MATH 5510 — Mathematical Models of Financial Derivatives

Topic 3 – Extended option models

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3.1 Exchange options

- An exchange option is an option that gives the holder the right but not the obligation to exchange one risky asset for another.

- Let $X_t$ and $Y_t$ be the price processes of the two risky assets.

- The terminal payoff of a European exchange option at maturity $T$ of exchanging $Y_T$ for $X_T$ is given by $\max(X_T - Y_T, 0)$.

Under the risk neutral measure $Q$, let $X_t$ and $Y_t$ be governed by

$$\frac{dX_t}{X_t} = (r - q_X) dt + \sigma_X dZ^Q_{X,t} \quad \text{and} \quad \frac{dY_t}{Y_t} = (r - q_X) dt + \sigma_Y dZ^Q_{Y,t},$$

where $r$ is the constant riskless interest rate, $\sigma_X$ and $\sigma_Y$ are the constant volatility of $X_t$ and $Y_t$, respectively, $q_X$ and $q_Y$ are the dividend yield of $X_t$ and $Y_t$, respectively. Also, the two standard Brownian motions are correlated with $dZ^Q_{X,t} dZ^Q_{Y,t} = \rho dt$, where $\rho$ is correlation coefficient.
Digression: Change of numeraire

The choice of the money market account $M(t)$ as the numeraire is not unique in order that the risk neutral valuation formula holds. Let $N(t)$ be a numeraire whereby we have the existence of an equivalent probability measure $Q_N$ such that all security prices normalized with respect to $N(t)$ are $Q_N$-martingale. In addition, if a contingent claim $Y$ is attainable under the numeraire-measure pair $(M(t), Q)$, then it is also attainable under an alternative pair $(N(t), Q_N)$.

The arbitrage price $V(t)$ of a contingent claim $Y$ as given by the risk neutral valuation approach under both measures should agree. Since $\frac{V(t)}{M(t)}$ and $\frac{V(t)}{N(t)}$ are $Q$-martingale and $Q_N$-martingale, respectively, we should expect

$$V(t) = M(t)E_Q \left[ \frac{Y}{M(T)} \mid \mathcal{F}_t \right] = N(t)E_{Q_N} \left[ \frac{Y}{N(T)} \mid \mathcal{F}_t \right].$$

**Question:** How to find the appropriate $\frac{dQ_N}{dQ}\mid \mathcal{F}_t$ such that the above formula holds?
Determination of the Radon-Nikodym derivative

Suppose we adopt the change of measure from $Q_N$ to $Q$ by the choice of the following Radon-Nikodym derivative, where

$$\frac{dQ_N}{dQ}|_{\mathcal{F}_t} = \frac{N(T)}{N(t)} / \frac{M(T)}{M(t)}.$$ 

Observing that both $N(t)$ and $M(t)$ are measurable with respect to $\mathcal{F}_t$, we then have

$$N(t)E_{Q_N}\left[ \frac{Y}{N(T)} \mid \mathcal{F}_t \right] = N(t)E_Q \left[ \frac{Y}{N(T)} \frac{N(T)}{N(t)} / \frac{M(T)}{M(t)} \mid \mathcal{F}_t \right]$$

$$= M(t)E_Q \left[ \frac{Y}{M(T)} \mid \mathcal{F}_t \right],$$

which verifies with the earlier risk neutral valuation formula under the change of numeraire.
Use of the underlying asset as the numeraire (share measure) and the associated change of measure

Recall that the risk neutral measure uses the money market account as the numeraire. Let the starting time be time zero for notational convenience. Consider the Radon-Nikodym derivative that is defined by taking the ratio of the asset numeraire and the money market account

$$L_t = \left. \frac{dQ^S}{dQ} \right|_{\mathcal{F}_0} = e^{qt} \frac{S_t}{S_0} \bigg/ \frac{M_t}{M_0}, \quad t \in (0, T],$$

where $M_t = e^{rt}$ is the money market account and $q$ is the dividend yield of the underlying asset. The inclusion of the factor $e^{qt}$ means one unit of asset initially grows to $e^{qt}$ units after time $t$ if all dividends are invested into the purchase of new units of the risky asset.
We examine the change of measure from $Q$ to $Q^S$ as effected by $L_t$. Symbolically, we write

$$L_t = \left. \frac{dQ^S}{dQ} \right|_{\mathcal{F}_0} , \ t \in (0,T].$$

Under the risk neutral measure $Q$, the dynamics of $S_t$ is governed by

$$\frac{dS_t}{S_t} = (r - q) \, dt + \sigma \, dZ^Q_t , \quad Z^Q_t \text{ is } Q\text{-Brownian.}$$

