Mathematical Excalibur

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Olympiad Corner

The Sixth Hong Kong (China) Mathematical Olympiad took place on December 20, 2003. Here are the problems. Time allowed: 3 hours

Problem 1. Find the greatest real K such that for every positive u, v and w with $u^2 > 4vw$, the inequality

$$(u^2 - 4vw)^2 > K(2v^2 - uw)(2w^2 - uv)$$

holds. Justify your claim.

Problem 2. Let $ABCDEF$ be a regular hexagon of side length 1, and O be the center of the hexagon. In addition to the sides of the hexagon, line segments are drawn from O to each vertex, making a total of twelve unit line segments. Find the number of paths of length 2003 along these line segments that start at O and terminate at O.

Problem 3. Let $ABCD$ be a cyclic quadrilateral. K, L, M, N are the midpoints of sides AB, BC, CD and DA respectively. Prove that the orthocentres of triangles AKN, BKL, CLM, DMN are the vertices of a parallelogram.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance and to Lee Man Fui and Poon Ming Fung for typesetting.

On-line: http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***May 25, 2004***.

For individual subscription for the next five issues for the 03-04 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Geometry via Complex Numbers

Kin Y. Li

Complex numbers are wonderful. In this article we will look at some applications of complex numbers to solving geometry problems. If a problem involves points and chords on a circle, often we can without loss of generality assume it is the unit circle. In the following discussion, we will use the same letter for a point to denote the same complex number in the complex plane. To begin, we will study the equation of lines through points. Suppose Z is an arbitrary point on the line through W_1 and W_2. Since the vector from W_1 to Z is a multiple of the vector from W_1 to W_2, so in terms of complex numbers, we get $Z - W_1 = t(W_2 - W_1)$ for some real t. Now $t = \bar{t}$ and so

$$\frac{Z - W_1}{W_2 - W_1} = \frac{\overline{Z} - \overline{W_1}}{\overline{W_2} - \overline{W_1}}$$

Reversing the steps, we can see that every Z satisfying the equation corresponds to a point on the line through W_1 and W_2. So this is the equation of a line through two points in the complex variable Z.

Next consider the line passing through a point C and perpendicular to the line through W_1 and W_2. Let Z be on this line. Then the vector from C to Z is perpendicular to the vector from W_1 to W_2. In terms of complex numbers, we get $Z - C = it(W_2 - W_1)$ for some real t. So

$$\frac{Z - C}{i(W_2 - W_1)} = \frac{\overline{Z} - \overline{C}}{\bar{i}(\overline{W_2} - \overline{W_1})}.$$

Again reversing steps, we can conclude this is the equation of the line through C perpendicular to the line through W_1 and W_2.

In case the points W_1 and W_2 are on the unit circle, we have $W_1\overline{W_1} = 1 = W_2\overline{W_2}$. Multiplying the numerators and denominators of the right sides of the two displayed equations above by W_1W_2, we can simplify them to

$$Z + W_1W_2\overline{Z} = W_1 + W_2$$

and $$Z - W_1W_2\overline{Z} = C - W_1W_2\overline{C}$$

respectively.

By moving W_2 toward W_1 along the unit circle, in the limit, we will get the equation of the tangent line at W_1 to the unit circle. It is $Z + W_1{}^2\overline{Z} = 2W_1$.

Similarly, the equation of the line through C perpendicular to this tangent line is $Z - W_1{}^2\overline{Z} = C - W_1{}^2\overline{C}$.

For a given triangle $A_1A_2A_3$ with the unit circle as its circumcircle, in terms of complex numbers, its circumcenter is the origin O, its centroid is $G = (A_1 + A_2 + A_3)/3$, its orthocenter is $H = A_1 + A_2 + A_3$ (because $OH = 3OG$) and the center of its nine point circle is $N = (A_1 + A_2 + A_3)/2$ (because N is the midpoint of OH).

Let us proceed to some examples.

