**Olympiad Corner**

The 2002 Canadian Mathematical Olympiad

**Problem 1.** Let \( S \) be a subset of \( \{1, 2, \ldots, 9\} \), such that the sums formed by adding each unordered pair of distinct numbers from \( S \) are all different. For example, the subset \( \{1, 2, 3, 5\} \) has this property, but \( \{1, 2, 3, 4, 5\} \) does not, since the pairs \( \{1, 4\} \) and \( \{2, 3\} \) have the same sum, namely \( 5 \). What is the maximum number of elements that \( S \) can contain?

**Problem 2.** Call a positive integer \( n \) practical if every positive integer less than or equal to \( n \) can be written as the sum of distinct divisors of \( n \).

For example, the divisors of 6 are 1, 2, 3, 6. Since
\[
1 = 1, \quad 2 = 2, \quad 3 = 3, \quad 4 = 1 + 3, \quad 5 = 2 + 3, \quad 6 = 6
\]
we see that 6 is practical.

Prove that the product of two practical numbers is also practical.

(continued on page 4)
易事，Pepin在1877年找到費馬數是否質數的一個判斷：若$N > 3$是一個形如$2^{2^n} + 1$的費馬數，則$N$是質數的一個充分必要條件是$3^{N-1} = 1 \mod N$。考慮到$N = 2^{2^n} - 1$，因此是對3不斷取平方，然後對求$N$的模。近代對求費馬數是否質數的一個質數上，許多都以此為起點。也因此，曾經有一段時間，已經知道$F_7$不是質數，但它的任一因子都不知道。

再簡述一下近代的結果：現在已知由$F_5$至$F_11$，都是合數，並且已完全分解。$F_5, F_{12}, F_{13}, F_{15}$至$F_{19}$，是合數，並且知道部分因子。但$F_{14}, F_{20}, F_{22}$等，知道是合數，但一個因子也不知道。最大的費馬合數，並且找到一些因子的是$F_382597$，讀者可想像一下，如果以十進制形式寫下這個數，它是多少個位數。另外如$F_{33}, F_{34}, F_{35}$等，究竟是合數或質數，一點也不知道。有興趣的話，可參考網頁http://www.fermatsearch.org/status.htm。

由於費馬數和相關的數有特定的形式，而且具備很多有趣的性質，因此也常在競賽中出現。舉例如下：

例一：給定費馬數$F_0, F_1, ..., F_n$，有以下的關係$F_0 F_1 \cdots F_{n-1} + 2 = F_n$。

**證明**：事實上$F_n = 2^{2^n} + 1 = 2^{2^n} - 1 + 2 = (2^{2^n-1} + 1)(2^{2^n-1} - 1) + 2 = (2^{2^n-1} - 1)F_{n-1} + 2$。對於$2^{2^n-1} - 1$，可以再分解下去，就可以得到要求的結果。當然嚴格證明可以用歸納法。

例二：給定費馬數$F_m, F_n, m > n$，則$F_m, F_n$是互質的。

**證明**：因為$F_m = F_{m-1} \cdots F_n \cdots F_0 + 2$。設$d$整除$F_m$和$F_n$，則$d$也整除2，所以$d = 1$或2。但$d = 2$，因為$F_m, F_n$都是奇數，因此$d = 1$，即$F_m, F_n$互質。

例三：有無限多個$n$，使得$F_n + 2$不是質數。

**證明**：只要嘗試幾次就可以觀察到$F_1 + 2 = 3, F_2 + 2 = 5, F_3 + 2 = 29$，都是7的倍數。事實上，對於$n = 0, 1, 2, ..., 2^{2^n} = 2, 4, 2, 4, ...$ (mod 7)，因此對於奇數$n, F_n + 2 = 2^{2^n} + 1 + 2 = 4 + 1 + 2 = 0$ (mod 7)，所以不是質數。

