Mathematical Excalibur

Volume 8, Number 3 *June 2003 – July 2003*

Olympiad Corner

The XV Asia Pacific Mathematics Olympiad took place on March 2003. The time allowed was 4 hours. No calculators were to be used. Here are the problems.

Problem 1. Let a, b, c, d, e, f be real numbers such that the polynomial

$$P(x) = x^8 - 4x^7 + 7x^6 + ax^5 + bx^4 + cx^3 + dx^2 + ex + f$$

factorises into eight linear factors $x - x_i$, with $x_i > 0$ for $i = 1, 2, \ldots, 8$. Determine all possible values of f.

Problem 2. Suppose $ABCD$ is a square piece of cardboard with side length a. On a plane are two parallel lines ℓ_1 and ℓ_2, which are also a units apart. The square $ABCD$ is placed on the plane so that sides AB and AD intersect ℓ_1 at E and F respectively. Also, sides CB and CD intersect ℓ_2 at G and H respectively. Let the perimeters of ΔAEF and ΔCGH be m_1 and m_2 respectively. Prove that no matter how the square was placed, $m_1 + m_2$ remains constant.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line: http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***August 10, 2003***.

For individual subscription for the next five issues for the 03-04 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

Dr. Kin-Yin LI
Department of Mathematics
The Hong Kong University of Science and Technology
Clear Water Bay, Kowloon, Hong Kong

Fax: (852) 2358 1643
Email: makyli@ust.hk

容斥原則和 Turan 定理

梁達榮

設 A 為有限集，以 $|A|$ 表示它含元素的個數。如果有兩個有限集 A 和 B，以 $A\cup B$ 表示 A 和 B 的并集（它包含屬於 A 或 B 的元素），以 $A\cap B$ 表示 A 和 B 的交集（它包含同時屬於 A 和 B 的元素）。眾所周知，如果 A 和 B 之間沒有共同元素，則 $|A\cup B|=|A|+|B|$，但是如果 A 和 B 之間有共同元素 x，當數算 A 元素的數目時，x 被算了一次，但數算 B 元素的數目時，x 又再被算了一次。為了抵消這樣的重覆，在計算 $|A\cup B|$ 時，我們要減去重覆數算的次數，即 $|A\cap B|$。因此 $|A\cup B|=|A|+|B|-|A\cap B|$。

對於三個集的并集 $A\cup B\cup C$，我們可以先數算 A，B 和 C 的個數，相加起來，發覺是太大了，必須減去一些交集的個數，現在 A，B 和 C 中任兩個集的交集可以是 $A\cap B$，$A\cap C$ 和 $B\cap C$，當我們減去這些交集的元素個數時，發覺又變得太少了，最後我們還要加上三個集的交集的元素個數，最後得 $|A\cup B\cup C|=|A|+|B|+|C|-|A\cap B|-|A\cap C|-|B\cap C|+|A\cap B\cap C|$。

一般來說，如果有 n 個有限集 $A_1, A_2,\ldots,A_n$，則 $|A_1\cup A_2\cup\cdots\cup A_n| = \sum_{i=1}^{n}|A_i| - \sum_{1\le i<j\le n}|A_i\cap A_j| + \sum_{1\le i<j<k\le n}|A_i\cap A_j\cap A_k| - \cdots + (-1)^{n-1}|A_1\cap A_2\cap\cdots\cap A_n|$ 等式中右邊第一個和式代表 A_1 至 A_n 各集元素個數的總和，第二個和式代表任何兩個集的交集元素個數的總和，餘此類推，直到考慮 $A_1, A_2,\cdots$ 至 A_n 的交集為止。

