Olympiad Corner

The 2003 USA Mathematical Olympiad took place on May 1. Here are the problems.

Problem 1. Prove that for every positive integer $n$ there exists an $n$-digit number divisible by $5^n$ of all of whose digits are odd.

Problem 2. A convex polygon $P$ in the plane is dissected into smaller convex polygons by drawing all of its diagonals. The lengths of all sides and all diagonals of the polygons $P$ are rational numbers. Prove that the lengths of all sides of all polygons in the dissection are also rational numbers.

Problem 3. Let $n \neq 0$. For every sequence of integers $A = a_0, a_1, a_2, \ldots, a_n$ satisfying $0 \leq a_i \leq i$, for $i = 0, \ldots, n$, define another sequence $t = t(a_0), t(a_1), t(a_2), \ldots, t(a_n)$ by setting $t(a_i)$ to be the number of terms in the sequence $A$ that precede the terms $a_i$ and are different from $a_i$. Show that, starting from any sequence $A$ as above, fewer than $n$ applications of the transformation $t$ lead to a sequence $B$ such that $t(B) = B$.

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考慮另外一個集 \( B \)，如果 \( x \) 不在 \( B \) 內，由於 \( B \) 和 \( A_1, A_2, \ldots, A_{50} \) 都相交，\n且由條件所限，相交的元素都不同，則 \( B \) 最少有 51 個元素，這是不可能的。
所以 \( x \) 在 \( B \) 內，且 \( B \) 是任意的，\n所以 \( x \) 在任一個集內，證明。\n
這個結果可以這樣推廣，且證明完全\n相似：設有 \( n \) 個集，每一個集有 \( k \) 個\n元素，任意兩集剛好有 \( k \) 共同元素。如果 \( n > k^2 - k + 1 \)，則這 \( n \) 個集\n有一個共同元素。\n
考慮一個較為困難的例子：\n
例三：(俄羅斯數學競賽 1996) 由\n1600 個議員組成 16000 個委員會，\n每個委員會由 80 個委員組成。試證\n明：一定存在兩個委員會，它們之間\n至少有 4 個相同的議員。\n
證明：這一次我們不考慮每一個委員\n會組成委員的數，反過來考慮每一個\n議員所參加委員會的數。設議員\n1, 2, \ldots, 1600 分別參加了 \( k_1, k_2, \ldots, k_{1600} \) 個委員會，則總共有\n\( C_{k_1}^2 + C_{k_2}^2 + \cdots + C_{k_{1600}}^2 \) 個委員會對。\n如果委員會的數目是 \( N \)，則 \( k_1 + k_2 + \cdots + k_{1600} = 80N \)。（在題中 \( N = 16000 \) 且每個委員會由 80 人組成）\n
現在試圖估計這些委員會對\n
\[
C_{k_1}^2 + C_{k_2}^2 + \cdots + C_{k_{1600}}^2 = \sum_{i=1}^{1600} k_i^2 - \sum_{i=1}^{1600} k_i \geq \left( \frac{\sum_{i=1}^{1600} k_i^2}{3200} \right)^2 \frac{2}{\sum_{i=1}^{1600} k_i} = \frac{(80N)^2}{3200} - \frac{80N}{2} \geq 2N^2 - 40N = 2N(N-20) \]

如果任兩個委員會最多有 3 個共同\n議員，則最多有\n
\[
3C_2^k = \frac{3N(N-1)}{2}
\]

個委員會對。因此\n
\[
2N(N-20) \leq \frac{3}{2} N(N-1)
\]

即 \( N \leq 77 \)，與 \( N = 16000 \) 矛盾。\n
（留意在估計中用到 Cauchy-Schwarz\nInequality。）無獨有偶，我們有以下的\n例子：\n
例四：(IMO1998) 在一次比賽中，有 \( m \)\n個比賽員和 \( n \) 個評判，其中 \( n \geq 3 \) 是\n一個奇數。每一個評判對每一個比賽員進\n行評審為合格或不合格。如果任一對評\n判最多對 \( k \) 個比賽員的評審一致，試證\n明\n
\[
\frac{k}{m} \geq \frac{n-1}{2n}
\]

