Mathematical Excalibur

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Olympiad Corner

The Final Round of the 51st Czech and Slovak Mathematical Olympiad was held on April 7-10, 2002. Here are the problems.

Problem 1. Solve the system

$$(4x)_5 + 7y = 14,$$
$$(2y)_5 - (3x)_7 = 74,$$

in the domain of the integers, where $(n)_k$ stands for the multiple of the number k closest to the number n.

Problem 2. Consider an arbitrary equilateral triangle KLM, whose vertices K, L and M lie on the sides AB, BC and CD, respectively, of a given square $ABCD$. Find the locus of the midpoints of the sides KL of all such triangles KLM.

Problem 3. Show that a given natural number A is the square of a natural number if and only if for any natural number n, at least one of the differences

$$(A+1)^2 - A,\ (A+2)^2 - A,$$
$$(A+3)^2 - A,\ \ldots,\ (A+n)^2 - A$$

is divisible by n.

(continued on page 4)

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On-line: http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***April 26, 2003***.

For individual subscription for the next five issues for the 02-03 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Countability

Kin Y. Li

Consider the following two questions:

(1) Is there a nonconstant polynomial with integer coefficients which has every prime number as a root?
(2) Is every real number a root of some nonconstant polynomial with integer coefficients?

The first question can be solved easily. Since the set of roots of a nonconstant polynomial is finite and the set of prime numbers is infinite, the roots cannot contain all the primes. So the first question has a negative answer.

However, for the second question, both the set of real numbers and the set of roots of nonconstant polynomials with integer coefficients are infinite. So we cannot answer this question as quickly as the first one.

In number theory, a number is said to be *algebraic* if it is a root of a nonconstant polynomial with integer coefficients, otherwise it is said to be *transcendental*. So the second question asks if every real number is algebraic.

Let's think about the second question. For every rational number a/b, it is clearly the root of the polynomial $P(x) = bx - a$. How about irrational numbers? For numbers of the form $\sqrt[n]{a/b}$, it is a root of the polynomial $P(x) = bx^n - a$. To some young readers, at this point they may think, perhaps the second question has a positive answer. We should do more checking before coming to any conclusion. How about π and e? Well, they are hard to check. Are there any other irrational number we can check?

Recall $\cos(3\theta)=4\cos^3\theta-3\cos\theta$. So setting $\theta=20^\circ$, we get $1/2 = 4\cos^3 20^\circ - 3\cos 20^\circ$. It follows that $\cos 20^\circ$ is a root of the polynomial $P(x) = 8x^3 - 6x - 1$. With this, we seem to have one more piece of evidence to think the second question has a positive answer.

So it is somewhat surprising to learn that the second question turns out to have a negative answer. In fact, it is known that π and e are not roots of nonconstant polynomials with integer coefficients, i.e. they are transcendental. Historically, the second question was answered before knowing π and e were transcendental. In 1844, Joseph Liouville proved for the first time that transcendental numbers exist, using continued fractions. In 1873, Charles Hermite showed e was transcendental. In 1882, Ferdinand von Lindemann generalized Hermite's argument to show π was also transcendental. Nowadays we know almost all real numbers are transcendental. This was proved by Georg Cantor in 1874. We would like to present Cantor's countability theory used to answer the question as it can be applied to many similar questions.

Let $\mathbb{N}$ denote the set of all positive integers, $\mathbb{Z}$ the set of all integers, $\mathbb{Q}$ the set of all rational numbers and $\mathbb{R}$ the set of all real numbers.

Recall a *bijection* is a function $f: A \to B$ such that for every b in B, there is exactly one a in A satisfying $f(a) = b$. Thus, f provides a way to *correspond* the elements of A with those of B in a *one-to-one* manner.

We say a set S is *countable* if and only if S is a finite set or there exists a bijection $f: \mathbb{N} \to S$. For an infinite set, since such a bijection is a one-to-one correspondence between the positive integers and the elements of S, we have

$$1 \leftrightarrow s_1,\ 2 \leftrightarrow s_2,\ 3 \leftrightarrow s_3,\ 4 \leftrightarrow s_4,\ \ldots$$

and so the elements of S can be listed *orderly* as $s_1, s_2, s_3, \ldots$ without repetition or omission. Conversely, any such list of the elements of a set is equivalent to showing the set is countable since assigning $f(1) = s_1$, $f(2) = s_2$, $f(3) = s_3$, … readily provide a bijection.