上面的等式，一般稱為容斥原則 (Inclusion-Exclusion Principle)，其命意義相當明顯。證明可以採歸納法，但也可以利用二項式定理加以證明。過程大概如下。設 x 屬於 $A_1\cup A_2\cup\cdots\cup A_n$，則 x 屬於其中 k 個 A_i，$(k\ge 1)$，為方便計，設 x 屬於 $A_1, A_2,\ldots,$ A_k，但不屬於 $A_{k+1},\ldots,A_n$。這樣的話，x 在 $A_1\cup A_2\cup\cdots\cup A_n$ 的“貢獻”為 1。在右邊第一個和式中，x 的“貢獻”為 $k=C_1^k$。在第二個和式中，由於 x 在 $A_1, A_2,\ldots,A_k$ 中出現，則 x 在它們任兩個集的交集中出現，但不在其他兩個集的交集中出現，因此，x 在第二個和式中的“貢獻”為 C_2^k。這樣分析下去，我們發覺 x 在右邊的“貢獻”總和是 $C_1^k - C_2^k + C_3^k - \cdots + (-1)^{k+1}C_k^k = 1-(1-1)^k = 1$。

留意我們用到了二項式定理，由於 x 在兩邊的貢獻相等，我們獲得了容斥原則成立的證明。

再者二項式系數有以下的性質。C_m^k 在 $m\le\frac{k}{2}$ 時遞增，在 $m\ge\frac{k}{2}$ 時遞減。(例如 $k=5$，有 $C_0^5<C_1^5<C_2^5=C_3^5>C_4^5>C_5^5$，$C_m^5$ 在 $m=2, 3$ 時取最大值，$k=6$，$C_0^6<C_1^6<C_2^6<C_3^6>C_4^6>C_5^6>C_6^6$，$C_m^6$ 在 $m=3$ 時最大值。) 利用這個關係，讀者可以証明，如果在容斥原則的右邊，略去一個正項及它以後各項，則式的左邊大於右邊，這是因為 x 對於右邊的貢獻非正，或者被略去的貢獻非負。同理，如果在容斥原則的右邊略去一個負項及它以後各項，則式的左邊變為小於右邊。這是一個有用的估計。

容斥原則作為數算集的大小的用途上時常出現，應用廣泛。

例一： 這是一個經典的題目，將 1, 2, …, n 重新安排次序，得到一個排列，如果沒有一個數字在原先的位置上，則稱之為亂序，(例如，4321 是一個亂序，4213 不是)，現在問，有多少個亂序？

解答: 顯而易見，所有的排列數目是 $n!=n\times(n-1)\times\cdots\times 1$。但如果直接找尋亂序的數目，卻不是很容易。因此我們定義 A_i 為 i 在正確位置的排列，$1\le i\le n$。易見 $|A_i|=(n-1)!$，同理 $\left|A_i\cap A_j\right|=(n-2)!$，此處 $i\neq j$，等等。因此

$$\begin{aligned}&\left|A_1\cup A_2\cup\cdots\cup A_n\right|\\&=\sum_{i=1}^{n}\left|A_i\right|-\sum_{1\le i<j\le n}\left|A_i\cap A_j\right|\\&+\sum_{1\le i<j<k\le n}\left|A_i\cap A_j\cap A_k\right|-\cdots\\&+(-1)^{n-1}\left|A_1\cap A_2\cap\cdots\cap A_n\right|\\&=n(n-1)!-C_2^n(n-2)!\\&+C_3^n(n-3)!-\cdots+(-1)^{n-1}1\\&=n!-\tfrac{n!}{2!}+\tfrac{n!}{3!}-\cdots+(-1)^{n-1}\tfrac{n!}{n!}\end{aligned}$$

最後，亂序的數目是

$$\begin{aligned}&n!-\left|A_1\cup A_2\cup\cdots\cup A_n\right|\\&=n!\left(\tfrac{1}{2!}-\tfrac{1}{3!}+\cdots+(-1)^n\tfrac{1}{n!}\right)\end{aligned}$$ 。

例二： (IMO 1991) 設 $S=\{1, 2, \ldots, 280\}$。求最小的自然數 n，使得 S 的每一個 n 元子集都含有 5 個兩兩互素的數。