證明：題目已經提醒我們，我們考慮的\n是評判所成的“對”，這些“對”對某些比賽員\n的決定一致。對於比賽員 \( i, 1 \leq i \leq m \)，如果有 \( x_i \) 個評判認為他\n合格， \( y_i \) 個評判認為他不合格，則評判\n一致的對是\n
\[
C_{x_i}^k + C_{y_i}^k = \binom{x_i + y_i}{2} - (x_i + y_i) \geq \frac{(x_i + y_i)^2}{4} - \frac{(x_i + y_i)}{2} = \frac{1}{4}n^2 - \frac{n}{2} = \frac{1}{4}(n-1)^2 - 1
\]

因為 \( n \) 是奇數，而 \( C_{y_i}^k + C_{x_i}^k \) 是整數，\n因為 \( C_{y_i}^k + C_{x_i}^k \) 最少是 \( \frac{1}{4}(n-1)^2 \)。現在\n因為有 \( n \) 個評判，而任一對評判最多對 \( k \) 個比賽員意見一致，因為一致的評判\n最多是 \( kC_n^k \)。所以

\[
kC_n^k \geq \sum_{i} \left[ C_{x_i}^k + C_{y_i}^k \right] \geq \frac{m(n-1)^2}{4}
\]

，化簡結果即為所求。\n
現在考慮一個形式略為不同的題\n目。我們的對象是一些長為 \( n \) 的數\n列，這些數列只包括 0 或 1，兩個這\n樣的數列的“距離”定義為對應位\n置數字不同的個數。例如 1101011 和\n10111000 為兩個長為 7 的數列，它們\n在位置 2, 3, 6, 7 的數字不同，因此它\n們的距離是 4。用集的言語來描述\n是，有 7 個元素 1, 2, 3, 4, 5, 6 和 7 的一\n個集，數列一直在位置 1, 2, 3, 6, 7 非零，\n因此可像是包括 1, 2, 4, 6, 7 的一個\n子集，數列二是包括 1, 3 和 4 的子\n集，屬於數列一或數列二，但不同時\n屬於兩個數列的子集包括 2, 3, 6, 7，稱\n為兩個子集的對稱差，而“距離”正\n好是對稱差所含元素的數目。現在可以\n考慮的是給定 \( n \) 和距離的限制，\n這樣的數列最多是多少。\n
例五：有 \( m \) 個包括 0 或 1，長為 \( n \) 的\n數列，如果任兩個數列間的距離最少\n為 \( d \)，試證明

\[
m \leq \frac{2d}{2d-n}\n\]

證明：現在要考慮的是任兩個數列中\n“相異對”的數目，因為有 \( C_n^m \) 對數\n列，而任一對數列的“相異對”或\n“距離最少是 \( d \)，因此總距離最少是\n\( dC_n^m \)。將這些數列排起來成為 \( m \) 個\n橫行，每一直行 \( j, 1 \leq j \leq n \) 就對應著\n那些數列的 \( j \) 位置。如果 \( j \) 直行有 \( x_j \)\n個“0”，則有 \( m - x_j \) 個“1”，因此相\n異對有 \( x_j(m - x_j) \) 個。觀察到

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Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver’s name, home (or email) address and school affiliation. Please send submissions to Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon. The deadline for submitting solutions is February 28, 2004.

Problem 191. Solve the equation

\[ x^3 - 3x = \sqrt{x + 2}. \]

Problem 192. Inside a triangle \( ABC \), there is a point \( P \) satisfying \( \angle PAB = \angle PBC = \angle PCA = \varphi \). If the angles of the triangle are denoted by \( a, \beta, \gamma \), prove that

\[ \frac{1}{\sin^2 \varphi} = \frac{1}{\sin^2 \alpha} + \frac{1}{\sin^2 \beta} + \frac{1}{\sin^2 \gamma}. \]

Problem 193. Is there any perfect square, which has the same number of positive divisors of the form \( 3k + 1 \) as of the form \( 3k + 2 \)? Give a proof of your answer.

