth School)
Yu Hok Pun (SKH Bishop Baker Secondary School).

The team leader was *K. Y. Li* and the deputy leaders were *Chiang Kin Nam* and *Luk Mee Lin*.

The scores this year ranged from 0 to 42. The cutoffs for medals were 29 points for gold, 24 points for silver and 14 points for bronze. The Hong Kong team received 1 gold medal (*Yu Hok Pun*), 2 silver medals (*Leung Wai Ying* and *Cheng Kei Tsi*) and 2 bronze medals (*Chao Khek Lun* and *Ip Chi Ho*). There were 3 perfect scores, two from China and one from Russia. After the 3 perfect scores, the scores dropped to 36 with 9 students! This was due to the tough marking schemes, which intended to polarize the students' performance to specially distinguish those who had close to complete solutions from those who should only deserve partial points.

The top five teams are China (212), Russia (204), USA (171), Bulgaria (167) and Vietnam (166). Hong Kong came in 24th (120), ahead of Australia, United Kingdom, Singapore, New Zealand, but behind Canada, France and Thailand this year.

One piece of interesting coincidence deserved to be pointed out. Both Hong Kong and New Zealand joined the IMO in 1988. Both won a gold medal for the first time this year and both gold medallists scored 29 points.

The IMO will be hosted by Japan next year at Keio University in Tokyo and the participants will stay in the Olympic village. Then Greece, Mexico, Slovenia will host in the following years.

Addendum. After the IMO, the German leader Professor Gronau sent an email to inform all leaders about his updated webpage
http://www.Mathematik-Olympiaden.de/
which contains IMO news and facts. Clicking *Internationale Olympiaden* on the left, then on that page, scrolling down and clicking *Top-Mathematikern*, *Die erfolgreichsten IMO-Teilnehmer* in blue on the right, we could find the following past IMO participants who have also won the Fields medals, the Nevanlinna prizes and the Wolf prizes:

Richard Borcherds (1977 IMO silver, 1978 IMO gold, 1998 Fields medal)

Vladmir Drinfeld (1969 IMO gold, 1990 Fields medal)

Tim Gowers (1981 IMO gold, 1998 Fields medal)

Laurent Lafforgue (1984 IMO silver, 1985 IMO silver, 2002 Fields medal)

Gregori Margulis (1959 IMO member, 1962 IMO silver, 1978 Fields medal)

Jean-Christoph Yoccoz (1974 IMO gold, 1994 Fields medal)

Alexander Razborov (1979 IMO gold, 1990 Nevanlinna prize)

Peter Shor (1977 IMO silver, 1998 Nevanlinna prize)

László Lovász (1963 IMO silver, 1964 IMO gold, 1965 IMO gold, 1966 IMO gold, 1999 Wolf prize)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. Solutions should be preceded by the solver's name, home (or **email**) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is ***November 2, 2002***.

Problem 156. If $a, b, c > 0$ and $a^2+b^2+c^2=3$, then prove that

$$\frac{1}{1+ab}+\frac{1}{1+bc}+\frac{1}{1+ca}\geq\frac{3}{2}.$$

Problem 157. In base 10, the sum of the digits of a positive integer n is 100 and of $44n$ is 800. What is the sum of the digits of $3n$?

Problem 158. Let ABC be an isosceles triangle with $AB = AC$. Let D be a point on BC such that $BD = 2DC$ and let P be a point on AD such that $\angle BAC = \angle BPD$. Prove that

$$\angle BAC = 2\angle DPC.$$

Problem 159. Find all triples (x, k, n) of positive integers such that

$$3^k - 1 = x^n.$$

Problem 160. We are given 40 balloons, the air pressure inside each of which is unknown and may differ from balloon to balloon. It is permitted to choose up to k of the balloons and equalize the pressure in them (to the arithmetic mean of their respective pressures.) What is the smallest k for which it is always possible to equalize the pressures in all of the balloons?

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Solutions

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Problem 151. Every integer greater than 2 can be written as a sum of distinct positive integers. Let $A(n)$ be the maximum number of terms in such a sum for n. Find $A(n)$. (*Source: 1993 German Math Olympiad*)

Solution. **CHEUNG Yun Kuen** (Hong Kong Chinese Women's Club College, Form 5), **Antonio LEI** (Colchester Royal Grammar School, UK, Year 12), **LEUNG Chi Man** (Cheung Sha Wan Catholic Secondary School, Form 6), **Poon Ming Fung** (STFA Leung Kau Kui College, Form 5), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 7), **Tsui Ka Ho** (CUHK, Year 1), **Tak Wai Alan WONG** (University of Toronto) and **WONG Wing Hong** (La Salle College, Form 5).

