Mathematical Excalibur

Volume 9, Number 4 — *October 2004 – December 2004*

Olympiad Corner

The Czech-Slovak-Polish Match this year took place in Bilovec on June 21-22, 2004.Here are the problems.

Problem 1. Show that real numbers p, q, r satisfy the condition

$$p^4(q-r)^2+2p^2(q+r)+1=p^4$$

if and only if the quadratic equations

$$x^2+px+q=0 \text{ and } y^2-py+r=0$$

have real roots (not necessarily distinct) which can be labeled by x_1, x_2 and y_1, y_2, respectively, in such way that the equality $x_1y_1-x_2y_2=1$ holds.

Problem 2. Show that for each natural number k there exist at most finitely many triples of mutually distinct primes p, q, r for which the number $qr-k$ is a multiple of p, the number $pr-k$ is a multiple of q, and the number $pq-k$ is a multiple of r.

Problem 3. In the interior of a cyclic quadrilateral $ABCD$, a point P is given such that $|\angle BPC|=|\angle BAP|+|\angle PDC|$. Denote by E, F and G the feet of the perpendiculars from the point P to the lines AB, AD and DC, respectively. Show that the triangles FEG and PBC are similar.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
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The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***January 20, 2005***.

For individual subscription for the next five issues for the 03-04 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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Homothety

Kin Y. Li

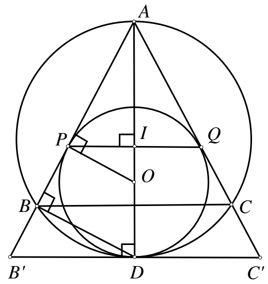
A <u>*geometric transformation*</u> of the plane is a function that sends every point on the plane to a point in the same plane. Here we will like to discuss one type of geometric transformations, called *homothety*, which can be used to solve quite a few geometry problems in some international math competitions.

A <u>*homothety with center O and ratio k*</u> is a function that sends every point X on the plane to the point X' such that

$$\overrightarrow{OX'}=k\overrightarrow{OX}.$$

So if $|k|>1$, then the homothety is a magnification with center O. If $|k|<1$, it is a reduction with center O. *A homothety sends a figure to a similar figure*. For instance, let D, E, F be the midpoints of sides BC, CA, AB respectively of ΔABC. The homothety with center A and ratio 2 sends ΔAFE to ΔABC. The homothety with center at the centroid G and ratio $-1/2$ sends ΔABC to ΔDEF.

<u>***Example 1***</u>. (*1978 IMO*) In ΔABC, $AB = AC$. A circle is tangent internally to the circumcircle of ABC and also to the sides AB, AC at P, Q, respectively. Prove that the midpoint of segment PQ is the center of the incircle of ΔABC.

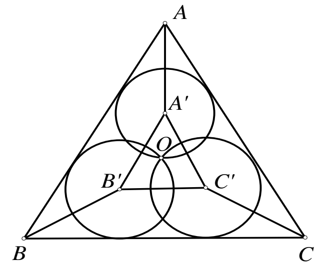


Solution. Let O be the center of the circle. Let the circle be tangent to the circumcircle of ΔABC at D. Let I be the midpoint of PQ. Then A, I, O, D are collinear by symmetry. Consider the homothety with center A that sends ΔABC to $\Delta AB'C'$ such that D is on $B'C'$. Thus, $k=AB'/AB$. As right triangles AIP, ADB', ABD, APO are similar, we have

$$AI/AO=(AI/AP)(AP/AO)$$
$$=(AD/AB')(AB/AD)=AB/AB'=1/k.$$

Hence the homothety sends I to O. Then O being the incenter of $\Delta AB'C'$ implies I is the incenter of ΔABC.

