Mathematical Excalibur

Volume 9, Number 2 *May 2004 – July 2004*

Olympiad Corner

The XVI Asian Pacific Mathematical Olympiad took place on March 2004. Here are the problems. Time allowed: 4 hours.

Problem 1. Determine all finite nonempty sets S of positive integers satisfying

$\frac{i+j}{(i,j)}$ is an element of S for all i, j in S,

where (i, j) is the greatest common divisor of i and j.

Problem 2. Let O be the circumcenter and H the orthocenter of an acute triangle ABC. Prove that the area of one of the triangles AOH, BOH, COH is equal to the sum of the areas of the other two.

Problem 3. Let a set S of 2004 points in the plane be given, no three of which are collinear. Let $\mathcal{L}$ denote the set of all lines (extended indefinitely in both directions) determined by pairs of points from the set. Show that it is possible to color the points of S with at most two colors, such that for any points p, q of S, the number

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The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***August 9, 2004***.

For individual subscription for the next five issues for the 03-04 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Inversion

Kin Y. Li

In algebra, the method of logarithm *transforms* tough problems involving multiplications and divisions into simpler problems involving additions and subtractions. For every positive number x, there is a unique real number $\log x$ in base 10. This is a one-to-one correspondence between the positive numbers and the real numbers.

In geometry, there are also transformation methods for solving problems. In this article, we will discuss one such method called *inversion*. To present this, we will introduce the *extended plane*, which is the plane together with a point that we would like to think of as infinity. Also, we would like to think of *all* lines on the plane will go through *this point at infinity*! To understand this, we will introduce the *stereographic projection*, which can be described as follow.

Consider a sphere sitting on a point O of a plane. If we remove the north pole N of the sphere, we get a *punctured* sphere. For every point P on the plane, the line NP will intersect the punctured sphere at a *unique* point S_P. So this gives a one-to-one correspondence between the plane and the punctured sphere. If we consider the points P on a circle in the plane, then the S_P points will form a circle on the punctured sphere. However, if we consider the points P on *any* line in the plane, then the S_P points will form a punctured circle on the sphere with N as the point removed from the circle. *If we move a point P on any line on the plane toward infinity, then S_P will go toward the same point N!* Thus, in this model, all lines can be thought of as going to the same infinity.

Now for the method of inversion, let O be a point on the plane and r be a positive number. The *inversion* with center O and radius r is the function on the extended plane that sends a point $X \neq O$ to the *image* point X' on the ray $\overrightarrow{OX}$ such that

$$OX \cdot OX' = r^2.$$

When $X = O$, X' is taken to be the point at infinity. When X is infinity, X' is taken to be O. The circle with center O and radius r is called the *circle of inversion*.

The method of inversion is based on the following facts.

(1) The function sending X to X' described above is a one-to-one correspondence between the extended plane with itself. (This follows from checking $(X')' = X$.)

(2) If X is on the circle of inversion, then $X' = X$. If X is outside the circle of inversion, then X' is the midpoint of the chord formed by the tangent points T_1, T_2 of the tangent lines from X to the circle of inversion. (This follows from

$$OX \cdot OX' = (r \sec \angle T_1OX)(r \cos \angle T_1OX) = r^2.)$$

(3) A circle not passing through O is sent to a circle not passing through O. In this case, the images of concyclic points are concyclic. The point O, the centers of the circle and the image circle are collinear. *However, the center of the circle is not sent to the center of the image circle!*

(4) A circle passing through O is sent to a line which is not passing through O and is parallel to the tangent line to the circle at O. Conversely, a line not passing through O is sent to a circle passing through O with the tangent line at O parallel to the line.

(5) A line passing through O is sent to itself.

(6) If two curves intersect at a certain angle at a point $P \neq O$, then the image curves will also intersect at the same angle at P'. If the angle is a right angle, the curves are said to be orthogonal. So in particular, orthogonal curves at P are sent to orthogonal curves at P'. A circle orthogonal to the circle of inversion is sent to itself. Tangent curves at P are sent to tangent curves at P'.

(7) If points A, B are different from O and points O, A, B are not collinear, then the equation $OA \cdot OA' = r^2 = OB \cdot OB'$ implies $OA/OB = OB'/OA'$. Along with $\angle AOB = \angle B'OA'$, they imply ΔOAB, $\Delta OB'A'$ are similar. Then

$$\frac{A'B'}{AB} = \frac{OA'}{OB} = \frac{r^2}{OA \cdot OB}$$

so that

$$A'B' = \frac{r^2}{OA \cdot OB} AB .$$

The following are some examples that illustrate the powerful method of inversion. In each example, when we do inversion, it is often that we take the point that plays the most significant role and where many circles and lines intersect.