Problem 194. \textit{(Due to Achilles Pavlos PORFYRIADIS, American College of Thessaloniki “Anatolia”, Thessaloniki, Greece)} A circle with center \( O \) is internally tangent to two circles inside it, with centers \( O_1 \) and \( O_2 \), at points \( S \) and \( T \) respectively. Suppose the two circles inside intersect at points \( M \), \( N \) with \( N \) closer to \( ST \). Show that \( S, \ N, \ T \) are collinear if and only if \( SO_1TO_2 = 0O_2T \).

Problem 195. \textit{(Due to Fei Zhenpeng, Yongfeng High School, Yancheng City, Jiangsu Province, China)} Given \( n \in \mathbb{N} \) (where \( n \) is greater than 3) points on a plane, no three of them are collinear, \( x \) pairs of these points are connected by line segments. Prove that if

\[ x \geq n(n - 1)(n - 2) + \frac{3}{3(n - 2)}, \]

then there is at least one triangle having these line segments as edges.

Find all possible values of integers \( n > 3 \) such that

\[ \frac{n(n - 1)(n - 2) + 3}{3(n - 2)} \]

is an integer and the minimum number of line segments guaranteeing a triangle in the above situation is this integer.

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Solutions

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Problem 186. \textit{(Due to Fei Zhenpeng, Yongfeng High School, Yancheng City, Jiangsu Province, China)} Let \( a, \beta, \gamma \) be complex numbers such that

\[ a + \beta + \gamma = 1, \]

\[ a^2 + \beta^2 + \gamma^2 = 3, \]

\[ a^3 + \beta^3 + \gamma^3 = 7. \]

Determine the value of \( a^{21} + \beta^{21} + \gamma^{21} \).

\textbf{Solution.} Helder Oliveira de CASTRO (Colegio Objetivo, 3rd Grade, Sao Paulo, Brazil), CHEUNG Yun Kuen (Hong Kong Chinese Women’s Club College, Form 6), Murray KLAMKIN (University of Alberta, Edmonton, Canada) and Achilles Pavlos PORFYRIADIS (American College of Thessaloniki “Anatolia”, Thessaloniki, Greece).

Assume \( a \) is rational. Then its decimal representation will eventually be periodic. Suppose the period has \( k \) digits. Then for every \( n > 10^k \), \( f(n) \) is nonzero and ends in at least \( k \) zeros, which imply the period cannot have \( k \) digits. We got a contradiction.

Problem 188. The line \( S \) is tangent to the circumcircle of acute triangle \( ABC \) at \( B \). Let \( K \) be the projection of the orthocenter of triangle \( ABC \) onto line \( S \) (i.e. \( K \) is the foot of perpendicular from the orthocenter of triangle \( ABC \) to \( S \)). Let \( L \) be the midpoint of side \( AC \). Show that triangle \( BKL \) is isosceles. \textsc{(Source: 2000 Saint Petersburg City Math Olympiad)}

\textbf{Solution.} SIU Ho Chung (Queen’s College, Form 5).

Let \( O, G \) and \( H \) be the circumcenter, centroid and orthocenter of triangle \( ABC \) respectively. Let \( T \) and \( B \) be the projections of \( G \) and \( L \) onto line \( S \). From the Euler line theorem (cf. Math Excalibur, vol. 3, no. 1, p.1), we know that \( O, G, H \) are collinear, \( G \) is between \( O \) and \( H \) and \( 2OG = GH \). Then \( T \) is between \( B \) and \( K \) and \( 2BT = TK \).

Also, \( G \) is on the median \( BL \) and \( 2LG = BG \). So \( T \) is between \( B \) and \( R \) and \( 2BT = TR \). Then \( 2BR = 2( BT + RT ) = TK + TB = BK \). So \( BR = RK \). Since \( LR \) is perpendicular to line \( S \), by Pythagorean theorem, \( BL = LK \).