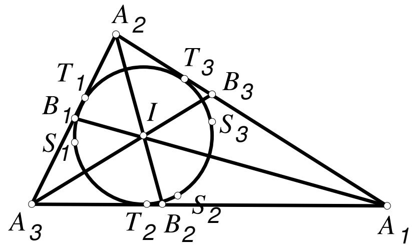
<u>***Example 2***</u>. (*1981 IMO*) Three congruent circles have a common point O and lie inside a given triangle. Each circle touches a pair of sides of the triangle. Prove that the incenter and the circumcenter of the triangle and the point O are collinear.



Solution. Consider the figure shown. Let A', B', C' be the centers of the circles. Since the radii are the same, so $A'B'$ is parallel to AB, $B'C'$ is parallel to BC, $C'A'$ is parallel to CA. Since AA', BB' CC' bisect $\angle A$, $\angle B$, $\angle C$ respectively, they concur at the incenter I of ΔABC. Note O is the circumcenter of $\Delta A'B'C'$ as it is equidistant from A', B', C'. Then the homothety with center I sending $\Delta A'B'C'$ to ΔABC will send O to the circumcenter P of ΔABC. Therefore, I, O, P are collinear.

<u>***Example 3***</u>. (*1982 IMO*) A non-isosceles triangle $A_1A_2A_3$ is given with sides a_1, a_2, a_3 (a_i is the side opposite A_i). For all i=1, 2, 3, M_i is the midpoint of side a_i, and T_i is the point where the incircle touchs side a_i. Denote by S_i the reflection of T_i in the interior bisector of angle A_i.

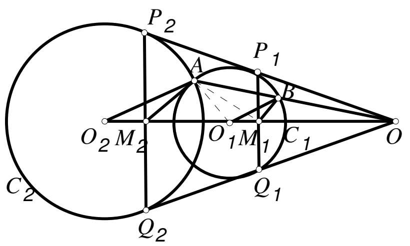
Prove that the lines M_1S_1, M_2S_2 and M_3S_3 are concurrent.



Solution. Let I be the incenter of $\Delta A_1A_2A_3$. Let B_1, B_2, B_3 be the points where the internal angle bisectors of $\angle A_1$, $\angle A_2$, $\angle A_3$ meet a_1, a_2, a_3 respectively. We will show S_iS_j is parallel to M_iM_j. With respect to A_1B_1, the reflection of T_1 is S_1 and the reflection of T_2 is T_3. So $\angle T_3IS_1 = \angle T_2IT_1$. With respect to A_2B_2, the reflection of T_2 is S_2 and the reflection of T_1 is S_3. So $\angle T_3IS_2 = \angle T_1IT_2$. Then $\angle T_3IS_1 = \angle T_3IS_2$. Since IT_3 is perpendicular to A_1A_2, we get S_2S_1 is parallel to A_1A_2. Since A_1A_2 is parallel to M_2M_1, we get S_2S_1 is parallel to M_2M_1. Similarly, S_3S_2 is parallel to M_3M_2 and S_1S_3 is parallel to M_1M_3.

Now the circumcircle of $\Delta S_1S_2S_3$ is the incircle of $\Delta A_1A_2A_3$ and the circumcircle of $\Delta M_1M_2M_3$ is the nine point circle of $\Delta A_1A_2A_3$. Since $\Delta A_1A_2A_3$ is not equilateral, these circles have different radii. Hence $\Delta S_1S_2S_3$ is not congruent to $\Delta M_1M_2M_3$ and there is a homothety sending $\Delta S_1S_2S_3$ to $\Delta M_1M_2M_3$. Then M_1S_1, M_2S_2 and M_3S_3 concur at the center of the homothety.

Example 4. (*1983 IMO*) Let A be one of the two distinct points of intersection of two unequal coplanar circles C_1 and C_2 with centers O_1 and O_2 respectively. One of the common tangents to the circles touches C_1 at P_1 and C_2 at P_2, while the other touches C_1 at Q_1 and C_2 at Q_2. Let M_1 be the midpoint of P_1Q_1 and M_2 be the midpoint of P_2Q_2. Prove that $\angle O_1AO_2 = \angle M_1AM_2$.