Example 1. *(2000 St. Petersburg City Math Olympiad, Problem Corner 188)* The line S is tangent to the circumcircle of acute triangle ABC at B. Let K be the projection of the orthocenter of triangle ABC onto line S (i.e. K is the foot of perpendicular from the orthocenter of triangle ABC to S). Let L be the midpoint of side AC. Show that triangle BKL is isosceles.

Solution. (*Due to POON Ming Fung, STFA Leung Kau Kui College, Form 6*) Without loss of generality, let the circumcircle of triangle ABC be the unit circle on the plane. Let $A = a + bi$, $B = -i$, $C = c + di$. Then the orthocenter is $H = A + B + C$ and $K = (a + c) - i$, $L = (a + c)/2 + (b + d)i/2$. Since

$$LB = \frac{1}{2}\sqrt{(a+c)^2 + (b+d+2)^2} = KL,$$

triangle BKL is isosceles.

Example 2. Consider triangle *ABC* and its circumcircle *S*. Reflect the circle with respect to *AB*, *AC* and *BC* to get three new circles S_{AB}, S_{AC} and S_{BC} (with the same radius as *S*). Show that these three new circles intersect at a common point. Identify this point.

Solution. Without loss of generality, we may assume *S* is the unit circle. Let the center of S_{AB} be O', then O' is the mirror image of *O* with respect to the segment *AB*. So $O' = A + B$ (because segments OO' and *AB* bisect each other). Similarly, the centers of S_{AC} and S_{BC} are $A + C$ and $B + C$ respectively. We need to show there is a point *Z* such that *Z* is on all three new circles, i.e.

$$|Z - (A + B)| = |Z - (A + C)| = |Z - (B + C)| = 1.$$

We easily see that the orthocenter of triangle *ABC*, namely $Z = H = A + B + C$, satisfies these equations. Therefore, the three new circles intersect at the orthocenter of triangle *ABC*.

Example 3. A point *A* is taken inside a circle. For every chord of the circle passing through *A*, consider the intersection point of the two tangents at the endpoints of the chord. Find the locus of these intersection points.

Solution. Without loss of generality we may assume the circle is the unit circle and *A* is on the real axis. Let *WX* be a chord passing through *A* with *W* and *X* on the circle. The intersection point *Z* of the tangents at *W* and *X* satisfies $Z + W^2\overline{Z} = 2W$ and $Z + X^2\overline{Z} = 2X$. Solving these equations together for *Z*, we find $Z = 2/(\overline{W} + \overline{X})$.

Since *A* is on the chord *WX*, the real number *A* satisfies the equation for line *WX*, i.e. $A + WXA = W + X$. Using $W\overline{W} = 1 = X\overline{X}$, we see that

$$\operatorname{Re} Z = \frac{1}{\overline{W} + \overline{X}} + \frac{1}{W + X} = \frac{WX + 1}{W + X} = \frac{1}{A}.$$

So the locus lies on the vertical line through $1/A$.

Conversely, for any point *Z* on this line, draw the two tangents from *Z* to the unit circle and let them touch the unit circle at the point *W* and *X*. Then the above equations are satisfied by reversing the argument. In particular, $A + WXA = W + X$ and so *A* is on the chord *WX*. Therefore, the locus is the line perpendicular to *OA* at a distance $1/OA$ from *O*.

Example 4. Let A_1, A_2, A_3 be the midpoints of W_2W_3, W_3W_1, W_1W_2 respectively. From A_i drop a perpendicular to the tangent line to the circumcircle of triangle $W_1W_2W_3$ at W_i. Prove that these perpendicular lines are concurrent. Identify this point of concurrency.