***Example* 1**. (*Ptolemy's Theorem*) For coplanar points A, B, C, D, if they are concyclic, then

$$AB \cdot CD + AD \cdot BC = AC \cdot BD.$$

Solution. Consider the inversion with center D and any radius r. By fact (4), the circumcircle of ΔABC is sent to the line through A', B', C'. Since $A'B' + B'C' = A'C'$, we have by fact (7) that

$$\frac{r^2}{AD \cdot BD} AB + \frac{r^2}{BD \cdot CD} BC = \frac{r^2}{AD \cdot CD} AC.$$

Multiplying by $(AD \cdot BD \cdot CD)/r^2$, we get the desired equation.

Remarks. The steps can be reversed to get the converse statement that if

$$AB \cdot CD + AD \cdot BC = AC \cdot BD,$$

then A,B,C,D are concyclic.

***Example* 2**. (*1993 USAMO*) Let $ABCD$ be a convex quadrilateral such that diagonals AC and BD intersect at right angles, and let O be their intersection point. Prove that the reflections of O across AB, BC, CD, DA are concyclic.

Solution. Let P, Q, R, S be the feet of perpendiculars from O to AB, BC, CD, DA, respectively. The problem is equivalent to showing P, Q, R, S are concyclic (since they are the midpoints of O to its reflections). Note $OSAP$, $OPBQ$, $OQCR$, $ORDS$ are cyclic quadrilaterals. Let their circumcircles be called C_A, C_B, C_C, C_D, respectively.

Consider the inversion with center O and any radius r. By fact (5), lines AC and BD are sent to themselves. By fact (4), circle C_A is sent to a line L_A parallel to BD, circle C_B is sent to a line L_B parallel to AC, circle C_C is sent to a line L_C parallel to BD, circle C_D is sent to a line L_D parallel to AC.

Next C_A intersects C_B at O and P. This implies L_A intersects L_B at P'. Similarly, L_B intersects L_C at Q', L_C intersects L_D at R' and L_D intersects L_A at S'.

Since $AC \perp BD$, $P'Q'R'S'$ is a rectangle, hence cyclic. Therefore, by fact (3), P, Q, R, S are concyclic.

***Example* 3.** (*1996 IMO*) Let P be a point inside triangle ABC such that

$$\angle APB - \angle ACB = \angle APC - \angle ABC.$$

Let D, E be the incenters of triangles APB, APC, respectively. Show that AP, BD, CE meet at a point.

Solution. Let lines AP, BD intersect at X, lines AP, CE intersect at Y. We have to show $X = Y$. By the angle bisector theorem, $BA/BP = XA/XP$. Similarly, $CA/CP = YA/YP$. As X, Y are on AP, we get $X = Y$ if and only if $BA/BP = CA/CP$.

Consider the inversion with center A and any radius r. By fact (7), ΔABC, $\Delta AC'B'$ are similar, ΔAPB, $\Delta AB'P'$ are similar and ΔAPC, $\Delta AC'P'$ are similar. Now

$$\begin{aligned}\angle B'C'P' &= \angle AC'P' - \angle AC'B' \\ &= \angle APC - \angle ABC \\ &= \angle APB - \angle ACB \\ &= \angle AB'P - \angle AB'C' \\ &= \angle C'B'P'.\end{aligned}$$

So $\Delta B'C'P'$ is isosceles and $P'B' = P'C'$. From ΔAPB, $\Delta AB'P'$ similar and ΔAPC, $\Delta AC'P'$ similar, we get

$$\frac{BA}{BP} = \frac{P'A}{P'B'} = \frac{P'A}{P'C'} = \frac{CA}{CP}.$$

Therefore, $X = Y$.

***Example* 4.** (*1995 Israeli Math Olympiad*) Let PQ be the diameter of semicircle H. Circle O is internally tangent to H and tangent to PQ at C. Let A be a point on H and B a point on PQ such that $AB \perp PQ$ and is tangent to O. Prove that AC bisects $\angle PAB$.

Solution. Consider the inversion with center C and any radius r. By fact (7), ΔCAP, $\Delta CP'A'$ similar and ΔCAB, $\Delta CB'A'$ similar. So AC bisects PAB if and only if $\angle CAP = \angle CAB$ if and only if $\angle CP'A' = \angle CB'A'$.