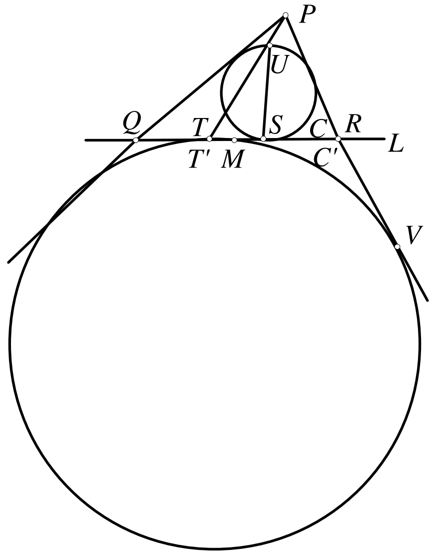
Solution. By symmetry, lines O_2O_1, P_2P_1, Q_2Q_1 concur at a point O. Consider the homothety with center O which sends C_1 to C_2. Let OA meet C_1 at B, then A is the image of B under the homothety. Since ΔBM_1O_1 is sent to ΔAM_2O_2, so $\angle M_1BO_1 = \angle M_2AO_2$.

Now ΔOP_1O_1 similar to ΔOM_1P_1 implies $OO_1/OP_1 = OP_1/OM_1$. Then

$$OO_1 \cdot OM_1 = OP_1^2 = OA \cdot OB,$$

which implies points A, B, M_1, O_1 are concyclic. Then $\angle M_1BO_1 = \angle M_1AO_1$. Hence $\angle M_1AO_1 = \angle M_2AO_2$. Adding $\angle O_1AM_2$ to both sides, we have $\angle O_1AO_2 = \angle M_1AM_2$.

Example 5. (*1992 IMO*) In the plane let C be a circle, L a line tangent to the circle C, and M a point on L. Find the locus of all points P with the following property: there exist two points Q, R on L such that M is the midpoint of QR and C is the inscribed circle of ΔPQR.



Solution. Let L be the tangent to C at S. Let T be the reflection of S with respect to M. Let U be the point on C diametrically opposite S. Take a point P on the locus. The homothety with center P that sends C to the excircle C' will send U to T', the point where QR touches C'. Let line PR touch C' at V. Let s be the semiperimeter of ΔPQR, then

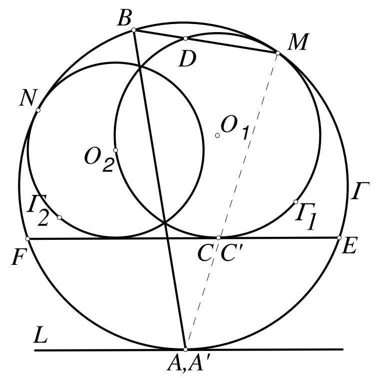
$$TR = QS = s - PR = PV - PR = VR = T'R$$

so that P, U, T are collinear. Then the locus is on the part of line UT, opposite the ray $\overrightarrow{UT}$.

Conversely, for any point P on the part of line UT, opposite the ray $\overrightarrow{UT}$, the homothety sends U to T and T', so $T = T'$. Then $QS = s - PR = PV - PR = VR = T'R = TR$ and $QM = QS - MS = TR - MT = RM$. Therefore, P is on the locus.

For the next example, the solution involves the concepts of power of a point with respect to a circle and the radical axis. We will refer the reader to the article "*Power of Points Respect to Circles*," in *Math Excalibur*, *vol.* 4, *no.* 3, *pp.* 2, 4.

Example 6. (*1999 IMO*) Two circles Γ_1 and Γ_2 are inside the circle Γ, and are tangent to Γ at the distinct points M and N, respectively. Γ_1 passes through the center of Γ_2. The line passing through the two points of intersection of Γ_1 and Γ_2 meets Γ at A and B. The lines MA and MB meet Γ_1 at C and D, respectively. Prove that CD is tangent to Γ_2.