另一個容易看到的事實是：

例四：對於$n > 1, F_n$的最尾的數字是7。

**證明**：對於$n > 1, 2^n$是4的倍數，設$2^n = 4k$，得$F_n = 2^{2^n} + 1 = 2^{4k} + 1 = (2^4)^k + 1 = 1^k + 1 = 2 \mod 5$。因此$F_n$的最尾的數字是2或7，它不可以是2，因為$F_n$不是偶數。

例五：證明存在一個正整數$k$，使得對任何正整數$n, k \cdot 2^n + 1$都不是質數。

(如果$n$固定，但容許$k$在正整數中變動，由一個重要的定理(Dirichlet)知道在序列中存在無限多個質數。但若果$k$固定，而$n$變動，在序列中究竟有多少個質數，是否無限多個，一般都不大清楚。

事實上，反可以找到一個$k$，對於任何正整數$n, k \cdot 2^n + 1$都不是質數。這原是波蘭數學家Sierpinski (1882-1969)的一個結果，後來演變成美國數學奧林匹克(1982)的一個題目，直到現在，基本是只有一種證明方法，並且與費馬數有關。)

(續於第四頁)
Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. Solutions should be preceded by the solver’s name, home (or email) address and school affiliation. Please send submissions to Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon.

Problem 161. Around a circle are written all of the positive integers from 1 to N, N ≥ 2, in such a way that any two adjacent integers have at least one common digit in their base 10 representations. Find the smallest N for which this is possible.

Problem 162. A set of positive integers is chosen so that among any 1999 consecutive positive integers, there is a chosen number. Show that there exist two chosen numbers, one of which divides the other.

Problem 163. Let a and n be integers. Let p be a prime number such that p > |a| + 1. Prove that the polynomial f(x) = x^n + ax + p cannot be a product of two nonconstant polynomials with integer coefficients.

Problem 164. Let O be the center of the excircle of triangle ABC opposite A. Let M be the midpoint of AC and let P be the intersection of lines MO and BC. Prove that if ∠BAC = 2∠ACB, then AB = BP.

Problem 165. For a positive integer n, let S(n) denote the sum of its digits. Prove that there exist distinct positive integers n₁, n₂, ..., nₙ₀ such that

\[ n₁ + S(n₁) = n₂ + S(n₂) = \cdots = nₙ₀ + S(nₙ₀). \]

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Solutions

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Problem 166. If a, b, c > 0 and \(a^2 + b^2 + c^2 = 3\), then prove that

\[ \frac{1}{1 + ab} + \frac{1}{1 + bc} + \frac{1}{1 + ca} \geq \frac{3}{2}. \]

(Source: 1999 Belarusian Olympiad)

Solution. SIU Tsz Hang (STFA Leung Kau Kui College, Form 7) and WONG Wing Hong (La Salle College, Form 5).

By the AM-GM and AM-HM inequalities, we have

\[ \frac{1}{1 + ab} + \frac{1}{1 + bc} + \frac{1}{1 + ca} \geq \frac{3}{2}. \]

Other commended solvers: CHAN Wai Hong (STFA Leung Kau Kui College, Form 7), CHAN Yat Fei (STFA Leung Kau Kui College, Form 6), CHEUNG Yun Kuen (Hong Kong Chinese Women’s Club College, Form 5), CHUNG Ho Yin (St. Joseph’s Anglo-Chinese School, Form 7), LAM Ho Yin (South Tuen Mun Government Secondary School, Form 6), LAM Wai Pui (STFA Leung Kau Kui College, Form 6), LEI Man Fui (STFA Leung Kau Kui College, Form 6), Antonio LEI (Colchester Royal Grammar School, UK, Year 12), LO Chi Fai (STFA Leung Kau Kui College, Form 7), POON Ming Fung (STFA Leung Kau Kui College, Form 5), TAM Choi Nang Julian (SKH Lam Kow Secondary School, teacher), TANG Ming Tak (STFA Leung Kau Kui College, Form 6), TANG Sze Ming (STFA Leung Kau Kui College, Form 5), YAU Chun Biu and YIP Wai Kiu (Jockey Club Ti-I College, Form 5) and Richard YEUNG Wing Fung (STFA Leung Kau Kui College, Form 5).