By fact (5), line PQ is sent to itself. Since circle O passes through C, circle O is sent to a line O' parallel to PQ. By fact (6), since H is tangent to circle O and is orthogonal to line PQ, H is sent to the semicircle H' tangent to line O' and has diameter $P'Q'$. Since segment AB is tangent to circle O and is orthogonal to PQ, segment AB is sent to arc $A'B'$ on the semicircle tangent to line O' and has diameter CB'. Now observe that arc $A'Q'$ and arc $A'C$ are symmetrical with respect to the perpendicular bisector of CQ' so we get $\angle CP'A' = \angle CB'A'$.

In the solutions of the next two examples, we will consider the nine-point circle and the Euler line of a triangle. Please consult Vol. 3, No. 1 of Mathematical Excalibur for discussion if necessary.

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Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong*. The deadline for submitting solutions is ***August 9, 2004.***

Problem 201. (*Due to Abderrahim OUARDINI, Talence, France*) Find which nonright triangles ABC satisfy

$$\tan A \tan B \tan C > [\tan A] + [\tan B] + [\tan C],$$

where $[t]$ denotes the greatest integer less than or equal to t. Give a proof.

Problem 202. (*Due to LUK Mee Lin, La Salle College*) For triangle ABC, let D, E, F be the midpoints of sides AB, BC, CA, respectively. Determine which triangles ABC have the property that triangles ADF, BED, CFE can be folded above the plane of triangle DEF to form a tetrahedron with AD coincides with BD; BE coincides with CE; CF coincides with AF.

Problem 203. (*Due to José Luis DÍAZ-BARRERO, Universitat Politecnica de Catalunya, Barcelona, Spain*) Let a, b and c be real numbers such that $a + b + c \neq 0$. Prove that the equation

$$(a+b+c)x^2 + 2(ab+bc+ca)x + 3abc = 0$$

has only real roots.

Problem 204. Let n be an integer with $n > 4$. Prove that for every n distinct integers taken from 1, 2, …, $2n$, there always exist two numbers whose least common multiple is at most $3n + 6$.

Problem 205. *(Due to HA Duy Hung, Hanoi University of Education, Vietnam)* Let a, n be integers, both greater than 1, such that $a^n - 1$ is divisible by n. Prove that the greatest common divisor (or highest common factor) of $a - 1$ and n is greater than 1.

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Solutions

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Problem 196. (*Due to John PANAGEAS, High School "Kaisari", Athens, Greece*) Let $x_1, x_2, ..., x_n$ be positive real numbers with sum equal to 1. Prove that for every positive integer m,

$$n \le n^m (x_1^m + x_2^m + ... + x_n^m).$$

Solution. **CHENG Tsz Chung** (La Salle College, Form 5), **Johann Peter Gustav Lejeune DIRICHLET** (Universidade de Sao Paulo – Campus Sao Carlos), **KWOK Tik Chun** (STFA Leung Kau Kui College, Form 6), **POON Ming Fung** (STFA Leung Kau Kui College, Form 6), **Achilleas P. PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece), **SIU Ho Chung** (Queen's College, Form 5) and **YU Hok Kan** (STFA Leung Kau Kui College, Form 6).

Applying Jensen's inequality to $f(x) = x^m$ on [0, 1] or the power mean inequality, we have

$$\left(\frac{x_1 + \cdots + x_n}{n}\right)^m \le \frac{x_1^m + \cdots + x_n^m}{n}.$$

Using $x_1 + \cdots + x_n = 1$ and multiplying both sides by n^{m+1}, we get the desired inequality.

Other commended solvers: **TONG Yiu Wai** (Queen Elizabeth School, Form 6), **YEUNG Wai Kit** (STFA Leung Kau Kui College, Form 3) and **YEUNG Yuen Chuen** (La Salle College, Form 4).

Problem 197. In a rectangular box, the lengths of the three edges starting at the same vertex are prime numbers. It is also given that the surface area of the box is a power of a prime. Prove that exactly one of the edge lengths is a prime number of the form $2^k - 1$. (*Source: KöMaL Gy.3281*)

Solution. **CHAN Ka Lok** (STFA Leung Kau Kui College, Form 4), **KWOK Tik Chun** (STFA Leung Kau Kui College, Form 6), **John PANAGEAS** (Kaisari High School, Athens, Greece), **POON Ming Fung** (STFA Leung Kau Kui College, Form 6), **Achilleas P. PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece), **SIU Ho Chung** (Queen's College, Form 5), **TO Ping Leung** (St. Peter's Secondary School), **YEUNG Wai Kit** (STFA Leung Kau Kui College, Form 3), **YEUNG Yuen Chuen** (La Salle College, Form 4) and **YU Hok Kan** (STFA Leung Kau Kui College, Form 6).