Solution. (*Official Solution*) Let EF be the chord of Γ which is the common tangent to Γ_1 and Γ_2 on the same side of line O_1O_2 as A. Let EF touch Γ_1 at C'. The homothety with center M that sends Γ_1 to Γ will send C' to some point A' and line EF to the tangent line L of Γ at A'. Since lines EF and L are parallel, A' must be the midpoint of arc $FA'E$. Then $\angle A'EC' = \angle A'FC' = \angle A'ME$. So $\Delta A'EC$ is similar to $\Delta A'ME$. Then the power of A' with respect to Γ_1 is $A'C' \cdot A'M = A'E^2$. Similar, the power of A' with respect to Γ_2 is $A'F^2$. Since $A'E = A'F$, A' has the same power with respect to Γ_1 and Γ_2. So A' is on the radical axis AB. Hence, $A' = A$. Then $C' = C$ and C is on EF.

Similarly, the other common tangent to Γ_1 and Γ_2 passes through D. Let O_i be the center of Γ_i. By symmetry with respect to O_1O_2, we see that O_2 is the midpoint of arc CO_2D. Then

$$\angle DCO_2 = \angle CDO_2 = \angle FCO_2.$$

This implies O_2 is on the angle bisector of $\angle FCD$. Since CF is tangent to Γ_2, therefore CD is tangent to Γ_2.

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Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong*. The deadline for submitting solutions is ***January 20, 2005.***

Problem 211. For every a, b, c, d in [1,2], prove that

$$\frac{a+b}{b+c}+\frac{c+d}{d+a}\le 4\frac{a+c}{b+d}.$$

(*Source: 32nd Ukranian Math Olympiad*)

Problem 212. Find the largest positive integer N such that if S is any set of 21 points on a circle C, then there exist N arcs of C whose endpoints lie in S and each of the arcs has measure not exceeding 120°.

Problem 213. Prove that the set of all positive integers can be partitioned into 100 nonempty subsets such that if three positive integers a, b, c satisfy $a + 99b = c$, then at least two of them belong to the same subset.

Problem 214. Let the inscribed circle of triangle ABC be tangent to sides AB, BC at E and F respectively. Let the angle bisector of $\angle CAB$ intersect segment EF at K. Prove that $\angle CKA$ is a right angle.

Problem 215. Given a 8×8 board. Determine all squares such that if each one is removed, then the remaining 63 squares can be covered by 21 3×1 rectangles.

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Solutions

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Problem 206. (*Due to Zdravko F. Starc, Vršac, Serbia and Montenegro*) Prove that if a, b are the legs and c is the hypotenuse of a right triangle, then

$$(a+b)\sqrt{a}+(a-b)\sqrt{b}<\sqrt{2\sqrt{2}c}\sqrt{c}.$$

Solution. **Cheng HAO** (The Second High School Attached to Beijing Normal University), **HUI Jack** (Queen's College, Form 5), **D. Kipp JOHNSON** (Valley Catholic School, Teacher, Beaverton, Oregon, USA), **POON Ming Fung**(STFA Leung Kau Kui College, Form 7), **Achilleas P. PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece), **Problem Group Discussion Euler-Teorema**(Fortaleza, Brazil), **Anna Ying PUN** (STFA Leung Kau Kui College, Form 6), **TO Ping Leung** (St. Peter's Secondary School) and **YIM Wing Yin** (South Tuen Mun Government Secondary School, Form 4).