Problem 167. In base 10, the sum of the digits of a positive integer n is 100 and of 44n is 800. What is the sum of the digits of 3n? (Source: 1999 Russian Math Olympiad)

Solution. CHAN Wai Hong (STFA Leung Kau Kui College, Form 7), CHAN Yat Fei (STFA Leung Kau Kui College, Form 6), Antonio LEI (Colchester Royal Grammar School, UK, Year 12), LO Chi Fai (STFA Leung Kau Kui College, Form 7), POON Ming Fung (STFA Leung Kau Kui College, Form 5), SIU Tsz Hang (STFA Leung Kau Kui College, Form 7), TANG Ming Tak (STFA Leung Kau Kui College, Form 6), WONG Wing Hong (La Salle College, Form 5).

Let S(x) be the sum of the digits of x in base 10. For digits a and b, if a + b > 9, then S(a + b) = S(a) + S(b) − 9. Hence, if we have to carry in adding x and y, then S(x + y) < S(x) + S(y). So in general, S(x + y) ≤ S(x) + S(y).

By induction, we have S(kx) ≤ ks(x) for every positive integer k. Now

\[ 800 = S(44n) = S(40n + n) \leq S(40n) + S(4n) = 2S(4n) \leq 8S(n) = 800. \]

Hence equality must hold throughout and there can be no carry in computing 4n = n + n + n + n. So there is no carry in 3n = n + n + n and S(3n) = 300.

Other commended solvers: CHU Tsz Ying (St. Joseph’s Anglo-Chinese School, Form 7).

Problem 158. Let ABC be an isosceles triangle with AB = AC. Let D be a point on BC such that BD = 2DC and let P be a point on AD such that ∠BAC = ∠BPD. Prove that ∠BAC = ∠BPD.

(Source: 1999 Turkish Math Olympiad)

Solution. LAM Wai Pui (STFA Leung Kau Kui College, Form 6), POON Ming Fung (STFA Leung Kau Kui College, Form 5), SIU Tsz Hang (STFA Leung Kau Kui College, Form 7), WONG Wing Hong (La Salle College, Form 5) and Richard YEUNG Wing Fung (STFA Leung Kau Kui College, Form 5).

Let E be a point on AD extended so that PE = PB. Since ∠CAB = ∠EPB and CA/AB = 1 = EP/PB, triangles CAB and EPB are similar. Then ∠ACB = ∠EPB, which implies A, C, E, B are concyclic. So ∠AEC = ∠ABC = ∠AEF. Therefore, AE bisects ∠CEB.

Let M be the midpoint of BE. By the angle bisector theorem, CE/EB = CD/DB = 1/2. So CE = ½EB = ME. Also, PE = PE and PE bisects ∠CEM. It follows triangles CEP and MEP are congruent. Then ∠BAC = ∠BPE = 2∠MPE = 2∠CPE = 2∠DPC.

Other commended solvers: CHAN Yat Fei (STFA Leung Kau Kui College, Form 6), CHEUNG Yun Kuen (Hong Kong Chinese Women’s Club College, Form 5) and Antonio LEI (Colchester Royal Grammar School, UK, Year 13).
Problem 159. Find all triples \((x, k, n)\) of positive integers such that
\[
3^k - 1 = x^n.
\]
(Source: 1999 Italian Math Olympiad)

Solution. (Official Solution)
For \(n = 1\), the solutions are \((x, k, n) = (3^k - 1, k, 1)\), where \(k\) is for any positive integer.

For \(n = 2\), if \(n\) is even, then \(x^n + 1 = 2\) (mod 3) and hence cannot be \(3^k \equiv 0\) (mod 3). So \(n\) must be odd. Now \(x^n + 1\) can be factored as
\[
(x + 1)(x^{n-1} - x^{n-2} + \cdots + 1).
\]
If \(3^k = x^2 + 1\), then both of these factors are powers of 3, say they are \(3^i\), \(3^j\), respectively.