Let the prime numbers x, y, z be the lengths of the three edges starting at the same vertex. Then $2(xy + yz + zx) = p^n$ for some prime p and positive integer n. Since the left side is even, we get $p = 2$. So $xy + yz + zx = 2^{n-1}$. Since x, y, z are at least 2, the left side is at least 12, so n is at least 5. If none or exactly one of x, y, z is even, then $xy + yz + zx$ would be odd, a contradiction. So at least two of x, y, z are even and prime, say $x = y = 2$. Then $z = 2^{n-3} - 1$. The result follows.

Other commended solvers: **NGOO Hung Wing** (Valtorta College).

Problem 198. In a triangle ABC, $AC = BC$. Given is a point P on side AB such that $\angle ACP = 30°$. In addition, point Q outside the triangle satisfies $\angle CPQ = \angle CPA + \angle APQ = 78°$. Given that all angles of triangles ABC and QPB, measured in degrees, are integers, determine the angles of these two triangles. (*Source: KöMaL C. 524*)

Solution. **CHAN On Ting Ellen** (True Light Girls' College, Form 4), **CHENG Tsz Chung** (La Salle College, Form 5), **POON Ming Fung** (STFA Leung Kau Kui College, Form 6), **TONG Yiu Wai** (Queen Elizabeth School, Form 6), **YEUNG Yuen Chuen** (La Salle College, Form 4) and **YU Hok Kan** (STFA Leung Kau Kui College, Form 6).

As $\angle ACB > \angle ACP = 30°$, we get

$$\angle CAB = \angle CBA < (180° - 30°)/2 = 75°.$$

Hence $\angle CAB \le 74°$. Then

$$\angle CPB = \angle CAB + \angle ACP \le 74° + 30° = 104°.$$

Now

$$\angle QPB = 360° - \angle QPC - \angle CPB \ge 360° - 78° - 104° = 178°.$$

Since the angles of triangle QPB are positive integers, we must have

$$\angle QPB = 178°,\ \angle PBQ = 1° = \angle PQB$$

and all less-than-or-equal signs must be equalities so that

$$\angle CAB = \angle CBA = 74° \text{ and } \angle ACB = 32°.$$

Other commended solvers: **CHAN Ka Lok** (STFA Leung Kau Kui College, Form 4), **KWOK Tik Chun** (STFA Leung Kau Kui College, Form 6), **Achilleas P. PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece), **SIU Ho Chung** (Queen's College, Form 5), **YEUNG Wai Kit** (STFA Leung Kau Kui College, Form 3), **Richard YEUNG Wing Fung** (STFA Leung Kau Kui College, Form 6) and **YIP Kai Shing** (STFA Leung Kau Kui College, Form 4).

Problem 199. Let R^+ denote the positive real numbers. Suppose $f : R^+ \to R^+$ is a strictly decreasing function such that for all $x, y \in R^+$,

$$f(x+y) + f(f(x) + f(y)) = f(f(x + f(y)) + f(y + f(x))).$$

Prove that $f(f(x)) = x$ for every $x > 0$. (*Source: 1997 Iranian Math Olympiad*)

Solution. **Johann Peter Gustav Lejeune DIRICHLET** (Universidade de Sao Paulo – Campus Sao Carlos) and **Achilleas P. PORFYRIADIS** (American College of

Thessaloniki "Anatolia", Thessaloniki, Greece).

Setting $y = x$ gives

$$f(2x) + f(2f(x)) = f(2f(x + f(x))).$$

Setting both x and y to $f(x)$ in the given equation gives

$$f(2f(x)) + f(2f(f(x))) = f(2f(f(x) + f(f(x)))).$$

Subtracting this equation from the one above gives

$$f(2f(f(x))) - f(2x) = f(2f(f(x) + f(f(x)))) - f(2f(x + f(x))).$$

Assume $f(f(x)) > x$. Then $2f(f(x)) > 2x$. Since f is strictly decreasing , we have $f(2f(f(x))) < f(2x)$. This implies the left side of the last displayed equation is negative. Hence,

$$f(2f(f(x) + f(f(x)))) < f(2f(x + f(x))).$$

Again using f strictly decreasing, this inequality implies

$$2f(f(x) + f(f(x))) > 2f(x + f(x)),$$

which further implies

$$f(x) + f(f(x)) < x + f(x).$$

Canceling $f(x)$ from both sides leads to the contradiction that $f(f(x)) < x$.