Solution. Without loss of generality, let the circumcircle of triangle $W_1W_2W_3$ be the unit circle. The line perpendicular to the tangent at W_1 through $A_1 = (W_2 + W_3)/2$ has equation

$$Z - W_1^2\overline{Z} = \frac{W_2 + W_3}{2} - W_1^2\,\frac{\overline{W_2} + \overline{W_3}}{2}.$$

Using $W_1\overline{W_1} = 1$, we may see that the right side is the same as

$$\frac{W_1 + W_2 + W_3}{2} - W_1^2\,\frac{\overline{W_1} + \overline{W_2} + \overline{W_3}}{2}.$$

From this we see that $N = (W_1 + W_2 + W_3)/2$ satisfies the equation of the line and so *N* is on the line. Since the expression for *N* is symmetric with respect to W_1, W_2, W_3, we can conclude that *N* will also lie on the other two lines. Therefore, the lines concur at *N*, the center of the nine point circle of triangle $W_1W_2W_3$.

Example 5. *(Simson Line Theorem)* Let *W* be on the circumcircle of triangle $Z_1Z_2Z_3$ and *P*, *Q*, *R* be the feet of the perpendiculars from *W* to Z_3Z_1, Z_1Z_2, Z_2Z_3 respectively. Prove that P, Q, R are collinear. (This line is called the *Simson line* of triangle $Z_1Z_2Z_3$ from *W*.)

Solution. Without loss of generality, we may assume the circumcircle of triangle $Z_1Z_2Z_3$ is the unit circle.
Then $|Z_1| = |Z_2| = |Z_3| = |W| = 1$. Now *P* is on the line Z_3Z_1 and the line through *W* perpendicular to Z_3Z_1. So *P* satisfies the equations $P + Z_1Z_3\overline{P} = Z_1 + Z_3$ and $P - Z_1Z_3\overline{P} = W - Z_1Z_3\overline{W}$. Solving these together for *P*, we get

$$P = \frac{Z_1 + Z_3 + W - Z_1Z_3\overline{W}}{2}.$$

Similarly,

$$Q = \frac{Z_1 + Z_2 + W - Z_1Z_2\overline{W}}{2}$$

and

$$R = \frac{Z_2 + Z_3 + W - Z_2Z_3\overline{W}}{2}.$$

To show *P*, *Q*, *R* are collinear, it suffices to check that

$$\frac{P-R}{Q-R} = \frac{\overline{P}-\overline{R}}{\overline{Q}-\overline{R}}.$$

Now the right side is

$$\frac{\overline{Z_1} - \overline{Z_2} - \overline{Z_1Z_3}W + \overline{Z_2Z_3}W}{\overline{Z_1} - \overline{Z_3} - \overline{Z_1Z_2}W + \overline{Z_2Z_3}W}.$$

Multiplying the numerator and denominator by $Z_1Z_2Z_3\overline{W}$ and using $Z_i\overline{Z_i} = 1 = W\overline{W}$, we get

$$\frac{Z_2Z_3\overline{W} - Z_1Z_3\overline{W} - Z_2 + Z_1}{Z_2Z_3\overline{W} - Z_1Z_2\overline{W} - Z_3 + Z_1}.$$

This equals the left side $(P - R)/(Q - R)$ and we complete the checking.

Example 6. (*2003 IMO, Problem 4*) Let *ABCD* be a cyclic quadrilateral. Let *P*, *Q* and *R* be the feet of the perpendiculars from *D* to the lines *BC*, *CA* and *AB* respectively. Show that $PQ = QR$ if and only if the bisectors of $\angle ABC$ and $\angle ADC$ meet on *AC*.

Solution. (*Due to SIU Tsz Hang, 2003 Hong Kong IMO team member*) Without loss of generality, assume *A*, *B*, *C*, *D* lies on the unit circle and the perpendicular bisector of AC is the real axis. Let $A = \cos\theta + i\sin\theta$, then $C = \overline{A} = \cos\theta - i\sin\theta$ so that $AC = 1$ and $A + C = 2\cos\theta$. Since the bisectors of $\angle ABC$ and $\angle ADC$ pass through the midpoints of the major and minor arc *AC*, we may assume the bisectors of $\angle ABC$ and $\angle ADC$ pass through 1 and −1 respectively. Let *AC* intersect the bisector of $\angle ABC$ at *Z*, then *Z* satisfies $Z + AC\overline{Z} = A + C$, (which is $Z + \overline{Z} = 2\cos\theta$), and $Z + B\overline{Z} = B + 1$. Solving for *Z*, we get