The solution to $S_t$ is given by

$$S_t = S_0 e^{\left( r - q - \frac{\sigma^2}{2} \right) t + \sigma Z^Q_t}$$

so that

$$L_t = e^{qt \frac{S_t}{S_0}} = e^{rt} = e^{-\frac{\sigma^2}{2} t + \sigma Z^Q_t} , \ t \in (0,T].$$

From the Girsanov theorem, we deduce that

$$Z^Q_t = Z^Q_t - \sigma t \text{ is a } Q^S\text{-Brownian.}$$

We commonly call $Q^S$ to be the share measure with respect to $S_t$. 
As a check, we consider

\[ V_0 = e^{-rT} E_Q [V_T(S_T)] = e^{-rT} E_{QS} \left[ \frac{V_T(S_T)}{L_T} \right] = S_0 e^{-qT} E_{QS} \left[ \frac{V_T(S_T)}{S_T} \right], \]

so that

\[ \frac{V_0}{\hat{S}_0} = E_{QS} \left[ \frac{V_T(S_T)}{\hat{S}_T} \right], \quad \text{where } \hat{S}_t = e^{qt} S_t. \]

Hence, \( V_t/\hat{S}_t \) is \( Q^S \)-martingale.

Write \( Q^X \) as the share measure with respect to the asset price process \( X_t \). We deduce that

\[ Z_{X,t}^{Q^X} = Z_{X,t}^Q - \sigma_X t \]

is \( Q^X \)-Brownian.
Digression: \( Z_{Y,t}^{Q^X} = Z_{Y,t}^Q - \rho \sigma_X t \) is \( Q^X \)-Brownian

By considering the moment generating function, it suffices to show that

\[
E_{Q^X}[\exp(\alpha Z_{Y}^{Q^X}(T))] = E_{Q^X}[\exp(\alpha Z_{Y}^{Q}(T) - \alpha \rho \sigma_X T)] = \exp \left( \frac{\alpha^2}{2} T \right).
\]

Recall

\[
LT = \frac{dQ^X}{dQ} \bigg|_{F_0} = \exp \left( -\frac{\sigma_X^2}{2} T + \sigma_X Z_X(T) \right),
\]

so

\[
E_{Q^X}[\exp(\alpha Z_{Y}^{Q^X}(T))]
= E_Q \left[ \exp(\alpha Z_{Y}^{Q}(T) - \alpha \rho \sigma_X T) \exp \left( -\frac{\sigma_X^2}{2} T + \sigma_X Z_X(T) \right) \right]
= \exp \left( -\frac{\sigma_X^2}{2} T - \alpha \rho \sigma_X T \right) E_Q \left[ \exp \left( \alpha Z_{Y}^{Q}(T) + \sigma_X Z_X^{Q}(T) \right) \right]
= \exp \left( -\frac{\sigma_X^2}{2} T - \alpha \rho \sigma_X T \right) \exp \left( \frac{\alpha^2 + 2 \rho \alpha \sigma_X + \sigma_X^2}{2} T \right)
= \exp \left( \frac{\alpha^2}{2} T \right).
\]
Application of the numeraire change to derivation of the price formula of an exchange option

Suppose we choose $e^{qX_t}X_t$ as the numeraire, the corresponding Radon-Nikodym derivative that effects the change from $Q$ to $Q^X$ is given by

$$L_T = e^{(qX-r)T}x_TX_0.$$ 

The price function of the exchange option with maturity $T$ and initial asset values $X_0$ and $Y_0$ is given by

$$V(X_0, Y_0; T) = e^{-rT}E_Q[\max(X_T - Y_T, 0)]$$

$$= e^{-rT}E_{Q^X} \left[ \frac{X_0e^{(r-qX)T}}{X_T}X_T \left( 1 - \frac{Y_T}{X_T} \right) 1\{Y_T/X_T<1\} \right].$$

Setting $W_T = Y_T/X_T$, then

$$\frac{V(X_0, Y_0; T)}{X_0} = e^{-qX^T}E_{Q^X}[(1 - W_T)1_{\{W_T<1\}}].$$
From Ito's lemma, the dynamics of $W_t$ under $Q$ is given by

$$\frac{dW_t}{W_t} = [(r - q_Y) - (r - q_X) - \rho \sigma_X \sigma_Y + \sigma_X^2] dt + \sigma_Y dZ_{Y,t}^Q - \sigma_X dZ_{X,t}^Q.$$ 