Certainly, $\mathbb{N}$ is countable as the identity function $f:\mathbb{N}\to\mathbb{N}$ defined by $f(n)=n$ is a bijection. This provides the usual listing of $\mathbb{N}$ as 1, 2, 3, 4, 5, 6, …. Next, for $\mathbb{Z}$, the usual listing would be

$$\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots.$$

However, to be in a one-to-one correspondence with $\mathbb{N}$, there should be a first element, followed by a second element, etc. So we can try listing $\mathbb{Z}$ as

$$0, 1, -1, 2, -2, 3, -3, 4, -4, \ldots.$$

From this we can construct a bijection $g:\mathbb{N}\to\mathbb{Z}$, namely define g as follow:

$$g(n) = (1-n)/2 \text{ if } n \text{ is odd}$$

and

$$g(n) = n/2 \text{ if } n \text{ is even.}$$

For $\mathbb{Q}$, there is no usual listing. So how do we proceed? Well, let's consider listing the set of all positive rational numbers $\mathbb{Q}^+$ first. Here is a table of $\mathbb{Q}^+$.

$$\begin{array}{ccccccc}
\frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \cdots \\
\frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \frac{2}{4} & \frac{2}{5} & \frac{2}{6} & \cdots \\
\frac{3}{1} & \frac{3}{2} & \frac{3}{3} & \frac{3}{4} & \frac{3}{5} & \frac{3}{6} & \cdots \\
\frac{4}{1} & \frac{4}{2} & \frac{4}{3} & \frac{4}{4} & \frac{4}{5} & \frac{4}{6} & \cdots \\
\frac{5}{1} & \frac{5}{2} & \frac{5}{3} & \frac{5}{4} & \frac{5}{5} & \frac{5}{6} & \cdots \\
\frac{6}{1} & \frac{6}{2} & \frac{6}{3} & \frac{6}{4} & \frac{6}{5} & \frac{6}{6} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

In the m-th row, the numerator is m and in the n-th column, the denominator is n.

Consider the southwest-to-northeast diagonals. The first one has 1/1, the second one has 2/1 and 1/2, the third one has 3/1, 2/2, 1/3, etc. We can list $\mathbb{Q}^+$ by writing down the numbers on these diagonals one after the other. However, this will repeat numbers, for example, 1/1 and 2/2 are the same. So to avoid repetitions, we will write down only numbers whose numerators and denominators are relatively prime. This will not omit any positive rational numbers because we can cancel common factors in the numerator and denominator of a positive rational number to arrive at a number in the table that we will not skip. Here is the list we will get for $\mathbb{Q}^+$:

1/1, 2/1, 1/2, 3/1, 1/3, 4/1, 3/2, 2/3, 1/4, 5/1, 1/5, 6/1, 5/2, 4/3, 3/4, … .

Once we have a listing of $\mathbb{Q}^+$, we can list $\mathbb{Q}$ as we did for $\mathbb{Z}$ from $\mathbb{N}$, i.e.

0, 1/1, –1/1, 2/1, –2/1, 1/2, –1/2, 3/1, –3/1, 1/3, –1/3, 4/1, –4/1, 3/2, ….

This shows $\mathbb{Q}$ is countable, although the bijection behind this listing is difficult to write down.

If a bijection $h:\mathbb{N}\to\mathbb{Q}$ is desired, then we can do the following. Define $h(1) = 0$. For an integer $n > 1$, write down the prime factorization of $g(n)$, where g is the function above. Suppose

$$g(n) = \pm\, 2^a\, 3^b\, 5^c\, 7^d \ldots.$$

Then we define

$$h(n) = \pm\, 2^{g(a+1)}\, 3^{g(b+1)}\, 5^{g(c+1)}\, 7^{g(d+1)} \ldots$$

with $g(n)$, $h(n)$ taking the same sign.