$$Z = \frac{2B\cos\theta - B - 1}{B - 1}.$$

Similarly, the intersection point Z' of *AC* with the bisector of $\angle ADC$ is

$$Z' = \frac{2D\cos\theta + D - 1}{D + 1}.$$

Next, *R* is on the line *AB* and the line through *D* perpendicular to *AB*. So $R + AB\overline{R} = A + B$ and $R - AB\overline{R} = D - AB\overline{D}$. Solving for *R*, we find

$$R = \frac{A + B + D - AB\overline{D}}{2}.$$

Similarly,

$$P = \frac{B + C + D - BC\overline{D}}{2}$$

and

$$Q = \frac{C + A + D - CA\overline{D}}{2}.$$

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Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for submitting solutions is ***May 25, 2004.***

Problem 196. (*Due to John PANAGEAS, High School "Kaisari", Athens, Greece*) Let $x_1, x_2, ..., x_n$ be positive real numbers with sum equal to 1. Prove that for every positive integer m,

$$n \le n^m (x_1^m + x_2^m + ... + x_n^m).$$

Problem 197. In a rectangular box, the length of the three edges starting at the same vertex are prime numbers. It is also given that the surface area of the box is a power of a prime. Prove that exactly one of the edge lengths is a prime number of the form $2^k - 1$.

Problem 198. In a triangle ABC, $AC = BC$. Given is a point P on side AB such that $\angle ACP = 30^\circ$. In addition, point Q outside the triangle satisfies $\angle CPQ = \angle CPA + \angle APQ = 78^\circ$. Given that all angles of triangles ABC and QPB, measured in degrees, are integers, determine the angles of these two triangles.

Problem 199. Let R^+ denote the positive real numbers. Suppose $f: R^+ \to R^+$ is a strictly decreasing function such that for all $x, y \in R^+$,

$$f(x+y) + f(f(x) + f(y)) = f(f(x + f(y)) + f(y + f(x))).$$

Prove that $f(f(x)) = x$ for every $x > 0$. (*Source: 1997 Iranian Math Olympiad*)

Problem 200. Aladdin walked all over the equator in such a way that each moment he either was moving to the west or was moving to the east or applied some magic trick to get to the opposite point of the Earth. We know that he travelled a total distance less than half of the length of the equator altogether during his westward moves. Prove that there was a moment when the difference between the distances he had covered moving to the east and moving to the west was at least half of the length of the equator.

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Solutions

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Due to an editorial mistake in the last issue, solutions to problems 186, 187, 188 by **POON Ming Fung** (*STFA Leung Kau Kui College, Form 6*) were overlooked and his name was not listed among the solvers. We express our apology to him and point out that his clever solution to problem 188 is printed in example 1 of the article "Geometry via Complex Numbers" in this issue.

Problem 191. Solve the equation

$$x^3 - 3x = \sqrt{x+2}.$$

Solution. **Helder Oliveira de CASTRO (**ITA-Aeronautic Institue of Technology, Sao Paulo, Brazil) and **Yufei ZHAO** (Don Mills Collegeiate Institute, Toronoto, Canada, Grade 10).

If $x < -2$, then the right side of the equation is not defined. If $x > 2$, then

$$x^3 - 3x = \frac{x^3 + 3x(x+2)(x-2)}{4}$$
$$> \frac{x^3}{4} > \sqrt{x+2}.$$

So the solution(s), if any, must be in [-2, 2]. Write $x = 2\cos a$, where $0 \le a \le \pi$. The equation becomes

$$8\cos^3 a - 6\cos a = \sqrt{2\cos a + 2}.$$

Using the triple angle formula on the left side and the half angle formula on the right side, we get

$$2\cos 3a = 2\cos\frac{a}{2} (\ge 0).$$

Then $3a \pm (a/2) = 2n\pi$ for some integer n. Since $3a \pm (a/2) \in [-\pi/2, 7\pi/2]$, we get $n = 0$ or 1. We easily checked that $a = 0$, $4\pi/5$, $4\pi/7$ yield the only solutions $x = 2$, $2\cos(4\pi/5)$, $2\cos(4\pi/7)$.