We observe that $Z_{X,t}^Q$ and $Z_{Y,t}^Q$ as defined by

$$dZ_{X,t}^Q = dZ_{X,t}^Q - \sigma_X dt \quad \text{and} \quad dZ_{Y,t}^Q = dZ_{Y,t}^Q - \rho \sigma_X dt$$

are $Q^X$-Brownian motions. The dynamics of $W_t$ under $Q^X$ becomes

$$\frac{dW_t}{W_t} = (q_X - q_Y) dt + \sigma_Y dZ_{Y,t}^Q - \sigma_X dZ_{X,t}^Q.$$ 

We deduce that $W_t$ remains to be a Geometric Brownian motion, and $\sigma_W^2 = \sigma_Y^2 - 2\rho \sigma_X \sigma_Y + \sigma_X^2$ and $\mu_W = q_X - q_Y$ under $Q^X$. We may write

$$\frac{dW_t}{W_t} = (q_X - q_Y) dt + \sigma_W dZ_{W,t}^Q,$$

where $Z_{W,t}^Q$ is $Q_X$-Brownian.
The payoff \((1 - W_T)^1_{\{W_T < 1\}}\) resembles a put payoff with unit strike and underlying \(W_t\). Using the put price formula, we deduce
\[
E_{Q_X}[(1 - W_T)^1_{\{W_T < 1\}}] = N(d_X) - W_0 e^{(q_X - q_Y)T} N(d_Y), \quad W_0 = \frac{Y_0}{X_0},
\]
where
\[
d_X = \frac{\ln \frac{X_0}{Y_0} + (q_Y - q_X)T + \frac{\sigma_W^2 T}{2}}{\sigma_W \sqrt{T}}, \quad d_Y = \frac{\ln \frac{X_0}{Y_0} + (q_Y - q_X)T - \frac{\sigma_W^2 T}{2}}{\sigma_W \sqrt{T}}.
\]

Finally, the price function of the exchange option is given by
\[
V(X_0, Y_0; T) = e^{-q_X T} X_0 N(d_X) - e^{-q_Y T} Y_0 N(d_Y).
\]
\[
= e^{-r T} \left[ e^{(r-q_X)T} X_0 N \left( \frac{\ln \frac{X_0}{Y_0} + \left[ (r - q_X) - (r - q_Y) + \frac{\sigma_W^2}{2} \right] T}{\sigma_W \sqrt{T}} \right) \right. \\
\left. - e^{(r-q_Y)T} Y_0 N \left( -\frac{\ln \frac{X_0}{Y_0} + \left[ (r - q_Y) - (r - q_X) + \frac{\sigma_W^2}{2} \right] T}{\sigma_W \sqrt{T}} \right) \right].
\]

Suppose we take \(Y_t\) to be the risk free asset so that \(Y_0\) is the strike price, \(q_Y = r\) and \(\sigma_W^2 = \sigma_X^2\) (since \(\sigma_Y = 0\)), we recover the usual call price formula.
4.3 Quanto option – equity options with exchange rate risk exposure

- A quanto option is an option on a foreign currency denominated asset but the payoff is in domestic currency.
- The holder of a quanto option is exposed to both exchange rate risk and equity risk.

Some examples of quanto call options are listed below:

1. Foreign equity call struck in foreign currency

\[ c_1(S_T, F_T, T) = F_T \max(S_T - X_f, 0). \]

Here, \( F_T \) is the terminal exchange rate, \( S_T \) is the terminal price of the underlying foreign currency denominated asset and \( X_f \) is the strike price in foreign currency.
2. Foreign equity call struck in domestic currency

\[ c_2(S_T, T) = \max(F_T S_T - X_d, 0) \]

Here, \( X_d \) is the strike price in domestic currency.

3. Fixed exchange rate foreign equity call

\[ c_3(S_T, T) = F_0 \max(S_T - X_f, 0) \]

Here, \( F_0 \) is some predetermined fixed exchange rate.

4. Equity-linked foreign exchange call

\[ c_4(S_T, T) = S_T \max(F_T - X_F, 0). \]

Here, \( X_F \) is the strike price on the exchange rate. The holder plans to purchase the foreign asset any way but wishes to place a floor value \( X_F \) on the exchange rate. If it happens that the terminal exchange rate \( F_T \) shoots beyond \( X_F \), she receives compensation from the positive payoff received through holding the foreign exchange call.
Quanto prewashing techniques

- Let $S_t$ and $F_t$ denote the stochastic process of the foreign asset price and exchange rate, respectively.

- Define $S^*_t = F_t S_t$, which is the foreign asset price in domestic currency.