Other commended solvers: CHEUNG Yun Kuen (Hong Kong Chinese Women’s Club College, Form 6) and Achilles Pavlos PORFYRIADIS (American College of Thessaloniki “Anatolia”, Thessaloniki, Greece).

Problem 189. \( 2n + 1 \) segments are marked on a line. Each of the segments intersects at least \( n \) other segments. Prove that one of these segments representation of \( a \) write the numbers \( f(1), f(2), f(3), \ldots \) in base 10 in a row. Is \( a \) rational? Give a proof. \textsc{(Source: Israeli Math Olympiad)}

\textbf{Solution.} Helder Oliveira de CASTRO (Colegio Objetivo, 3rd Grade, Sao Paulo, Brazil), CHEUNG Yun Kuen (Hong Kong Chinese Women’s Club College, Form 6), Murray KLAMKIN (University of Alberta, Edmonton, Canada) and Achilles Pavlos PORFYRIADIS (American College of Thessaloniki “Anatolia”, Thessaloniki, Greece).
intersect all other segments. (Source 2000 Russian Math Olympiad)

**Solution.** Achileas Pavlos PORFYRIADIS (American College of Thessaloniki “Anatolia”, Thessaloniki, Greece).

We imagine the segments on the line as intervals on the real axis. Going from left to right, let $I_i$ be the $i$-th segment we meet with $i = 1, 2, \ldots, 2n + 1$. Let $I'_1$ and $I'_2$ be the left and right endpoints of $I_i$ respectively. Now $I_i$ contains $I'_1, \ldots, I'_n$ and so on. Therefore the segments $I_1, I_2, \ldots, I_{2n+1}$ intersect each other.

Next let $I'_n$ be the rightmost endpoint among $I'_1, I'_2, \ldots, I'_{2n+1}$. For each of the $n$ remaining intervals $I_{2n+2}, I_{2n+3}, \ldots, I_{2n+1}$, it must intersect at least one of $I_1, I_2, \ldots, I_n$ so it has to intersect at least $n$ intervals. This means for every $j \geq n + 2$, there is at least one $m \leq n$ such that $I'_m \leq I'_n \leq I'_j$, then $I_i$ intersects $I_j$ and hence every interval.

**Problem 190. (Due to Abderrahim Ouadi) For nonnegative integer $n$, let $[x]$ be the greatest integer less than or equal to $x$ and

$$f(n) = [\sqrt{n + \sqrt{n + 1 + \sqrt{n + 2}}} - \sqrt{n + n + 1}] .$$

Find the range of $f$ and for each $p$ in the range, find all nonnegative integers $n$ such that $f(n) = p$.

**Combined Solution by the Proposer and CHEUNG Yun Kuen (Hong Kong Chinese Women’s Club, Form 6).

For positive integer $n$, we claim that

$$\sqrt{n + 8} < g(n) < \sqrt{n + 9} .$$

where $g(n) = \sqrt{n + \sqrt{n + 1 + \sqrt{n + 2}}}$. This follows from

$$g(n)^2 = 3n + 3 + 2(\sqrt{n(n + 1) + (n + 1)(n + 2) + (n + 2)n})$$

and the following readily verified inequalities for positive integer $n$,

$$(n + 0.4)^2 < n(n + 1) < (n + 0.5)^2,$$

$$(n + 1.4)^2 < (n + 1)(n + 2) < (n + 1.5)^2$$

and $$(n + 0.7)^2 < (n + 2)n < (n + 1)^2 .$$

The claim implies the range of $f$ is a subset of nonnegative integers.

Suppose there is a positive integer $n$ such that $f(n) \geq 2$. Then

$$\sqrt{n + 9} > [g(n)] > 1 + \sqrt{n + 1} .$$

Squaring the two extremes and comparing, we see this is false for $n > 1$. Since $f(0) = 1$ and $f(1) = 1$, we have $f(n) = 0$ or 1 for all nonnegative integers $n$.

