Olympiad Corner

Problem 1. For $n \geq 2$, let $a_1, a_2, \ldots, a_n, a_{n+1}$ be positive and $a_2 = a_1 = a_3 = a_2 = \cdots = a_{n+1} = a_1 = 0$. Prove that
$$\frac{1}{a_1^2} + \frac{1}{a_2^2} + \cdots + \frac{1}{a_{n+1}^2} \leq \frac{n}{2} a_1 a_2 a_3 a_{n+1}.$$

Determine when equality holds.

Problem 2. In a school there are $b$ teachers and $c$ students. Suppose that (i) each teacher teaches exactly $k$ students; and (ii) for each pair of distinct students, exactly $h$ teachers teach both of them. Show that $b = \frac{c(c-1)}{k(k-1)}$.

Problem 3. On the sides $AB$ and $AC$ of triangle $ABC$, there are points $P$ and $Q$ respectively such that $\angle APC = \angle AQB = 45^\circ$. Let the perpendicular line to side $AB$ through $P$ intersect line $BQ$ at $S$. Let the perpendicular line to side $AC$ through $Q$ intersects line $CP$ at $R$. Let $D$ be on side $BC$ such that $AD \perp BC$.

(continued on page 4)
些隊員中有且僅有一人在場上，且

\[ A, A_1, A_2, A_3, \text{ 每人上場的總時間（以分}

離為單位）均被 \( 7 \) 整除，\( A, A_1, A_2, A_3 \)

\[ \text{ 每人上場的總時間（以分離為單位）均被} \]

\[ 13 \] 整除，如果每場換人次數不限，那麼按各隊員上場的總時間計

算，共有多少種不同的情況？

解答 令 \( a_i = 7k_i, a_1 = 13k_1, i, 5, 6, 7, \)

其中 \( k_i = 1, 2, \ldots, 7 \)\n
\[ \text{ 隨便選取} \]

\[ k_i \in \mathbb{Z}^+ \]

令 \( \sum_{i=1}^{4} k_i = m, \sum_{i=1}^{3} k_i = n \) 其中

\[ m \geq 4, \quad n \geq 3, \quad \text{且} \quad m,n \in \mathbb{Z}^+ \]

則 \( 7m + 13n = 270 \)，易得其一個整數解

\[ \begin{cases} m = \frac{33}{3} \times 3 & , \text{又因} (7, 13, 1) \text{，故其整數} \end{cases} \]

通解為

\[ \begin{cases} m = \frac{33 + 13t}{3} & , \text{再由} \end{cases} \]

\[ \begin{cases} 3 \leq t \leq 7 & \quad \text{得} \quad \frac{29}{13} \leq r \leq 0 \quad \text{枚整} \end{cases} \]

\[ \text{數} \leq 0, -1, -2 \]

\[ \text{從而其滿足條件的整數解} \]

\[ \begin{cases} m = 33, \quad m = 20, \quad m = 7, \quad m = 10, \quad m = 17 \end{cases} \]

\[ \text{對於} \sum_{i=1}^{4} k_i = 33 \text{的整數解，可以} \]

寫一數形計 33 個 1，在每相鄰兩個 1 間空之任選 3 個填入 "+" 號，再把 3 個 "+" 號分隔的

\[ \text{4 個部分的} \text{1 分別統計，就可得到} \]

\[ \text{其一個整數解，故} \sum_{i=1}^{4} k_i = 33 \] 有

\[ \text{C}_{12}^3 \text{個整數解} (k_1, k_2, k_3, k_4) \]；同理

\[ \sum_{i=1}^{4} k_i = 3 \] 有 \( \text{C}_{12}^3 \) 個整數解

\[ (k_1, k_2, k_3) \)；從而此時滿足條件的整數解

\[ (k_1, k_2, k_3, k_4, k_5, k_6, k_7, k_8, k_9, k_{10}) \]

\[ \text{有} \quad \text{C}_{12}^3 \]