- Let $r_d$ and $r_f$ denote the constant domestic and foreign interest rate, respectively, and let $q$ denote the dividend yield of the foreign asset.

- We assume that both $S_t$ and $F_t$ follow the Geometric Brownian motion.
• Under the domestic risk neutral measure $Q_d$, the drift rate of $S^*$ and $F$ are

$$\delta_{S^*}^d = r_d - q \quad \text{and} \quad \delta_F^d = r_d - r_f.$$  

• The reciprocal of $F$ can be considered as the foreign currency price of one unit of domestic currency.

• The drift rate of $S$ and $1/F$ under the foreign risk neutral measure $Q_f$ are given by

$$\delta_S^f = r_f - q \quad \text{and} \quad \delta_{1/F}^f = r_f - r_d,$$

respectively. Note that the dividend yield is the same for the foreign asset in the two-currency world.

• “Quanto prewashing” means finding $\delta_S^d$, that is, the drift rate in the stochastic price process of the foreign currency denominated asset $S$ under the domestic risk neutral measure $Q_d$. 

Let the dynamics of $S_t$ and $F_t$ under $Q_d$ be governed by

\[
\begin{align*}
\frac{dS_t}{S_t} &= \delta_S^d \, dt + \sigma_S \, dZ_S^d \\
\frac{dF_t}{F_t} &= \delta_F^d \, dt + \sigma_F \, dZ_F^d,
\end{align*}
\]

where $dZ_S^d \, dZ_F^d = \rho \, dt$, $\sigma_S$ and $\sigma_F$ are the volatility of $S_t$ and $F_t$, respectively. Since $S_t^* = F_t S_t$, we obtain from Ito’s lemma (see Problem 3 in HW3):

\[
\delta_{S^*}^d = \delta_{FS}^d = \delta_F^d + \delta_S^d + \rho \sigma_F \sigma_S.
\]

We then obtain

\[
\delta_S^d = \delta_{S^*}^d - \delta_F^d - \rho \sigma_F \sigma_S = (r_d - q) - (r_d - r_f) - \rho \sigma_F \sigma_S = r_f - q - \rho \sigma_F \sigma_S.
\]

Comparing with $\delta_S^f = r_f - q$, we need to add the quanto prewashing term $-\rho \sigma_F \sigma_S$ when we specify the dynamics of $S_t$ changing from $Q_f$ to $Q_d$. 

Siegel's paradox \[ \delta^d_{1/F} = r_f - r_d + \sigma_F^2 = \delta^f_{1/F} + \sigma_F^2 \]

Given that the dynamics of \( F_t \) under \( Q_d \) is

\[ \frac{dF_t}{F_t} = (r_d - r_f) \, dt + \sigma_F \, dZ_d, \]

then the process for \( 1/F_t \) under \( Q_d \) is (see Problem 3 in HW3)

\[ \frac{d(1/F_t)}{1/F_t} = (r_f - r_d + \sigma_F^2) \, dt - \sigma_F \, dZ_d. \]

This is seen as a puzzle to many people since the risk neutral drift rate for \( 1/F \) is expected to be \( r_f - r_d \) instead of \( r_f - r_d + \sigma_F^2 \).

We observe directly from the above SDE's that

\[ \sigma_F = \sigma_{1/F} \quad \text{and} \quad \rho_{F,1/F} = -1. \]

This is also consistent with the quanto prewashing technique when it is applied to \( 1/F \), where the added prewashing term \( -\rho \sigma_F \sigma_{1/F} \) becomes \( -(-1)\sigma_F^2 = \sigma_F^2 \).
An interesting application of Siegel's paradox

Suppose the terminal payoff of an exchange rate option is $F_T 1\{F_T > K\}$. Let $V^d(F, t)$ denote the value of the option in the domestic currency world. Define

$$V^f(F_t, t) = V^d(F_t, t)/F_t,$$

so that the terminal payoff of the exchange rate option in foreign currency world is $1\{F_T > K\}$. Now

$$V^f(F, t) = e^{-r_f(T-t)} E_t^Q [1\{F_T > K\} | F_t = F].$$
From \( \delta^d_{1/F} = \delta^f_{1/F} + \sigma^2_F \) and observing \( \sigma_F = \sigma_{1/F} \), we deduce that

\[
\delta^f_F = \delta^d_F + \sigma^2_F.
\]