Let $a_m = m(m+1)/2$. This is the sum of 1, 2, …, m and hence the sequence a_m is strictly increasing to infinity. So for every integer n greater than 2, there is a positive integer m such that $a_m \leq n < a_{m+1}$. Then n is the sum of the m positive integers

$$1, 2, \ldots, m-1,\ n-m(m-1)/2.$$

Assume $A(n) > m$. Then

$$a_{m+1} = 1+2+\cdots+(m+1)\leq n,$$

a contradiction. Therefore, $A(n) = m$. Solving the quadratic inequality

$$a_m = m(m+1)/2 \leq n,$$

we find m is the greatest integer less than or equal to $\left(-1+\sqrt{8n+1}\right)/2$.

Other commended solvers: **CHU Tsz Ying** (St. Joseph's Anglo-Chinese School, Form 7).

Problem 152. Let $ABCD$ be a cyclic quadrilateral with E as the intersection of lines AD and BC. Let M be the intersection of line BD with the line through E parallel to AC. From M, draw a tangent line to the circumcircle of $ABCD$ touching the circle at T. Prove that $MT = ME$. (*Source: 1957 Nanjing Math Competition*)

Solution. **CHEUNG Yun Kuen** (Hong Kong Chinese Women's Club College, Form 5), **CHU Tsz Ying** (St. Joseph's Anglo-Chinese School, Form 7), **Antonio LEI** (Colchester Royal Grammar School, UK, Year 12), **Poon Ming Fung** (STFA Leung Kau Kui College, Form 5), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 7), TANG Sze Ming (STFA Leung Kau Kui College), **Tsui Ka Ho** (CUHK, Year 1) and **WONG Wing Hong** (La Salle College, Form 5).

Since ME and AC are parallel, we have

$$\angle MEB = \angle ACB = \angle ADB = \angle MDE.$$

Also, $\angle BME = \angle EMD$. So triangles BME and EMD are similar. Then

$$MB / ME = ME / MD.$$

So $ME^2 = MD \cdot MB$. By the intersecting chord theorem, also $MT^2 = MD \cdot MB$. Therefore, $MT = ME$.

Problem 153. Let R denote the real numbers. Find all functions $f: R \to R$ such that the equality

$$f(f(x)+y) = f(x^2-y)+4f(x)y$$

holds for all pairs of real numbers x, y. (*Source: 1997 Czech-Slovak Match*)

Solution. **CHU Tsz Ying** (St. Joseph's Anglo-Chinese School, Form 7) and **Antonio LEI** (Colchester Royal Grammar School, UK, Year 12),

Setting $y = x^2$, we have

$$f(f(x)+x^2) = f(0)+4x^2 f(x).$$

Setting $y = -f(x)$, we have

$$f(0) = f(f(x)+x^2)+4f(x)^2.$$

Comparing these, we see that for each x, we must have $f(x) = 0$ or $f(x) = x^2$. Suppose $f(a) = 0$ for some nonzero a. Putting $x = a$ into the given equation, we get

$$f(y) = f(a^2 - y).$$

For $y \neq a^2/2$, we have

$$y^2 \neq (a^2-y)^2,$$

which will imply $f(y) = 0$. Finally, setting $x = 2a$ and $y = a^2/2$, we have

$$f(a^2/2) = f(7a^2/2) = 0.$$

So either $f(x) = 0$ for all x or $f(x) = x^2$ for all x. We can easily check both are solutions.

Comments: Many solvers submitted incomplete solutions. Most of them got $\forall x\ (f(x) = 0$ or $x^2)$, which is not the same as the desired conclusion that $(\forall x\ f(x) = 0)$ or $(\forall x\ f(x) = x^2)$.

Problem 154. For nonnegative numbers a, d and positive numbers b, c satisfying $b + c \geq a + d$, what is the minimum value of $\dfrac{b}{c+d}+\dfrac{c}{a+b}$?

(*Source: 1988 All Soviet Math Olympiad*)

Solution. Without loss of generality, we may assume that $a \geq d$ and $b \geq c$. From $b + c \geq a + d$, we get

$$b+c \geq (a+b+c+d)/2.$$

Now

$$\frac{b}{c+d}+\frac{c}{a+b}$$

$$=\frac{b+c}{c+d}-c\left(\frac{1}{c+d}-\frac{1}{a+b}\right)$$

$$\geq \frac{a+b+c+d}{2(c+d)}$$

$$-(c+d)\left(\frac{1}{c+d}-\frac{1}{a+b}\right)$$

$$=\frac{a+b}{2(c+d)}+\frac{c+d}{a+b}-\frac{1}{2}$$

$$\geq 2\sqrt{\frac{a+b}{2(c+d)}\cdot\frac{c+d}{a+b}}-\frac{1}{2}$$

$$=\sqrt{2}-\frac{1}{2},$$

where the AM-GM inequality was used to get the last inequality. Tracing the equality conditions, we need $b+c=a+d$, $c=c+d$ and $a+b=\sqrt{2}\,c$. So the minimum $\sqrt{2}-1/2$ is attained, for example, when $a=\sqrt{2}+1$, $b=\sqrt{2}-1$, $c=2$, $d=0$.