Next, how about $\mathbb{R}$? This is interesting. It turns out $\mathbb{R}$ is <u>*uncountable*</u> (i.e. not countable). To explain this, consider the function $u:(0,1)\to\mathbb{R}$ defined by $u(x) = \tan \pi(x-1/2)$. It has an inverse function $v(x) = 1/2 + (\operatorname{Arctan} x)/\pi$. So both u and v are bijections. Now assume there is a bijection $f:\mathbb{N}\to\mathbb{R}$. Then $F = v\circ f:\mathbb{N}\to(0,1)$ is also a bijection. Now we write the decimal representations of $F(1)$, $F(2)$, $F(3)$, $F(4)$, $F(5)$, … in a table.

$$\begin{aligned}
F(1) &= 0.a_{11}a_{12}a_{13}a_{14}\ldots \\
F(2) &= 0.a_{21}a_{22}a_{23}a_{24}\ldots \\
F(3) &= 0.a_{31}a_{32}a_{33}a_{34}\ldots \\
F(4) &= 0.a_{41}a_{42}a_{43}a_{44}\ldots \\
F(5) &= 0.a_{51}a_{52}a_{53}a_{54}\ldots \\
F(6) &= 0.a_{61}a_{62}a_{63}a_{64}\ldots
\end{aligned}$$

Consider the number

$$r = 0.\, b_1 b_2 b_3 b_4 b_5 b_6 \ldots,$$

where the digit $b_n = 2$ if $a_{nn} = 1$ and $b_n = 1$ if $a_{nn} \neq 1$. Then $F(n) \neq r$ for all n because $a_{nn} \neq b_n$. This contadicts F is a bijection. Thus, no bijection $f:\mathbb{N}\to\mathbb{R}$ can exist. Therefore, (0,1) and $\mathbb{R}$ are both uncountable.

We remark that the above argument shows no matter how the elements of (0,1) are listed, there will always be numbers omitted. The number r above is one such number.

So some sets are countable and some sets are uncountable.

For more complicated sets, we will use the following theorems to determine if they are countable or not.

<u>***Theorem 1.***</u> Let A be a subset of B. If B is countable, then A is countable.

<u>***Theorem 2.***</u> If for every integer n, S_n is a countable set, then their union is countable.

For the next theorem, we introduce some terminologies first. An object of the form $(x_1,\ldots,x_n)$ is called an <u>*ordered n-tuple*</u>. For sets $T_1, T_2, \ldots, T_n$, the <u>*Cartesian product*</u> $T_1\times\cdots\times T_n$ of these sets is the set of all ordered n-tuples $(x_1,\ldots,x_n)$, where each x_i is an element of T_i for $i = 1,\ldots, n$.

<u>***Theorem 3.***</u> If $T_1, T_2, \ldots, T_n$ are countable sets, then their Cartesian product is also countable.

We will give some brief explanations for these theorems. For theorem 1, if A is finite, then A is countable. So suppose A is infinite, then B is infinite. Since B is countable, we can list B as $b_1, b_2, b_3, \ldots$ without repetition or omission. Removing the elements b_i that are not in A, we get a list for A without repetition or omission.

For theorem 2, let us list the elements of S_n without repetition or omission in the n-th row of a table. (If S_n is finite, then the row contains finitely many elements.) Now we can list the union of these sets by writing down the diagonal elements as we have done for the positive rational numbers. To avoid repetition, we will not write the element if it has appeared before. Also, if some rows are finite, it is possible that as we go diagonally, we may get to a "hole". Then we simply skip over the hole and go on.

For theorem 3, we use mathematical induction. The case $n = 1$ is trivial. For the case $n = 2$, let $a_1, a_2, a_3, \ldots$ be a list of the elements of T_1 and $b_1, b_2, b_3, \ldots$ be a list of the elements of T_2 without repetition or omission. Draw a table with (a_i, b_j) in the i-th row and j-th column. Listing the diagonal elements as for the positive rational numbers, we get a list for $T_1\times T_2$ without repetition or omission. This takes care the case $n = 2$.

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Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is ***April 26, 2003***.