Other commended solvers: **CHUNG Ho Yin** (STFA Leung Kau Kui College, Form 7), **LEE Man Fui** (CUHK, Year 1), **LING Shu Dung**, **POON Ming Fung** (STFA Leung Kau Kui College, Form 6), **SINN Ming Chun** (STFA Leung Kau Kui College, Form 4), **SIU Ho Chung** (Queen's College, Form 5), **TONG Yiu Wai** (Queen Elizabeth School), **YAU Chi Keung** (CNC Memorial College, Form 7) and **YIM Wing Yin** (South Tuen Mun Government Secondary School, Form 4).

Problem 192. Inside a triangle ABC, there is a point P satisfies $\angle PAB = \angle PBC = \angle PCA = \varphi$. If the angles of the triangle are denoted by α, β and γ, prove that

$$\frac{1}{\sin^2\varphi} = \frac{1}{\sin^2\alpha} + \frac{1}{\sin^2\beta} + \frac{1}{\sin^2\gamma}.$$

Solution. **LEE Tsun Man Clement** (St. Paul's College), **POON Ming Fung** (STFA Leung Kau Kui College, Form 6), **SIU Ho Chung** (Queen's College, Form 5) and **Yufei ZHAO** (Don Mills Collegiate Institute, Tornoto, Canada, Grade 10).

Let AP meet BC at X. Since $\angle XBP = \angle BAX$ and $\angle BXP = \angle AXB$, triangles XPB and XBA are similar. Then $XB/XP = XA/XB$. Using the sine law and the last equation, we have

$$\frac{\sin^2\varphi}{\sin^2\beta} = \frac{\sin^2\angle XAB}{\sin^2\angle XBA} = \frac{XB^2}{XA^2}$$
$$= \frac{XP\cdot XA}{XA^2} = \frac{XP}{XA}$$

Using [ ] to denote area, we have

$$\frac{XP}{XA} = \frac{[XBP]}{[XBA]} = \frac{[XCP]}{[XCA]} = \frac{[BPC]}{[ABC]}$$

Combining the last two equations, we have $\sin^2\varphi/\sin^2\beta = [BPC]/[ABC]$. By similar arguments, we have

$$\frac{\sin^2\varphi}{\sin^2\alpha} + \frac{\sin^2\varphi}{\sin^2\phi} + \frac{\sin^2\varphi}{\sin^2\gamma}$$
$$= \frac{[APB]}{[ABC]} + \frac{[BPC]}{[ABC]} + \frac{[CPA]}{[ABC]}$$
$$= \frac{[ABC]}{[ABC]} = 1$$

The result follows.

Other commended solvers: **CHENG Tsz Chung** (La Salle College, Form 5), **LEE Man Fui** (CUHK, Year 1) and **Achilleas P. PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece).

Comments: Professor Murray KLAMKIN (University of Alberta, Edmonton, Canada) informed us that the result $\csc^2\varphi = \csc^2\alpha + \csc^2\beta + \csc^2\gamma$ in the problem is a known relation for the Brocard angle φ of a triangle. Also known is $\cot\varphi = \cot\alpha + \cot\beta + \cot\gamma$. He mentioned these relations and others are given in R.A. Johnson, *Advanced Euclidean Geometry*, Dover, N.Y., 1960, pp. 266-267. (For the convenience of interested readers, the Chinese translation of this book can be found in many bookstore.–Ed) LEE Man Fui and Achilleas PORFYRIADIS gave a proof of the cotangent relation and use it to derive the cosecant relation, which is the equation in the problem, by trigonometric manipulations.