Similarly, $f(f(x)) < x$ would also lead to a contradiction as can be seen by reversing all inequality signs above. Therefore, we must have $f(f(x)) = x$.

Problem 200. Aladdin walked all over the equator in such a way that each moment he either was moving to the west or was moving to the east or applied some magic trick to get to the opposite point of the Earth. We know that he travelled a total distance less than half of the length of the equator altogether during his westward moves. Prove that there was a moment when the difference between the distances he had covered moving to the east and moving to the west was at least half of the length of the equator. (*Source: KöMaL F. 3214*)

Solution.

Let us abbreviate Aladdin by A. At every moment let us consider a twin, say $\forall$, of A located at the opposite point of the position of A. *Now draw the equator circle.* Observe that at every moment either both are moving east or both are moving west. Combining the movement swept out by A and $\forall$, we get two continuous paths on the equator. At the same moment, each point in one path will have its opposite point in the other path.

Let N be the initial point of A in his travel and let $P(N)$ denote the path beginning with N. Let W be the westernmost point on $P(N)$. Let N' and W' be the opposite points of N and W respectively. By the westward travel condition on A, W cannot be as far as N'.

Assume the conclusion of the problem is false. Then the easternmost point reached by $P(N)$ cannot be as far as N'. So $P(N)$ will not cover the inside of minor arc WN' and the other path will not cover the inside of minor arc $W'N$. Since A have walked over all points of the equator (and hence A and $\forall$ together walked every point at least twice), $P(N)$ must have covered every point of the minor arc $W'N$ at least twice. Since $P(N)$ cannot cover the entire equator, every point of minor arc $W'N$ must be traveled westward at least once by A or $\forall$. Then A travelled westward at least a distance equal to the sum of lengths of minor arcs $W'N$ and NW, i.e. half of the equator. We got a contradiction.

Other commended solvers: **POON Ming Fung** (STFA Leung Kau Kui College, Form 6).

Olympiad Corner

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of lines in $\mathcal{L}$ which separate p from q is odd if and only if p and q have the same color.

Note: A line ℓ separates two points p and q if p and q lie on opposite sides of ℓ with neither point on ℓ.

Problem 4. For a real number x, let $\lfloor x \rfloor$ stand for the largest integer that is less than or equal to x. Prove that

$$\left\lfloor \frac{(n-1)!}{n(n+1)} \right\rfloor$$

is even for every positive integer n.

Problem 5. Prove that

$$(a^2+2)(b^2+2)(c^2+2) \ge 9(ab+bc+ca)$$

for all real numbers $a, b, c > 0$.

Inversion

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Example 5. (*1995 Russian Math Olympiad*) Given a semicircle with diameter AB and center O and a line, which intersects the semicircle at C and D and line AB at M ($MB < MA$, $MD < MC$). Let K be the second point of intersection of the circumcircles of triangles AOC and DOB. Prove that $\angle MKO = 90^\circ$.

Solution. Consider the inversion with center O and radius $r = OA$. By fact (2), A, B, C, D are sent to themselves. By fact (4), the circle through A, O, C is sent to line AC and the circle through D, O, B is sent to line DB. Hence, the point K is sent to the intersection K' of lines AC with DB and the point M is sent to the intersection M' of line AB with the circumcircle of ΔOCD. Then the line MK is sent to the circumcircle of $OM'K'$.

To solve the problem, note by fact (7), $\angle MKO = 90^\circ$ if and only if $\angle K'M'O = 90^\circ$.

Since $BC \perp AK'$, $AD \perp BK'$ and O is the midpoint of AB, so the circumcircle of ΔOCD is the nine-point circle of $\Delta ABK'$, which intersects side AB again at the foot of perpendicular from K' to AB. This point is M'. So $\angle K'M'O = 90^\circ$ and we are done.

Example 6. (*1995 Iranian Math Olympiad*) Let M, N and P be points of intersection of the incircle of triangle ABC with sides AB, BC and CA respectively. Prove that the orthocenter of ΔMNP, the incenter of ΔABC and the circumcenter of ΔABC are collinear.

Solution. Note the incircle of ΔABC is the circumcircle of ΔMNP. So the first two points are on the Euler line of ΔMNP.

Consider inversion with respect to the incircle of ΔABC with center I. By fact (2), A, B, C are sent to the midpoints A', B', C' of PM, MN, NP, respectively. The circumcenter of $\Delta A'B'C'$ is the center of the nine point circle of ΔMNP, which is on the Euler line of ΔMNP. By fact (3), the circumcircle of ΔABC is also on the Euler line of ΔMNP.