Other commended solvers: **CHEUNG Yun Kuen** (Hong Kong Chinese Women's Club College, Form 5) and **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 7).

Problem 155. We are given 1997 distinct positive integers, any 10 of which have the same least common multiple. Find the maximum possible number of pairwise relatively prime numbers among them. (*Source: 1997 Hungarian Math Olympiad*)

Solution. **Antonio LEI** (Colchester Royal Grammar School, UK, Year 12) and **WONG Wing Hong** (La Salle College, Form 5).

The answer is 9. Suppose there were 10 pairwise relatively prime numbers $a_1, a_2, \ldots, a_{10}$ among them. Being pairwise relatively prime, their least common multiple is their product *M*. Then the least common multiple of $b, a_2, \ldots, a_{10}$ for any other *b* in the set is also *M*. Since a_1 is relatively prime to each of $a_2, \ldots, a_{10}$, so *b* is divisible by a_1. Similarly, *b* is divisible by the other a_i. Hence *b* is divisible by *M*. Since *M* is a multiple of *b*, so *b* = *M*, a contradiction to having 1997 distinct integers.

To get an example of 9 pairwise relatively prime integers among them, let p_n be the *n*-th prime number, $a_i = p_i$ (for *i* = 1, 2, …, 8), $a_9 = p_9 p_{10} \cdots p_{1988}$ and

$$b_i = p_1 p_2 \cdots p_{1988} / p_i$$

for $i = 1, 2, \ldots, 1988$. It is easy to see that the a_i's are pairwise relatively prime and any 10 of these 1997 numbers have the same least common multiple.

Other commended solvers: **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 7).

Olympiad Corner

(continued from page 1)

Problem 3. (*cont.*)

$$\frac{a^m+a-1}{a^n+a^2-1}$$

is an integer.

Problem 4. Let *n* be an integer greater than 1. The positive divisors of *n* are $d_1, d_2, \ldots, d_k$ where

$$1=d_1<d_2<\cdots<d_k=n.$$

Define

$$D=d_1d_2+d_2d_3+\cdots+d_{k-1}d_k.$$

(a) Prove that $D<n^2$.
(b) Determine all *n* for which *D* is a divisor of n^2.

Problem 5. Find all functions *f* from the set ℝ of real numbers to itself such that

$$(f(x)+f(z))(f(y)+f(t)) = f(xy-zt)+f(xt+yz)$$

for all *x, y, z, t* in ℝ.

Problem 6. Let $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ be circles of radius 1 in the plane, where $n \geq 3$. Denote their centers by $O_1, O_2, \ldots, O_n$ respectively. Suppose that no line meets more than two of the circles. Prove that

$$\sum_{1\leq i<j\leq n}\frac{1}{O_iO_j}\leq\frac{(n-1)\pi}{4}.$$

Mathematical Games (I)

(Continued from page 2)

Remarks. This method of proving is known as the *method of infinite descent*. It showed that you cannot always decrease a quantity when it can only have finitely many possible values.

Example 7. (*1961 All-Russian Math Olympiad*) Real numbers are written in an $m \times n$ table. It is permissible to reverse the signs of all the numbers in any row or column. Prove that after a number of these operations, we can make the sum of the numbers along each line (row or column) nonnegative.

Solution. Let *S* be the sum of all the *mn* numbers in the table. Note that after an operation, each number stay the same or turns to its negative. Hence there are at most 2^{mn} tables. So *S* can only have finitely many possible values. To make the sum of the numbers in each line nonnegative, just look for a line whose numbers have a negative sum. If no such line exists, then we are done. Otherwise, reverse the sign of all the numbers in the line. Then *S* increases. Since *S* has finitely many possible values, *S* can increase finitely many times. So eventually the sum of the numbers in every line must be nonnegative.

Example 8. Given 2*n* points in a plane with no three of them collinear. Show that they can be divided into *n* pairs such that the *n* segments joining each pair do not intersect.

Solution. In the beginning randomly pair the points and join the segments. Let *S* be the sum of the lengths of the segments. (Note that since there are finitely many ways of connecting 2*n* points by *n* segments, there are finitely many possible values of *S*.) If two segments *AB* and *CD* intersect at *O*, then replace pairs *AB* and *CD* by *AC* and *BD*. Since

$$AB+CD=AO+OB+CO+OD > AC+BD$$

by the triangle inequality, whenever there is an intersection, doing this replacement will always decrease *S*. Since there are only finitely many possible values of *S*, so eventually there will not be any intersection.