Problem 193. Is there any perfect square, which has the same number of positive divisors of the form $3k + 1$ as of the form $3k + 2$? Give a proof of your answer.

Solution 1. **K.C. CHOW** (Kiangsu-Chekiang College Shatin, Teacher), **LEE Tsun Man Clement** (St. Paul's College), **SIU Ho Chung** (Queen's College, Form 5) and **Yufei ZHAO** (Don Mills Collegiate Institute, Toronto, Canada, Grade 10).

No. For a perfect square m^2, let $m = 3^a b$ with b not divisible by 3. Then $m^2 = 3^{2a}b^2$. Observe that divisors of the form $3k + 1$ or $3k + 2$ for m^2 and for b^2 consist of the same numbers because they cannot have any factor of 3. Since b^2 has an odd number of divisors and they can only be of the form $3k + 1$ or $3k + 2$, so the number of divisors of the form $3k + 1$ cannot be the same as the number of divisors of the form $3k + 2$. Therefore, the same is true for m^2.

Solution 2. **Helder Oliveira de CASTRO** (ITA-Aeronautic Institute of Technology, Sao Paulo, Brazil), **LEE Man Fui** (CUHK, Year 1), **LING Shu Dung, POON Ming Fung** (STFA Leung Kau Kui College, Form 6), **Achilleas P. PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece), **Alan T.W. WONG** (Markham, Ontario, Canada) and **YIM Wing Yin** (South Tuen Mun Government Secondary School, Form 4).

No. For a perfect square, its prime factorization is of the form $2^{2e_1}3^{2e_2}5^{2e_3}\cdots$. Let x, y, z be the number of divisors of the form $3k$, $3k + 1$, $3k + 2$ for this perfect square respectively. Then $x+y+z = (2e_1 + 1)(2e_2 + 1)(2e_3 + 1)\cdots$ is odd. Now divisor of the form $3k$ has at least one factor 3, so $x = (2e_1 + 1)(2e_2)(2e_3 + 1)\cdots$ is even. Then $y + z$ is odd. Therefore y cannot equal z.

Other commended solvers: **CHENG Tsz Chung** (La Salle College, Form 5) and **YEUNG Wai Kit** (STFA Leung Kau Kui College).

Problem 194. (*Due to Achilleas Pavlos PORFYRIADIS, American College of Thessaloniki "Anatolia", Thessaloniki, Greece*) A circle with center O is internally tangent to two circles inside it, with centers O_1 and O_2, at points S and T respectively. Suppose the two circles inside intersect at points M, N with N closer to ST. Show that S, N, T are collinear if and only if $SO_1/OO_1 = OO_2/TO_2$.

Solution. **CHENG Tsz Chung** (La Salle College, Form 5), **K. C. CHOW** (Kiangsu-Chekiang College Shatin, Teacher), **Helder Oliveira de CASTRO** (ITA-Aeronautic Institute of Technology, Sao Paulo, Brazil), **LEE Tsun Man** Clement (St. Paul's College), **LING Shu Dung, POON Ming Fung** (STFA Leung Kau Kui College, Form 6), **SIU Ho Chung** (Queen's College, Form 5), **YEUNG Wai Kit** (STFA Leung Kau Kui College), **Yufei ZHAO** (Don Mills Collegiate Institute, Toronto, Canada, Grade 10) and the proposer.