Problem 176. (*Proposed by Achilleas PavlosPorfyriadis,AmericanCollege of Thessaloniki "Anatolia", Thessaloniki, Greece*) Prove that the fraction

$$\frac{m(n+1)+1}{m(n+1)-n}$$

is irreducible for all positive integers m and n.

Problem 177. A locust, a grasshopper and a cricket are sitting in a long, straight ditch, the locust on the left and the cricket on the right side of the grasshopper. From time to time one of them leaps over one of its neighbors in the ditch. Is it possible that they will be sitting in their original order in the ditch after 1999 jumps?

Problem 178. Prove that if $x < y$, then there exist integers m and n such that

$$x < m + n\sqrt{2} < y.$$

Problem 179. Prove that in any triangle, a line passing through the incenter cuts the perimeter of the triangle in half if and only if it cuts the area of the triangle in half.

Problem 180. There are $n \geq 4$ points in the plane such that the distance between any two of them is an integer. Prove that at least 1/6 of the distances between them are divisible by 3.

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Solutions

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Problem 171. (*Proposed by Ha Duy Hung, Hanoi University of Education, Hanoi City, Vietnam*) Let a, b, c be positive integers, $[x]$ denote the greatest integer less than or equal to x and $\min\{x,y\}$ denote the minimum of x and y. Prove or disprove that

$$c\left[\frac{c}{ab}\right]-\left[\frac{c}{a}\right]\left[\frac{c}{b}\right]\leq c\min\left\{\frac{1}{a},\frac{1}{b}\right\}.$$

Solution. **LEE Man Fui** (STFA Leung Kau Kui College, Form 6) and **TANG Ming Tak** (STFA Leung Kau Kui College, Form 6).

Since the inequality is symmetric in a and b, without loss of generality, we may assume $a \geq b$. For every x, $bx \geq b[x]$. Since $b[x]$ is an integer, we get $[bx] \geq b[x]$. Let $x = c/(ab)$. We have

$$\begin{aligned} & c[c/(ab)] - [c/a][c/b] \\ &= c[x] - [bx][c/b] \\ &\leq (c/b)[bx] - [bx][c/b] \\ &= [bx]((c/b) - [c/b]) \\ &< bx\cdot 1 = c/a = c\min\{1/a,1/b\}. \end{aligned}$$

Other commended solvers: **CHEUNG Yun Kuen** (Hong Kong Chinese Women's Club College, Form5), **Antonio LEI** (Colchester Royal Grammar School, UK, Year 13), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 7), **Rooney TANG Chong Man** (Hong Kong Chinese Women's Club College, Form 5).

Problem 172. (*Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain*) Find all positive integers such that they are equal to the square of the sum of their digits in base 10 representation.

Solution. **D. Kipp JOHNSON** (Valley Catholic High School, Beaverton, Oregon, USA), **Antonio LEI** (Colchester Royal Grammar School, UK, Year 13), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 7) and and **WONG Wing Hong** (La Salle College, Form 5).

Suppose there is such an integer n and it has k digits. Then $10^{k-1} \leq n \leq (9k)^2$. However, for $k \geq 5$, we have

$$(9k)^2 = 81k^2 < (5^4/2)2^k \leq (5^{k-1}/2)2^k = 10^{k-1}.$$

So $k \leq 4$. Then $n \leq 36^2$. Since n is a perfect square, we check $1^2, 2^2, \ldots, 36^2$ and find only 1 and $9^2 = 81$ work.

Other commended solvers: **CHAN Yat Fei** (STFA Leung Kau Kui College, Form 6) and **Rooney TANG Chong Man** (Hong Kong Chinese Women's Club College, Form 5).

Problem 173. 300 apples are given, no one of which weighs more than 3 times any other. Show that the apples may be divided into groups of 4 such that no group weighs more than 3/2 times any other group. (*Source: 1997 Russian Math Olympiad*)

Solution. **CHEUNG Yun Kuen** (Hong Kong Chinese Women's Club College, Form 5) and **D. Kipp JOHNSON** (Valley Catholic High School, Beaverton, Oregon, USA).