解答: 首先利用容斥原則求得 $n\ge 217$。設 A_1,A_2,A_3,A_4 是 S 中分別為 2, 3, 5, 7 的倍數的集，則

$$\begin{aligned}&\left|A_1\right|=140,\left|A_2\right|=93,\left|A_3\right|=56,\left|A_4\right|=40,\\&\left|A_1\cap A_2\right|=46,\left|A_1\cap A_3\right|=28,\\&\left|A_1\cap A_4\right|=20,\left|A_2\cap A_3\right|=18,\\&\left|A_2\cap A_4\right|=13,\left|A_3\cap A_4\right|=8,\\&\left|A_1\cap A_2\cap A_3\right|=9,\left|A_1\cap A_2\cap A_4\right|=6,\\&\left|A_1\cap A_3\cap A_4\right|=4,\left|A_2\cap A_3\cap A_4\right|=2,\\&\left|A_1\cap A_2\cap A_3\cap A_4\right|=1\end{aligned}$$

因此

$$\begin{aligned}&\left|A_1\cap A_2\cap A_3\cap A_4\right|=140+93\\&+56+40-46-28-20-18-13\\&-8+9+6+4+2-1=216\end{aligned}$$

對於這個 216 元的集，任取 5 個數，必有兩個同時屬於 A_1,A_2,A_3 或 A_4，因此不互素。按題意，所以必須有 $n\ge 217$。現在要證明 S 中任一 217 元集必有 5 個互素的數，方法是要構造適當的“抽屜”。其中一個比較簡潔的構造是這樣的。設 A 是 S 的一個子集，並且 $|A|\ge 217$。定義

B_1={1 和 S 中的素數}，$\left|B_1\right|=60$，
$B_2=\{2^2,3^2,5^2,7^2,11^2,13^2\},\left|B_2\right|=6,$
$B_3=\{2\times131,3\times89,5\times53,7\times37,11\times23,13\times19\},\left|B_3\right|=6,$
$B_4=\{2\times127,3\times87,5\times47,7\times31,11\times19,13\times17\},\left|B_4\right|=6,$
$B_5=\{2\times113,3\times79,5\times43,7\times27,11\times17\},\left|B_5\right|=5,$
$B_6=\{2\times109,3\times73,5\times41,7\times23,11\times13\},\left|B_6\right|=5.$

易見 B_1 至 B_6 互不相交，並且 $|B_1\cup B_2\cup B_3\cup B_4\cup B_5\cup B_6|=88$。去掉這 88 個數，$S$ 中尚有 $280-88=192$ 個數。現在 A 最小有 217 個元素，$217-192=25$，即是說 A 中最小有 25 個元屬於 B_1 至 B_6。易見，不可能每個 B_i 只含 A 中 4 個或以下的元素，即是說最少有 5 個或以上的元素屬於同一個 B_i，因此互素。注意這裏我們用到另一個原則：抽屜原則。

例三： (1989 IMO) 設 n 是正整數。我們說集$\{1, 2, 3, \ldots, 2n\}$的一個排列 $(x_1, x_2,\ldots,x_{2n})$ 具有性質 P，如果在$\{1, 2, 3, \ldots, 2n-1\}$中至少有一個 i，使得 $|x_i-x_{i+1}|=n$ 成立。證明具有性質 P 的排列比不具有性質 P 的排列多。