Since
\[
x + 1 \leq x^n - x^{n-2} + \cdots + 1,
\]
so \(s \leq t\). Then
\[
0 \equiv 3^i \equiv (-1)^{n+1} - (-1)^{n-2} + \cdots + 1 = 1 \pmod{n}.
\]
Implying \(n\) is divisible by \(x + 1\) and hence also by 3. Let \(y = x^{n/3}\). Then
\[
3^k = y^2 + 1 = (y + 1)(y^2 - y + 1).
\]
So \(y + 1\) is also a power of 3, say it is \(3^s\). If \(r = 1\), then \(y = 2\) and \(x, k, n) = (2, 2, 3)\) is a solution. Otherwise, \(r > 1\) and
\[
3^k = y^2 + 1 = 3^{2r} - 3^{2r-1} + 3^{r+1}
\]
is strictly between \(3^{2r-1}\) and \(3^{2r}\), a contradiction.

Other commended solvers: LEE Pui Chung (Wah Yan College, Kowloon, Form 7), LEUNG Chi Man (Cheung Sha Wan Catholic Secondary School, Form 6), POON Ming Fung (STFA Leung Kau Kui College, Form 5) and SIU Tsz Hang (STFA Leung Kau Kui College, Form 7).

Problem 160. We are given 40 balloons, the air pressure inside each of which is unknown and may differ from balloon to balloon. It is permitted to choose up to \(k\) of the balloons and equalize the pressure in them (to the arithmetic mean of their respective pressures.) What is the smallest \(k\) for which it is always possible to equalize the pressures in all of the balloons?
(Source: 1999 Russian Math Olympiad)

Solution. CHEUNG Yun Kuen (Hong Kong Chinese Women’s Club College, Form 5) and Antonio LEI (Colchester Royal Grammar School, UK, Year 13).

For \(k = 5\), it is always possible. We equalize balloons 1 to 5, then 6 to 10, and so on (five at time). Now take one balloon from each of these 8 groups. We have eight balloons, say \(a, b, c, d, e, f, g, h\). We can equalize \(a, b, c, d\), then \(e, f, g, h\), followed by \(a, b, c, f\) and finally \(c, d, g, h\). This will equalize all 8 balloons. Repeat getting one balloon from each of the 8 groups for 4 more times, then equalize them similarly. This will make all 40 balloons having the same pressure.

For \(k < 5\), it is not always possible. If the \(i\)-th balloon has initial pressure \(p_i = \pi\), then after equalizing operations, their pressures will always have the form \(c_1p_1 + \cdots + c_mp_m\) for some rational numbers \(c_1, \ldots, c_m\). The least common multiple of the denominators of these rational numbers will always be of the form \(2^r 3^s\) as \(k \leq 1, 2, 3, 4\) implies we can only change the denominators by a factor of 2, 3 or 4 after an operation. So \(c_1, \ldots, c_m\) can never all be equal to \(1/40\).

Olympiad Corner
(continued from page 1)

Problem 3. Prove that for all positive real numbers \(a, b, c\),
\[
\frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ab} \geq a + b + c
\]
and determine when equality occurs.

Problem 4. Let \(G\) be a circle with radius \(r\). Let \(A\) and \(B\) be distinct points on \(G\) such that \(AB < \sqrt{3r}\). Let the circle with center \(A\) and radius \(AB\) meet \(G\) again at \(C\). Let \(P\) be the point inside \(G\) such that triangle \(ABP\) is equilateral. Finally, let the line \(CP\) meet \(G\) again at \(Q\). Prove that \(PQ = r\).

Problem 5. Let \(N = \{0, 1, 2, \ldots\}\). Determine all functions \(f : N \to N\) such that
\[
x f(y) + y f(x) = (x + y) f(x^2 + y^2)
\]
for all real \(x, y\) in \(N\).