By Pythagoras' theorem,

$$a+b\le\sqrt{(a+b)^2+(a-b)^2}=\sqrt{2}c.$$

Equality if and only if $a = b$. By the Cauchy-Schwarz inequality,

$$\begin{aligned}&(a+b)\sqrt{a}+(a-b)\sqrt{b}\\ \le\ &\sqrt{(a+b)^2+(a-b)^2}\sqrt{a+b}\\ \le\ &\sqrt{2}c\sqrt{\sqrt{2}c}.\end{aligned}$$

For equality to hold throughout, we need $a+b:a-b=\sqrt{a}:\sqrt{b}=1:1$, which is not possible for legs of a triangle. So we must have strict inequality.

Other commended solvers: **HUDREA Mihail** (High School "Tiberiu Popoviciu" Cluj-Napoca Romania) and **TONG Yiu Wai** (Queen Elizabeth School, Form 7).

Problem 207. Let $A = \{0, 1, 2, ..., 9\}$ and $B_1, B_2, ..., B_k$ be nonempty subsets of A such that B_i and B_j have at most 2 common elements whenever $i \ne j$. Find the maximum possible value of k.

Solution. **Cheng HAO** (The Second High School Attached to Beijing Normal University), **HUI Jack** (Queen's College, Form 5), **POON Ming Fung**(STFA Leung Kau Kui College, Form 7) and **Achilleas P. PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece).

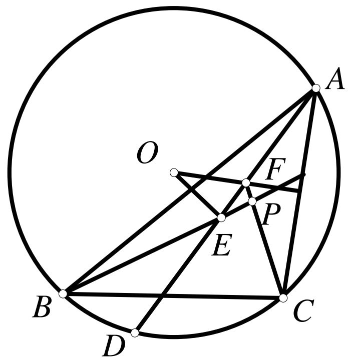
If we take all subsets of A with 1, 2 or 3 elements, then these 10 + 45 + 120 = 175 subsets satisfy the condition. So $k \ge 175$.

Let $B_1, B_2, ..., B_k$ satisfying the condition with k maximum. If there exists a B_i with at least 4 elements, then every 3 element subset of B_i cannot be one of the B_j, $j \ne i$, since B_i and B_j can have at most 2 common elements. So adding these 3 element subsets to $B_1, B_2, ..., B_k$ will still satisfy the conditions. Since B_i has at least four 3 element subsets, this will increase k, which contradicts maximality of k. Then every B_i has at most 3 elements. Hence, $k \le 175$. Therefore, the maximum k is 175.

Other commended solvers: **CHAN Wai Hung** (Carmel Divine Grace Foundation Secondary School, Form 6), **LI Sai Ki** (Carmel Divine Grace Foundation Secondary School, Form 6), **LING Shu Dung**, **Anna Ying PUN** (STFA Leung Kau Kui College, Form 6) and **YIM Wing Yin** (South Tuen Mun Government Secondary School, Form 4).

Problem 208. In ΔABC, $AB > AC > BC$. Let D be a point on the minor arc BC of the circumcircle of ΔABC. Let O be the circumcenter of ΔABC. Let E, F be the intersection points of line AD with the perpendiculars from O to AB, AC, respectively. Let P be the intersection of lines BE and CF. If $PB = PC + PO$, then find $\angle BAC$ with proof.

Solution. **Achilleas P. PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece), **Problem Group Discussion Euler - Teorema** (Fortaleza, Brazil) and **Anna Ying PUN** (STFA Leung Kau Kui College, Form 6).



Since E is on the perpendicular bisector of chord AB and F is on the perpendicular bisector of chord AC, $AE = BE$ and $AF = CF$. Applying exterior angle theorem,

$$\begin{aligned}\angle BPC &= \angle AEP + \angle CFD\\ &= 2(\angle BAD + \angle CAD)\\ &= 2\angle BAC = \angle BOC.\end{aligned}$$

Hence, B, C, P, O are concyclic. By Ptolemy's theorem,

$$PB\cdot OC = PC\cdot OB + PO\cdot BC.$$

Then $(PB - PC)\cdot OC = PO\cdot BC$. Since $PB - PC = PO$, we get $OC = BC$ and so ΔOBC is equilateral. Then

$$\angle BAC=\tfrac{1}{2}\angle BOC = 30°$$

Other commended solvers: **Cheng HAO** (The Second High School Attached to Beijing Normal University), **HUI Jack** (Queen's College, Form 5), **POON Ming Fung**(STFA Leung Kau Kui College, Form 7), **TONG Yiu Wai**

(Queen Elizabeth School, Form 7) and **YIM Wing Yin** (South Tuen Mun Government Secondary School, Form 4).