解答: 留意如果 $|x_i-x_{i+1}|=n$，其中一個 x_i 或 x_{i+1} 必小於 $n+1$。因此對於 $k=1, 2, \ldots, n$, 定義 A_k 為 k 與 $k+n$ 相鄰的排列的組合，易見 $|A_k|=2\times(2n-1)!$。(這是因為 k 與 $k+n$ 并合在一起，但位置可以互相交換，想像它是一個“數”，而另外有 $2n-2$ 個數，這$(2n-2)+1$ 個數位置隨意。) 同時 $|A_k\cap A_h|=2^2\times(2n-2)!$, $1\le k<h\le n$, (k 與 $k+n$ 合在一起成為一個“數”h 與 $h+n$ 合在一起成為一個“數”。)因此具性質 P 的排列的數目

$$\begin{aligned}&|A_1\cup A_2\cup\cdots\cup A_n|\ge\sum_{k=1}^{n}\left|A_k\right|\\&-\sum_{1\le k<h\le n}|A_k\cap A_h|\\&=2\times(2n-1)!\times n-C_2^n\times2^2\times(2n-2)!\\&=2n\times(2n-2)!\times n=(2n)!\times\tfrac{n}{2n-1}>(2n)!\times\tfrac{1}{2}\end{aligned}$$

這個數目超過$(2n)!$的一半，因此具性質 P 的排列比不具性質 P 的排列多。

(這一個問題，當年被視為一個難題，但如果看到它與容斥原則的關係，就變得很容易了。)

例四：設 n 和 k 為正整數， $n>3$, $\frac{n}{2}<k<n$。平面上有 n 個點，其中任意三點不共線，如果其中每個點至少與其它 k 個點用線連結，則連結的線段中至少有三條圍成一個三角形。

解答: 因為 $n>3, k>\frac{n}{2}$，則 $k\ge 2$，所以 n 個點中必中兩個點 v_1 和 v_2 相連結。考慮餘下的點，設與 v_1 相連結的點集為 A，與 v_2 相連結的點集為 B，則 $|A|\ge k-1,|B|\ge k-1$。另外

$$\begin{aligned}&n-2\ge|A\cup B|=|A|+|B|-|A\cap B|\\&\ge 2k-2-|A\cap B|\end{aligned}$$

即 $|A\cap B|\ge 2k-n>0$。因此，存在點 v_3 與 v_1 和 v_2 相連結，構成一個三角形。

例五：一次會議有 1990 位數學家參加，其中每人最少有 1327 位合作者。証明，可以找到 4 位數學家，他們中每兩人都合作過。

証明: 將數學家考慮為一個點集，曾經合作過的連結起來，得到一個圖。如上例，v_1 互 v_2 曾合作過，所以連結起來，餘下的，設 A 為和 v_1 合作過的點集，B 為和 v_2 合作過的點集，則 $|A|\ge 1326,|B|\ge 1326$，同樣，$|A\cup B|\le 1990-2=1998$，因此

$$\begin{aligned}&|A\cap B|=|A|+|B|-|A\cup B|\\&\ge 2\times1326-1998=664>0\end{aligned}$$

即是說，可以找到數學家 v_3，與 v_1 和 v_2 都合作過。設 C 為除 v_1 和 v_2 以外，與 v_3 合作過的數學家，即 $|C|\ge 1325$。同時

$1998\ge|(A\cap B)\cup C|=|A\cap B|+|C|-|A\cap B\cap C|$ 即

$$\begin{aligned}&|A\cap B\cap C|\ge|A\cap B|+|C|-1988\\&\ge 664+1325-1988=1>0\end{aligned}$$ 。

因此 $A\cap B\cap C$ 非空，取 $v_4\in A\cap B\cap C$，則 v_1,v_2,v_3,v_4 都曾經合作過。

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon.* The deadline for submitting solutions is ***August 10, 2003***.

Problem 181. (*Proposed by Achilleas PavlosPorfyriadis, AmericanCollege of Thessaloniki "Anatolia", Thessaloniki, Greece*) Prove that in a convex polygon, there cannot be two sides with no common vertex, each of which is longer than the longest diagonal.

Problem 182. Let a_0, a_1, a_2, … be a sequence of real numbers such that

$$a_{n+1} \ge a_n^2 + 1/5 \quad \text{for all } n \ge 0.$$

Prove that $\sqrt{a_{n+5}} \ge a_{n-5}^2$ for all $n \ge 5$.