Next observe that

$$\sqrt{n + 8} < [g(n)] < \sqrt{n + 9}$$

is impossible by squaring all expressions. So $[g(n)] = \lfloor \sqrt{n + 8} \rfloor$.

Now $f(n) = 1$ if and only if $p = [g(n)]$ satisfies $\lfloor \sqrt{n + 1} \rfloor = n + 1$ or $p = 1$ . i.e.

$$\sqrt{n + 1} < p \leq \sqrt{n + 8} .$$

Considering squares (mod 9), we see that $p^2 = 9n + 4$ or $9n + 7$.

If $p^2 = 9n + 4$, then $p = 9k + 3$ or $9k + 6$.

In the former case, $n = 9k^2 + 4k$ and $(9k + 2)^2 \leq 9n + 1 = 81k^2 + 36k + 1 < (9k + 2)^2$ so that $\lfloor \sqrt{n + 1} \rfloor = 9k + 1 = p - 1$. In the latter case, $n = 9k^2 + 14k + 5$ and $(9k + 6)^2 \leq 9n + 1 = 81k^2 + 126k + 46 < (9k + 7)^2$ so that $\lfloor \sqrt{n + 1} \rfloor = 9k + 6 = p - 1$.

If $p^2 = 9n + 7$, then $p = 9k + 4$ or $9k + 5$. In the former case, $n = 9k^2 + 8k + 1$ and $(9k + 3)^2 \leq 9n + 1 = 81k^2 + 72k + 10 < (9k + 4)^2$ so that $\lfloor \sqrt{n + 1} \rfloor = 9k + 3 = p - 1$. In the latter case, $n = 9k^2 + 10k + 2$ and $(9k + 4)^2 \leq 9n + 1 = 81k^2 + 90k + 19 < (9k + 5)^2$ so that $\lfloor \sqrt{n + 1} \rfloor = 9k + 4 = p - 1$.

Therefore, $f(n) = 1$ if and only if $n$ is of the form $9k^2 + 4k$ or $9k^2 + 14k + 5$ or $9k^2 + 8k + 1$ or $9k^2 + 10k + 2$.

**Olympiad Corner**

(continued from page 1)

**Problem 4.** Let $ABC$ be a triangle. A circle passing through $A$ and $B$ intersects segments $AC$ and $BC$ at $D$ and $E$, respectively. Rays $BA$ and $ED$ intersect at $F$ while lines $BD$ and $CF$ intersect at $M$. Prove that $MF = MC$ if and only if $MB \cdot MD = MC^2$.

**Problem 5.** Let $a, b, c$ be positive real numbers. Prove that

$$(2a + b + c)^2 + (2b + c + a)^2 \leq \frac{(2a + b + c)^2 + (2b + c + a)^2}{2} + \frac{(2a + b + c)^2}{2} \leq 8 .$$

**Problem 6.** At the vertices of a regular hexagon are written six nonnegative integers whose sum is 2003. Bert is allowed to make moves of the following form: he may pick a vertex and replace the number written there by the absolute value of the difference between the numbers written at the two neighboring vertices. Prove that Bert can make a sequence of moves, after which the numbers 0 appears at all six vertices.

集與子集族

(continued from page 2)

$$x_j(m - x_j) \leq \left(\frac{x_j + (m - x_j)}{2}\right)^2 = \frac{m^2}{4} ,$$

因此

$$dc^m_n \leq \sum_{j=1}^{n} x_j(m - x_j) \leq \sum_{j=1}^{n} \frac{n^2}{4} = \frac{nm^2}{4} .$$

化簡即得

$$m \leq \frac{2d}{2d - n} .$$

例如 $n = 7, d = 4$, 得 $\frac{2d}{2d - n} = 8$，

所以不可能構造 9 個長為 7, 而相互間最少距離為 4 的數列。(讀者可試圖構造 8 個這樣的數列。) 這個例子實際上是編碼理論一個結果的特殊情況，這個結果通常稱為 Plotkin 線 (Plotkin Bound)。

集與子集族還有許多有趣的結果, 有待研究和討論。