Let $a_1, a_2, \ldots, a_{300}$ be the weights of the apples in increasing order. For $j = 1, 2, \ldots, 75$, let the j-th group consist of the apples with weights $a_j, a_{75+j}, a_{150+j}, a_{225+j}$. Note the weights of the groups are increasing. Then the ratio of the weights of any two groups is at most

$$\begin{aligned} &\frac{a_{75}+a_{150}+a_{225}+a_{300}}{a_1+a_{76}+a_{151}+a_{226}} \\ &\leq \frac{a_{76}+a_{151}+a_{226}+3a_1}{a_1+a_{76}+a_{151}+a_{226}} \\ &= 1+\frac{2}{1+(a_{76}+a_{151}+a_{226})/a_1}. \end{aligned}$$

Since $3 \leq (a_{76}+a_{151}+a_{226})/a_1 \leq 9$, so the ratio of groups is at most $1+2/(1+3)=3/2$.

Other commended solvers: **CHAN Yat Fei** (STFA Leung Kau Kui College, Form 6), **Terry CHUNG Ho Yin** (STFA Leung Kau Kui College, Form 6), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 7) and **TANG Ming Tak** (STFA Leung Kau Kui College, Form 6).

Problem 174. Let M be a point inside acute triangle ABC. Let A', B', C' be the mirror images of M with respect to BC, CA, AB, respectively. Determine (with proof) all points M such that A, B, C, A', B', C' are concyclic.

Solution. **Achilleas Pavlos PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece).

For such M, note the points around the circle are in the order A, B', C, A', B, C'. Now $\angle ACC' = \angle ABC'$ as they are subtended by chord AC'. Also, $AB'=AC'$ because they both equal to AM by symmetry. So $\angle ABC' = \angle ACB'$ as they are subtended by chords AC' and AB' respectively. By symmetry, we also have $\angle ACB' = \angle ACM$. Therefore, $\angle ACC' = \angle ACM$ and so C, M, C' are collinear. Similarly, A, M, A' are collinear. Then $CM \perp AB$ and $AM \perp BC$. So M is the orthocenter of ΔABC.

Conversely, if M is the orthocenter, then $\angle ACB' = \angle ACM = 90°-\angle BAC = \angle ABB'$, which implies A, B, C, B' are concyclic. Similarly, A' and C' are on the circumcircle of ΔABC.

Other commended solvers: **CHEUNG**

Yun Kuen (Hong Kong Chinese Women's Club College, Form 5), **Antonio LEI** (Colchester Royal Grammar School, UK, Year 13), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 7) and **WONG Wing Hong** (La Salle College, Form 5).

Problem 175. A regular polygon with n sides is divided into n isosceles triangles by segments joining its center to the vertices. Initially, $n+1$ frogs are placed inside the triangles. At every second, there are two frogs in some common triangle jumping into the interior of the two neighboring triangles (one frog into each neighbor). Prove that after some time, at every second, there are at least $[\,(n+1)/2\,]$ triangles, each containing at least one frog. (*Source: 1993 Jiangsu Province Math Olympiad*)

Solution. (*Official Solution*)

By the pigeonhole principle, the process will go on forever. Suppose there is a triangle that never contains any frog. Label that triangle number 1. Then label the other triangles in the clockwise direction numbers 2 to n. For each frog in a triangle, label the frog the number of the triangle. Let S be the sum of the squares of all frog numbers. On one hand, $S \le (n+1)\,n^2$. On the other hand, since triangle 1 never contains any frog, then at every second, some two terms of S will change from $i^2 + i^2$ to $(i+1)^2 + (i-1)^2 = 2i^2 + 2$ with $i < n$. Hence, S will keep on increasing , which contradicts $S \le (n+1)\,n^2$. Thus, after some time T, every triangle will eventually contain some frog at least once.

By the jumping rule, for any pair of triangles sharing a common side, if one of them contains a frog at some second, then at least one of them will contain a frog from then on. If n is even, then after time T, the n triangles can be divided into $n/2 = [\,(n+1)/2]$ pairs, each pair shares a common side and at least one of the triangles in the pair has a frog. If n is odd, then after time T, we may remove one of the triangles with a frog and divide the rest into $(n-1)/2$ pairs. Then there will exist $1 + (n-1)/2 = [\,(n+1)/2]$ triangles, each contains at least one frog.