Problem 183. Do there exist 10 distinct integers, the sum of any 9 of which is a perfect square?

Problem 184. Let $ABCD$ be a rhombus with $\angle B = 60^\circ$. M is a point inside ΔADC such that $\angle AMC = 120^\circ$. Let lines BA and CM intersect at P and lines BC and AM intersect at Q. Prove that D lies on the line PQ.

Problem 185. Given a circle of n lights, exactly one of which is initially on, it is permitted to change the state of a bulb provided one also changes the state of every d-th bulb after it (where d is a divisor of n and is less than n), provided that all n/d bulbs were originally in the same state as one another. For what values of n is it possible to turn all the bulbs on by making a sequence of moves of this kind?

****

Solutions

****

Problem 176. (*Proposed by Achilleas PavlosPorfyriadis,AmericanCollege of Thessaloniki "Anatolia", Thessaloniki, Greece*) Prove that the fraction

$$\frac{m(n+1)+1}{m(n+1)-n}$$

is irreducible for all positive integers m and n.

Solution. **CHEUNG Yun Kuen** (Hong Kong Chinese Women's Club College, Form5), **TAM Choi Nang Julian** (Teacher, SKH Lam Kau Mow Secondary School), **Anderson TORRES** (Colegio Etapa, Brazil, 3rd Grade) and **Alan T. W. WONG** (Markham, ON, Canada).

If the fraction is reducible, then $m(n+1)+1$ and $m(n+1)-n$ are both divisible by a common factor $d>1$. So their difference $n+1$ is also divisible by d. This would lead to

$$1 = (m(n+1)+1) - m(n+1)$$

divisible by d, a contradiction.

Other commended solvers: **CHEUNG Tin** (STFA Leung Kau Kui College, Form 4), **CHUNG Ho Yin** (STFA Leung Kau Kui College, Form 6), **D. Kipp JOHNSON** (Teacher, Valley Catholic High School, Beaverton, Oregon, USA), **LEE Man Fui** (STFA Leung Kau Kui College, Form 6), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 7), **Alexandre THIERY** (Pothier High School, Orleans, France), **Michael A. VEVE** (Argon Engineering Associates, Inc., Virginia, USA) and **Maria ZABAR** (Trieste College, Trieste, Italy).

Problem 177. A locust, a grasshopper and a cricket are sitting in a long, straight ditch, the locust on the left and the cricket on the right side of the grasshopper. From time to time one of them leaps over one of its neighbors in the ditch. Is it possible that they will be sitting in their original order in the ditch after 1999 jumps?

Solution. **CHEUNG Yun Kuen** (Hong Kong Chinese Women's Club College, Form5), **D. Kipp JOHNSON** (Teacher, Valley Catholic High School, Beaverton, Oregon, USA), **Achilleas Pavlos PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 7) and **Anderson TORRES** (Colegio Etapa, Brazil, 3rd Grade).

Let L, G, C denote the locust, grasshopper, cricket, respectively. There are 6 orders:

$$LCG, CGL, GLC, CLG, GCL, LGC.$$

Let LCG, CGL, GLC be put in one group and CLG, GCL, LGC be put in another group. Note after one leap, an order in one group will become an order in the other group. Since 1999 is odd, the order LGC originally will change after 1999 leaps.

Problem 178. Prove that if $x < y$, then there exist integers m and n such that

$$x < m + n\sqrt{2} < y.$$

Solution. **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 7).