Problem 209. Prove that there are infinitely many positive integers n such that $2^n + 2$ is divisible by n and $2^n + 1$ is divisible by $n - 1$.

Solution. **D. Kipp JOHNSON** (Valley Catholic School, Teacher, Beaverton, Oregon, USA), **POON Ming Fung**(STFA Leung Kau Kui College, Form 7) and **Problem Group Discussion Euler-Teorema**(Fortaleza, Brazil).

As $2^2 + 2 = 6$ is divisible by 2 and $2^2 + 1 = 5$ is divisible by 1, $n = 2$ is one such number.

Next, suppose $2^n + 2$ is divisible by n and $2^n + 1$ is divisible by $n - 1$. We will prove $N = 2^n + 2$ is another such number. Since $N - 1 = 2^n + 1 = (n - 1)k$ is odd, so k is odd and n is even. Since $N = 2^n + 2 = 2(2^{n-1} + 1) = nm$ and n is even, so m must be odd. Recall the factorization

$$x^i + 1 = (x + 1)(x^{i-1} - x^{i-3} + \cdots + 1)$$

for odd positive integer i. Since k is odd, $2^N + 2 = 2(2^{N-1} + 1) = 2(2^{(n-1)k} + 1)$ is divisible by $2(2^{n-1} + 1) = 2^n + 2 = N$ using the factorization above. Since m is odd, $2^N + 1 = 2^{nm} + 1$ is divisible by $2^n + 1 = N - 1$. Hence, N is also such a number. As $N > n$, there will be infinitely many such numbers.

Problem 210. Let $a_1 = 1$ and

$$a_{n+1} = \frac{a_n}{2} + \frac{1}{a_n}$$

for $n = 1, 2, 3, \ldots$. Prove that for every integer $n > 1$,

$$\frac{2}{\sqrt{a_n^2 - 2}}$$

is an integer.

Solution. **G.R.A. 20 Problem Group** (Roma, Italy), **HUDREA Mihail** (High School "Tiberiu Popoviciu" Cluj-Napoca Romania), **Problem Group Discussion Euler – Teorema** (Fortaleza, Brazil), **TO Ping Leung** (St. Peter's Secondary School) and **YIM Wing Yin** (South Tuen Mun Government Secondary School, Form 4).

Note $a_n = p_n / q_n$, where $p_1 = q_1 = 1$, $p_{n+1} = p_n^2 + 2q_n^2$, $q_{n+1} = 2p_nq_n$ for $n = 1, 2, 3, \ldots$. Then

$$\frac{2}{\sqrt{a_n^2 - 2}} = \frac{2q_n}{\sqrt{p_n^2 - 2q_n^2}}.$$

It suffices to show by mathematical induction that $p_n^2 - 2q_n^2 = 1$ for $n > 1$. We have $p_2^2 - 2q_2^2 = 3^2 - 2\cdot2^2 = 1$. Assuming case n is true, we get

$$p_{n+1}^2 - 2q_{n+1}^2 = (p_n^2 + 2q_n^2)^2 - 2(2p_nq_n)$$
$$= (p_n^2 - 2q_n^2)^2 = 1.$$