Other commended solvers: **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 7).

Olympiad Corner

(continued from page 1)

Problem 4. Find all pairs of real numbers a, b for which the equation in the domain of the real numbers

$$\frac{ax^2 - 24x + b}{x^2 - 1} = x$$

has two solutions and the sum of them equals 12.

Problem 5. A triangle KLM is given in the plane together with a point A lying on the half-line opposite to KL. Construct a rectangle $ABCD$ whose vertices B, C and D lie on the lines KM, KL and LM, respectively. (We allow the rectangle to be a square.)

Problem 6. Let $\mathbb{R}^+$ denote the set of positive real numbers. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying for all $x, y \in \mathbb{R}^+$ the equality

$$f(x\,f(y)) = f(xy) + x.$$

Countability

(continued from page 2)

Assume the case $n = k$ is true. For $k+1$ countable sets $T_1, \ldots, T_k, T_{k+1}$, we apply the case $n = k$ to conclude $T_1 \times \cdots \times T_k$ is countable. Then $(T_1 \times \cdots \times T_k) \times T_{k+1}$ is countable by the case $n = 2$.

We should remark that for theorem 1, if C is uncountable and B is countable, then C cannot be a subset of B. As for theorem 2, it is also true for finitely many set $S_1, \ldots, S_n$ because we can set $S_{n+1}, S_{n+2}, \ldots$ all equal to S_1, then the union of $S_1, \ldots, S_n$ is the same as the union of $S_1, \ldots, S_n, S_{n+1}, S_{n+2}, \ldots$. However, for theorem 3, it only works for finitely many sets. Although it is possible to define ordered infinite tuples, the statement is not true for the case of infinitely many sets.

Now we go back to answer question 2 stated in the beginning of this article. We have already seen that $C = \mathbb{R}$ is uncountable. To see question 2 has a negative answer, it is enough to show the set B of all algebraic numbers is countable. By the remark for theorem 1, we can conclude that $C = \mathbb{R}$ cannot be a subset of B. Hence, there exists at least one real number which is not a root of any nonconstant polynomial with integer coefficients.

To show B is countable, we will first show the set D of all nonconstant polynomials with integer coefficients is countable.

Observe that every nonconstant polynomial is of degree n for some positive integer n. Let D_n be the set of all polynomials of degree n with integer coefficients. Let $\mathbb{Z}'$ denote the set of all nonzero integers. Since $\mathbb{Z}'$ is a subset of $\mathbb{Z}$, $\mathbb{Z}'$ is countable by theorem 1 (or simply deleting 0 from a list of $\mathbb{Z}$ without repetition or omission).

Note every polynomial of degree n is of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 \text{ (with } a_n \ne 0),$$

which is uniquely determined by its coefficients. Hence, if we define the function $w : \mathbb{Z}' \times \mathbb{Z} \times \cdots \times \mathbb{Z} \to D_n$ by

$$w(a_n, a_{n-1}, \ldots, a_0) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0,$$

then w is a bijection. By theorem 3, $\mathbb{Z}' \times \mathbb{Z} \times \cdots \times \mathbb{Z}$ is countable. So there is a bijection $q : \mathbb{N} \to \mathbb{Z}' \times \mathbb{Z} \times \cdots \times \mathbb{Z}$. Then $w \circ q : \mathbb{N} \to D_n$ is also a bijection. Hence, D_n is countable for every positive integer n. Since D is the union of $D_1, D_2, D_3, \ldots$, by theorem 2, D is countable.

Finally, let $P_1, P_2, P_3, \ldots$ be a list of all the elements of D. For every n, let R_n be the set of all roots of P_n, which is finite by the fundamental theorem of algebra. Hence R_n is countable. Since B is the union of $R_1, R_2, R_3, \ldots$, by theorem 2, B is countable and we are done.

Historically, the countability concept was created by Cantor when he proved the rational numbers were countable in 1873. Then he showed algebraic numbers were also countable a little later. Finally in December 1873, he showed real numbers were uncountable and wrote up the results in a paper, which appeared in print in 1874. It was this paper of Cantor that also introduced the one-to-one correspondence concept into mathematics for the first time!