Note $0 < \sqrt{2} - 1 < 1$. For a positive integer

$$k > \frac{\log(b-a)}{\log(\sqrt{2}-1)},$$

we get $0 < (\sqrt{2}-1)^k < b - a$. By the binomial expansion,

$$x = (\sqrt{2}-1)^k = p + q\sqrt{2}$$

for some integers p and q. Next, there is an integer r such that

$$r - 1 \le \frac{a - [a]}{x} < r.$$

Then a is in the interval

$$I = [[a] + (r-1)x, [a] + rx).$$

Since the length of I is $x < b - a$, we get

$$a < [a] + rx = ([a] + rp) + rq\sqrt{2} < b.$$

Other commended solvers: **CHEUNG Yun Kuen** (Hong Kong Chinese Women's Club College, Form 5), **D. Kipp JOHNSON** (Teacher, Valley Catholic High School, Beaverton, Oregon, USA), **Alexandre THIERY** (Pothier High School, Orleans, France) and **Anderson TORRES** (Colegio Etapa, Brazil, 3rd Grade).

Problem 179. Prove that in any triangle, a line passing through the incenter cuts the perimeter of the triangle in half if and only if it cuts the area of the triangle in half.

Solution. **CHEUNG Yun Kuen** (Hong Kong Chinese Women's Club College, Form 5), **LEE Man Fui** (STFA Leung Kau Kui College, Form 6), **Achilleas Pavlos PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece), **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 7), **TAM Choi Nang Julian** (Teacher, SKH Lam Kau Mow Secondary School), and **Alexandre THIERY** (Pothier High School, Orleans, France).

Let ABC be the triangle, s be its semiperimeter and r be its inradius. Without loss of generality, we may assume the line passing through the incenter cuts AB and AC at P and Q respectively. (If the line passes through a vertex of ΔABC, we may let $Q = C$.)

Let $[XYZ]$ denote the area of ΔXYZ. The line cuts the perimeter of ΔABC in half if and only if $AP + AQ = s$, which is equivalent to

$$[APQ] = [API] + [AQI] = (r\cdot AP)/2 + (r\cdot AQ)/2 = rs/2 = [ABC]/2.$$

i.e. the line cuts the area of ΔABC in half.

Problem 180. There are $n \ge 4$ points in the plane such that the distance between any two of them is an integer. Prove that at least 1/6 of the distances between them are divisible by 3.

Solution. **CHEUNG Yun Kuen** (Hong Kong Chinese Women's Club College, Form 5), **D. Kipp JOHNSON** (Teacher, Valley Catholic High School, Beaverton, Oregon, USA) and **SIU Tsz Hang** (STFA Leung Kau Kui College, Form 7).

We will first show that for any 4 of the points, there is a pair with distance divisible by 3. Assume A, B, C, D are 4 of the points such that no distance between any pair of them is divisible by 3. Since $x \equiv 1$ or 2 (mod 3) implies $x^2 \equiv 1$ (mod 3), AB^2, AC^2, AD^2, BC^2, BD^2 and CD^2 are all congruent to 1 (mod 3).

Without loss of generality, we may assume that $\angle ACD = \alpha + \beta$, where $\alpha = \angle ACB$ and $\beta = \angle BCD$. By the cosine law,

$$AD^2 = AC^2 + CD^2 - 2AC\cdot CD\cos\angle ACD.$$

Now

$$\cos\angle ACD = \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

By cosine law, we have

$$\cos\alpha = \frac{AC^2 + BC^2 - AB^2}{2AC\cdot BC} \quad \text{and}$$

$$\cos\beta = \frac{BC^2 + CD^2 - BD^2}{2BC\cdot CD}.$$

Using $\sin x = \sqrt{1-\cos^2 x}$, we can also find $\sin\alpha$ and $\sin\beta$. Then

$$2BC^2\cdot AD^2 = 2BC^2(AC^2 + CD^2) - (2AC\cdot BC)(2BC\cdot CD)\cos\angle ACD = P + Q,$$

where

$$P = 2BC^2(AC^2 + CD^2) - (AC^2+BC^2-AB^2)(BC^2+CD^2-BD^2)$$

and

$$Q^2 = (4AC^2\cdot BC^2 - (AC^2 + BC^2 - AB^2)^2) \times (4BC^2\cdot CD^2 - (BC^2 + CD^2 - BD^2)^2).$$

However, $2BC^2\cdot AD^2 \equiv 2 \pmod 3$, $P \equiv 0 \pmod 3$ and $Q \equiv 0 \pmod 3$. This lead to a contradiction.