Other commended solvers: **Ellen CHAN On Ting** (True Light Girls' College, Form 5), **Cheng HAO** (The Second High School Attached to Beijing Normal University), **HUI Jack** (Queen's College, Form 5), **D. Kipp JOHNSON** (Valley Catholic School, Teacher, Beaverton, Oregon, USA), **LAW Yau Pui** (Carmel Divine Grace Foundation Secondary School, Form 6), **Asger OLESEN** (Toender Gymnasium (grammar school), Denmark), **POON Ming Fung**(STFA Leung Kau Kui College, Form 7), **Achilleas P. PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece), **Anna Ying PUN** (STFA Leung Kau Kui College, Form 6), **Steve ROFFE**, **TONG Yiu Wai** (Queen Elizabeth School, Form 7) and **YEUNG Wai Kit** (STFA Leung Kau Kui College, Form 4).

Olympiad Corner

(continued from page 1)

Problem 4. Solve the system of equations

$$\frac{1}{xy} = \frac{x}{z} + 1,\ \frac{1}{yz} = \frac{y}{x} + 1,\ \frac{1}{zx} = \frac{z}{y} + 1$$

in the domain of real numbers.

Problem 5. In the interiors of the sides AB, BC and CA of a given triangle ABC, points K, L and M, respectively, are given such that

$$\frac{|AK|}{|KB|} = \frac{|BL|}{|LC|} = \frac{|CM|}{|MA|}.$$

Show that the triangles ABC and KLM have a common orthocenter if and only if the triangle ABC is equilateral.

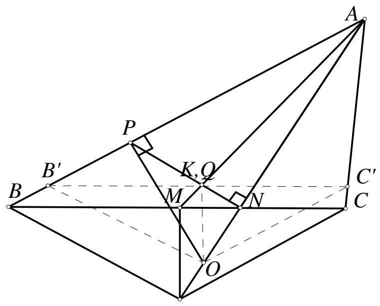
Problem 6. On the table there are k heaps of 1, 2, …, k stones, where $k \geq 3$. In the first step, we choose any three of the heaps on the table, merge them into a single new heap, and remove 1 stone (throw it away from the table) from this new heap. In the second step, we again merge some three of the heaps together into a single new heap, and then remove 2 stones from this new heap. In general, in the i-th step we choose any three of the heaps, which contain more than i stones when combined, we merge them into a single new heap, then remove i stones from this new heap. Assume that after a number of steps, there is a single heap left on the table, containing p stones. Show that the number p is a perfect square if and only if the numbers $2k+2$ and $3k+1$ are perfect squares. Further, find the least number k for which p is a perfect square.

Homothety

(continued from page 2)

Example 7. (*2000APMO*) Let ABC be a triangle. Let M and N be the points in which the median and the angle bisector, respectively at A meet the side BC. Let Q and P be the points in which the perpendicular at N to NA meets MA and BA respectively and O the point in which the perpendicular at P to BA meets AN produced.

Prove that QO is perpendicular to BC.



Solution (*due to Bobby Poon*). The case $AB = AC$ is clear.

Without loss of generality, we may assume $AB > AC$. Let AN intersect the circumcircle of ΔABC at D. Then

$$\angle DBC = \angle DAC = \tfrac{1}{2}\angle BAC$$
$$= \angle DAB = \angle DCB.$$

So $DB = DC$ and MD is perpendicular to BC.

Consider the homothety with center A that sends ΔDBC to $\Delta OB'C'$. Then $OB' = OC'$ and BC is parallel to $B'C'$. Let $B'C'$ intersect PN at K. Then

$$\angle OB'K = \angle DBC = \angle DAB$$
$$= 90° - \angle AOP = \angle OPK.$$

So points P, B', O, K are concyclic. Hence $\angle B'KO = \angle B'PO = 90°$ and $B'K = C'K$. Since $BC \parallel B'C'$, this implies K is on MA. Hence, $K = Q$. Now $\angle B'KO = 90°$ implies $QO = KO \perp B'C'$. Finally, $BC \parallel B'C'$ implies QO is perpendicular to BC.