個．

因此滿足條件的所有整數解

\[ (k_1, k_2, k_3, k_4, k_5, k_6, k_7, k_8, k_9, k_{10}) \]

\[ \text{有} \quad \text{C}_{12}^3 \]

\[ \text{個} \]

\[ \text{例 4} \text{由數字} 1, 2, 3 \text{組成} \text{n} \text{位數，且在} \]

\[ \text{這個} \text{n} \text{位數中} 1, 2, 3 \text{的每個至少出現一次，問這樣的} \text{n} \text{位數有多少個？} \]

解答 令 U 是由 1, 2, 3, 3 成為的 3 位元數

的集合，\( A_i \) 是 U 中不含數字 1 的 1 位元數

的集合，\( A_i \) 是 U 中不含數字 2 的 1 位元

數的集合，\( A_i \) 是 U 中不含數字 3 的 1 位元

數的集合，則

\[ \text{card}(A) = \text{card}(A) = \text{card}(A) = 2, \]

\[ \text{card}(A_1 \cap A_2 \cap A_3) = \text{card}(A_1 \cap A_2 \cap A_3) = 1, \]

\[ \text{card}(A_1 \cap A_2 \cap A_3) = 0. \]

因此

\[ \text{card}(U) - \text{card}(A_1 \cup A_2 \cup A_3) \]

\[ = 3 \times 2 \times 3 - 1 = 0 = 3 \times 2 \times 3 \]

[\text{即符合題意的} \text{n} \text{位數的個數為} \text{3} \times 2 \times 3 \times 3 \]

\[ \text{下面，我們再來看一個關於容斥原} \]

\[ \text{理應用的異問題：} \]

\[ \text{例 5} \text{參加大型團體表演的學生共 240} \]

\[ \text{名，他們面對教練站成一行，自左至} \]

\[ \text{右按} 1, 2, 3, 4, 5, \ldots \text{依次報數，} \]

教練要求全體學生牢記各自所報的

\[ \text{數，並做下列動作：先讓報的數是} 3 \]

\[ \text{的倍數的全體同學後轉} ; \text{接著報數} \]

\[ \text{的數是} 5 \text{的倍數的全體同學向後轉} ; \text{最後報數的數是} 7 \text{的倍數的全體} \]

\[ \text{同學向後轉} \text{。問：} \]

\[ (1) \text{此時還有多少名同學面對教} \]

\[ (2) \text{面教練的同學中，自左至右第} \]

\[ 66 \text{位同學所報的數是？} \]

解答 令 U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}, A 表示

\[ \text{由 U 中所有的倍數組成的集合：} \]

\[ \text{card}(U) = 240, \text{card}(A) = \frac{240}{3} = 80, \]

\[ \text{card}(A_1) = \frac{240}{7} = 34, \]

\[ \text{card}(A_2) = \frac{240}{5} = 48, \text{card}(A_3) = \frac{240}{9} = 34 \]

\[ \text{card}(A_1 \cap A_2 \cap A_3) = \frac{240}{15} = 16, \text{card}(A) = \frac{240}{21} = 11, \]

\[ \text{card}(A_1 \cap A_2 \cap A_3) = \frac{240}{33} = 6, \text{card}(A_9) = \frac{240}{105} = 2. \]

\[ \text{從而此時有} \]

\[ \text{card}(U) - \text{card}(A_1 \cup A_2 \cup A_3) \]

\[ + 2[\text{card}(A_1) + \text{card}(A_2) + \text{card}(A_3)] \]

\[ - 4\text{card}(A) = 136 \]

名同學面對教練．

如果我們借助威斯林圖進行分

析，利用上面所得數據分別繪入圖

1，注意按從內到外的順序填，

\[ \text{圖 1} \]

如圖 1，此時面對教練的同學一目了

然，應有 109 + 14 + 4 + 9 = 136 名，

\[ (\text{continued on page 4}) \]
Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver’s name, home (or email) address and school affiliation. Please send submissions to Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong. The deadline for submitting solutions is March 31, 2005.