For $n \ge 4$, there are C_4^n groups of 4 points. By the reasoning above, each of these groups has a pair of points with distance divisible by 3. This pair of points is in a total of C_2^{n-2} groups. Since $C_4^n / C_2^{n-1} = \frac{1}{6}C_2^n$, the result follows.

Olympiad Corner

(continued from page 1)

Problem 3. Let $k \ge 14$ be an integer, and let p_k be the largest prime number which is strictly less than k. You may assume that $p_k \ge 3k/4$. Let n be a composite integer. Prove:

(a) if $n = 2p_k$, then n does not divide $(n-k)!$;

(b) if $n > 2p_k$, then n divides $(n-k)!$.

Problem 4. Let a, b, c be the sides of a triangle, with $a + b + c = 1$, and let $n \ge 2$ be an integer. Show that

$$\sqrt[n]{a^n + b^n} + \sqrt[n]{b^n + c^n} + \sqrt[n]{c^n + a^n} < 1 + \frac{\sqrt[n]{2}}{2}.$$

Problem 5. Given two positive integers m and n, find the smallest positive integer k such that among any k people, either there are $2m$ of them who form m pairs of mutually acquainted people or there are $2n$ of them forming n pairs of mutually unacquainted people.

容斥原則和 Turan 定理

(continued from page 2)

套用圖論的語言，例四和例五的意義正如，給定一個 n 點的圖，最少有多少條線，才可以保證有一個三角形 (K_3) 或一個 K_4（四點的圖，任兩點都相連），或者換另一種說法，設有一個 n 點的圖沒有三角形，則該圖最多有多少條線段，等等。這一範圍的圖論稱為極端圖論。最先的結果是這樣的：

Mantel 定理 (1907): 設 n 點的簡單圖不含 K_3，則其邊數最大值為 $\left[\frac{n^2}{4}\right]$。

(此處 $[x]$ 是小於或等於 x 的最大整數。在例四中，邊數和多於 $\binom{n}{2}\times n\times\frac{1}{2} > \left[\frac{n^2}{4}\right]$，因此結果立即成立。)

比較精緻的命題是這樣的。

__定理__： 如果 n 點的圖有 q 條邊，則圖至少有 $\dfrac{4q(q-\frac{n^2}{4})}{3n}$ 個三角形。

__例六__： 在圓周上有 21 個點，由其中二點引伸至圓心所成的圓心角度，最多有 110 個大於 120°。

__解答__： 如果兩點與圓心形成的圓心角度大於 120°，則將兩點連結起來，得到一個圖，這個圖沒有三角形，因此邊數最多有 $\left[\frac{21^2}{4}\right] = \left[\frac{441}{4}\right] = 110$ 條，或者最多有 110 個引伸出來的圓心角度大於 120°。

如上所說，定義 K_p 是一個 p 個點的完全圖，即 p 個點任兩點都相連，對於一個 n 點的圖 G，如果沒有包含 K_p，則 G 最多有多少條邊呢?

Turan 定理(1941): 如果一個 n 點的圖 G 不含 K_p，則該圖最多有

$$\frac{p-2}{2(p-1)}n^2 - \frac{r(p-1-r)}{2(p-1)}$$

條邊，其中 r 是由 $n = k(p-1) + r$, $0 \le r < p - 1$ 所定義的。如 Mantel 定理的情況，這個定理是極端圖論的一個起點。

Paul Turan (1910-1976) 猶太裔匈牙利人，當他在考慮這一類問題時，還是被關在一個集中營內的呢！