If S, N, T are collinear, then triangles SO_1N and SOT are isosceles and share the common angle OST, which imply they are similar. Thus $\angle SO_1N = \angle SOT$ and so lines O_1N and OT are parallel. Similarly, lines O_2N and OS are parallel. Hence, OO_1NO_2 is a parallelogram and $OO_2 = O_1N = O_1S$, $OO_1 = O_2N = O_2T$. Therefore, $SO_1/OO_1 = OO_2/TO_2$. Conversely, if $SO_1/OO_1 = OO_2/TO_2$, then using $OO_1 = OS - O_1S$ and $OO_2 = OT - O_2T$, we get

$$\frac{O_1S}{OS - O_1S} = \frac{OT - O_2T}{O_2T},$$

which reduces to $O_1S + O_2T = OS$. Then $OO_1 = OS - O_1S = O_2T = O_2N$ and $OO_2 = OT - O_2T = O_1S = O_1N$. Hence OO_1NO_2 is again a parallelogram. Then

$$\begin{aligned}&\angle O_1NS + \angle O_1NO_2 + \angle O_2NT\\ &= \angle O_1SN + \angle O_1NO_2 + \angle O_2TN\\ &= \frac{1}{2}\angle OO_1N + \angle O_1NO_2 + \frac{1}{2}\angle OO_2N\\ &= 180^\circ.\end{aligned}$$

Therefore, S, N, T are collinear.

Other commended solver: **TONG Yiu Wai** (Queen Elizabeth School).

Problem 195. (*Due to Fei Zhenpeng, Yongfeng High School, Yancheng City, Jiangsu Province, China*) Given n ($n > 3$) points on a plane, no three of them are collinear, x pairs of these points are connected by line segments. Prove that if

$$x \geq \frac{n(n-1)(n-2)+3}{3(n-2)},$$

then there is at least one triangle having these line segments as edges. Find all possible values of integers $n > 3$ such that $\frac{n(n-1)(n-2)+3}{3(n-2)}$ is an integer and the minimum number of line segments guaranteeing a triangle in the above situation is this integer.

Solution. **SIU Ho Chung** (Queen's College, Form 5), **Yufei ZHAO** (Don Mills Collegiate Institute, Toronto, Canada, Grade 10) and the proposer.

For every three distinct points A, B, C, form a pigeonhole containing the three segments AB, BC, CA. (Each segment may be in more than one pigeonholes.) There are C_3^n pigeonholes. For each segment joining a pair of endpoints, that segment will be in $n - 2$ pigeonholes. So if $x(n-2) \geq 2C_3^n + 1$, that is

$$x \geq \frac{2C_3^n + 1}{n-2} = \frac{n(n-1)(n-2)+3}{3(n-2)},$$

then by the pigeonhole principle, there is at least one triangle having these line segments as edges.

If $f(n) = (n(n - 1)(n - 2) + 3) / (3(n - 2))$ is an integer, then $3(n - 2)f(n) = n(n - 1)(n - 2) + 3$ implies 3 is divisible by $n - 2$. Since $n > 3$, we must have $n = 5$. Then $f(5) = 7$. For the five vertices A, B, C, D, E of a regular pentagon, if we connected the six segments BC, CD, DE, EA, AC, BE, then there is no triangle. So a minimum of $f(5) = 7$ segments is needed to get a triangle.

Other commended solvers: **K. C. CHOW** (Kiangsu-Chekiang College Shatin, Teacher) and **POON Ming Fung** (STFA Leung Kau Kui College, Form 6).

Olympiad Corner

(continued from page 1)

Problem 4. Find, with reasons, all integers a, b, and c such that

$$\frac{1}{2}(a+b)(b+c)(c+a) + (a+b+c)^3 = 1 - abc.$$

Geometry via Complex Numbers

(continued from page 2)

By Simson's theorem, P, Q, R are collinear. So $PQ = QR$ if and only if $Q = (P + R)/2$. In terms of A, B, C, D, this may be simplified to

$$C + A - 2B = (2CA - AB - BC)\overline{D}.$$

In terms of B, D, θ, this is equivalent to $(2\cos\theta - 2B)D = 2 - 2B\cos\theta$. This is easily checked to be the same as

$$\frac{2\cos\theta - B - 1}{B - 1} = \frac{2D\cos\theta + D - 1}{D + 1},$$

i.e. $Z = Z'$.

Comments: The official solution by pure geometry is shorter, but it takes a fair amount of time and cleverness to discover. Using complex numbers as above reduces the problem to straight computations.