Problem 216. (Due to Alfred Eckstein, Arad, Romania) Solve the equation
\[4x^5 - 6x^2 + 2\sqrt{2} = 0.\]

Problem 217. Prove that there exist infinitely many positive integers which cannot be represented in the form
\[x_1^3 + x_2^3 + x_3^2 + x_4^9 + x_5^{11},\]
where \(x_1, x_2, x_3, x_4, x_5\) are positive integers. (Source: 2002 Belarussian Mathematical Olympiad, Final Round)

Problem 218. Let \(O\) and \(P\) be distinct points on a plane. Let \(ABCD\) be a parallelogram on the same plane such that its diagonals intersect at \(O\). Suppose \(P\) is not on the reflection of line \(AB\) with respect to line \(CD\). Let \(M\) and \(N\) be the midpoints of segments \(AP\) and \(BP\) respectively. Let \(Q\) be the intersection of lines \(MC\) and \(ND\). Prove that \(P, Q, O\) are collinear and the point \(Q\) does not depend on the choice of parallelogram \(ABCD\). (Source: 2004 National Math Olympiad in Slovenia, First Round)

Problem 219. (Due to Dorin Mărghidanu, Coleg. Nat. “A.I. Cuza”, Corabia, Romania) The sequences \(a_0, a_1, a_2, \ldots\) and \(b_0, b_1, b_2, \ldots\) are defined as follows: \(a_0 = 0 > 0\) and
\[a_{n+1} = a_n + \frac{1}{2b_n}, \quad b_{n+1} = b_n + \frac{1}{2a_n}\]
for \(n = 1, 2, 3, \ldots\). Prove that
\[\max\{a_{2004}, b_{2004}\} > \sqrt{2005}.\]

Problem 220. (Due to Cheng HAO, The Second High School Attached to Beijing Normal University) For \(i = 1, 2, \ldots, n\) and \(k \geq 4\), let \(A_i = (a_1, a_2, \ldots, a_k)\) with \(a_j = 0\) or 1 and every \(A_i\) has at least 3 of the \(k\) coordinates equal 1.

Define the distance between \(A_i\) and \(A_j\) to be
\[\sum_{m=1}^{\infty} |a_{im} - a_{jm}|.\]
If the distance between any \(A_i\) and \(A_j\) \((i \neq j)\) is greater than 2, then prove that
\[n \leq 2^{3^k} - 1.\]

Solution

Problem 211. For every \(a, b, c, d\) in [1, 2], prove that
\[\frac{a + b}{b + c} + \frac{c + d}{d + a} \leq 4 \frac{a + c}{b + d}.\]
(Source: 32nd Ukrainian Math Olympiad)

Solution.

Problem 212. Find the largest positive integer \(N\) such that if \(N\) is any set of 21 points on a circle \(C\), then there exist \(N\) arcs of \(C\) whose endpoints lie in \(S\) and each of the arcs has measure not exceeding \(120^\circ\).

Solution.

We will \(N = 100\). To see that \(N \leq 100\), consider a diameter \(AB\) of \(C\). Place 11 points close to \(A\) and 10 points close to \(B\). The number of desired arcs is then
\[\binom{11}{2} + \binom{10}{2} = 100.\]
To see that \(N \geq 100\), we need to observe that for every set \(T\) of \(k = 21\) points on \(C\), there exists a point \(X\) in \(T\) such that there are at least \([k - 1/2]\) arcs \(XY\) with \(Y \neq X\) having measure not exceeding \(120^\circ\). This is because we can divide the circle \(C\) into three arcs \(C_1, C_2, C_3\) of \(120^\circ\) (only overlapping at endpoints) such that the common endpoint of \(C_1\) and \(C_2\) is a point \(X\) of \(T\). If \(X\) does not have the required property, then there are \(1 + [(k - 1/2)]\) points of \(T\) lies on \(C_1\) and any of them can serve as \(X\).

Next we remove \(X\) and apply the same argument to \(k = 20\), then remove that point, and repeat with \(k = 19, 18, \ldots, 3\). We get a total of \(10 + 9 + 8 + 8 + \ldots + 1 + 1 = 100\) arcs.

Problem 213. Prove that the set of all positive integers can be partitioned into 100 nonempty subsets such that if three positive integers \(a, b, c\) satisfy \(a + 99b = c\), then at least two of them belong to the same subset.


Summary statistics:

<table>
<thead>
<tr>
<th>Name</th>
<th>Position</th>
<th>School</th>
<th>Year</th>
</tr>
</thead>
<tbody>
<tr>
<td>CHUNG Yun Kuen</td>
<td>(HKUST, Math Major, Year 1)</td>
<td>School “Tiberiu Popoviciu” Cluj-Napoca</td>
<td>2005</td>
</tr>
<tr>
<td>Achilleas P. POFYRiadis</td>
<td>(American College of Thessaloniki “Anatolia”, Thessaloniki, Greece)</td>
<td>2005</td>
<td></td>
</tr>
<tr>
<td>Dr. Kin Y. Li</td>
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</tbody>
</table>

Solution. 1. Let \(f(n)\) be the largest nonnegative integer \(k\) for which \(n\) is divisible by \(2^k\). Then given three positive integers \(a, b, c\) satisfying \(a + 99b = c\) at least two of \(f(a), f(b), f(c)\) are equal. To prove this, if \(f(a) = f(b)\), then we are done. If \(f(a) < f(b)\), then \(f(c) > f(b)\). If \(f(a) > f(b)\), then \(f(c) = f(b)\).

Therefore, the following partition suffices:
\[S_i = \{n | f(n) \equiv i \pmod{100}\}\]
for \(1 \leq i \leq 100\).

Problem 214. Let the inscribed circle of triangle \(ABC\) be tangent to sides \(AB\), \(BC\) at \(E\) and \(F\) respectively. Let the angle bisector of \(\angle CAB\) intersect segment \(EF\) at \(K\). Prove that \(\angle KCA\) is a right angle.

Solution. 1. CHENG Hei (Tsuen Wan Government Secondary School, Form 5); HUdREA MIHALI (High School “Tiberiu Popoviciu” Cluj-Napoca Romania); Achilleas P. POFYRiadis (American College of Thessaloniki “Anatolia”, Thessaloniki, Greece); YIM Wing Yin (South Tuen Mun Government Secondary School, Form 5).
Let us number the squares of the board from 1 to 64, with 1 to 8 on the first row, 9 to 16 on the second row and so on.

Using this numbering, a $3 \times 1$ rectangle will cover three numbers with a sum divisible by 3. Since 64 = 1 (mod 3), only squares with numbers congruent to 1 (mod 3) need to be considered for our problem.

However, the only such square and its image squares having this property are the squares with numbers 19, 22, 43 and 46.

Finally square 19 has the required property (and hence also squares 22, 43, 46 by symmetry) by putting 3 squares with numbers 19, 22, 43 and 46.

The board, the images of a desired square under these symmetries are also desired. Hence they must also have numbers congruent to 1 (mod 3) in them.

Example: Let $A$, $I$, $K$ be the center of the inscribed circle. Then $A$, $I$, $K$ are collinear. Now $\angle CIK = \frac{1}{2}(\angle BAC + \angle ACB)$. Next, $BE = BF$ implies that $\angle BFE = 90^\circ - \frac{1}{2} \angle CBA = \frac{1}{2}(\angle BAC + \angle ACB) = \angle CIK$. (In the second figure, we have $\angle CKF = \angle BFE = \angle CIK$.) Hence $C, I, K, F$ are concyclic.

Therefore, $\angle CKI = \angle CFI = 90^\circ$.

Other commended solvers: CHEUNG Yun Kuen (HKUST, Math Major, Year 1).

Problem 215. Given a 8×8 board. Determine all squares such that if each one is removed, then the remaining 63 squares can be covered by 21 $3 \times 1$ rectangles.

Solution. CHEUNG Yun Kuen (HKUST, Math Major, Year 1).

Let $I$ be the center of the inscribed circle. Then $A$, $I$, $K$ are collinear. Now $\angle CIK = \frac{1}{2}(\angle BAC + \angle ACB)$. Next, $BE = BF$ implies that $\angle BFE = 90^\circ - \frac{1}{2} \angle CBA = \frac{1}{2}(\angle BAC + \angle ACB) = \angle CIK$. (In the second figure, we have $\angle CKF = \angle BFE = \angle CIK$.) Hence $C, I, K, F$ are concyclic.

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