This result is consistent with the Siegel formula if we interexchange the foreign and domestic currency worlds. We obtain

\[
V^d(F, t) = FV^f(F, t) = e^{-r_f(T-t)} F N(d) = e^{-r_d \tau} e^{\delta^d_{F \tau}} F N(d)
\]

where

\[
d = \ln \frac{F}{K} + \frac{\left( \delta^f_F - \frac{\sigma^2_F}{2} \right)}{\sigma \sqrt{\tau}} \tau
\]

\[
= \ln \frac{F}{K} + \frac{\left( r_d - r_f + \frac{\sigma^2_F}{2} \right)}{\sigma \sqrt{\tau}} \tau.
\]
Price formulas of various quanto options

1. Foreign equity call struck in foreign currency

Let $c^f_1(S, \tau)$ denote the usual vanilla call option on the foreign currency asset in the foreign currency world. The terminal payoff is

$$c^f_1(S, 0) = \max(S - X_f, 0).$$

We treat this call as if it is structured in the foreign currency world. Its value can always be converted into domestic currency using the prevailing exchange rate.
\[ c_1(S, F, \tau) = Fc_1^f(S, \tau) = F \left[ Se^{-q\tau} N(d_1^{(1)}) - X_f e^{-r_f \tau} N(d_2^{(1)}) \right], \]

where

\[
d_1^{(1)} = \frac{\ln \frac{S}{X_f} + \left( \delta^f + \frac{\sigma^2_S}{2} \right) \tau}{\sigma_S \sqrt{\tau}}, \quad d_2^{(1)} = d_1^{(1)} - \sigma_S \sqrt{\tau}.
\]

Note that both the correlation risk \( \rho \) and exchange rate risk \( \sigma_F \) do not appear in the price formula! This is reasonable since we do not set the exchange rate to some fixed value \( F_0 \).
2. Foreign equity call struck in domestic currency

The terminal payoff in domestic currency is

\[ c_2(S, F, 0) = \max(S^* - X_d, 0), \]

where \( S^* = FS \) is the price of a domestic currency denominated asset. Note that

\[ \delta^d_{S^*} = r_d - q \quad \text{and} \quad \sigma^2_{S^*} = \sigma^2_S + 2\rho\sigma_S\sigma_F + \sigma^2_F. \]

The price formula of the foreign equity call is then given by

\[ c_2(S, F, \tau) = S^*e^{-q\tau}N(d_1^{(2)}) - X_d e^{-r_d\tau}N(d_2^{(2)}), \]

where

\[ d_1^{(2)} = \frac{\ln \frac{S^*}{X_d} + \left(\delta^d_{S^*} + \frac{\sigma^2_{S^*}}{2}\right)\tau}{\sigma_{S^*}\sqrt{\tau}}, \quad d_2^{(2)} = d_1^{(2)} - \sigma_{S^*}\sqrt{\tau}. \]
3. Fixed exchange rate foreign equity call

The terminal payoff is denominated in the domestic currency world, so the drift rate $\delta^d_S$ of the foreign asset in $Q_d$ should be used. The price function of the fixed exchange rate foreign equity call is given by

$$c_3(S, \tau) = F_0e^{-r_d\tau}\left[Se^{\delta^d_S\tau}N(d^{(3)}_1) - X_FN(d^{(3)}_2)\right],$$

where

$$d^{(3)}_1 = \frac{\ln\frac{S}{X_F} + \left(\delta^d_S + \frac{\sigma^2_S}{2}\right)\tau}{\sigma_S\sqrt{\tau}}, \quad d^{(3)}_2 = d^{(3)}_1 - \sigma_S\sqrt{\tau}.$$  

- The price formula does not depend on the exchange rate $F$ since the exchange rate has been chosen to be the fixed value $F_0$.
- The currency exposure of the call is embedded in the quanto-prewashing term $-\rho\sigma_S\sigma_F$ in $\delta^d_S$. This call option has exposure to both correlation risk and exchange rate risk.
4. Equity-linked foreign exchange call

Write the terminal payoff in the form of an exchange option

\[ c_4(S, F, 0) = \max(S^* - XS, 0). \]

Taking the two assets to be an exchange \( XS \) for \( S^* \), the ratio of the two assets is \( \frac{S^*}{XS} = \frac{F}{X} \) and the difference of the drift rates under \( Q_d \) is \( \delta_{S*}^d - \delta_S^d = r_d - r_f + \rho \sigma_F \sigma_S \).

\[ c_4(S, \tau) = e^{-r_d \tau} \left[ S^* e^{\delta_{S*}^d \tau} N(d_1^{(4)}) - XS e^{\delta_S^d \tau} N(d_2^{(4)}) \right] \]

\[ = S e^{-q \tau} \left[ FN(d_1^{(4)}) - X e^{(r_f - r_d - \rho \sigma_F \sigma_S) \tau} N(d_2^{(4)}) \right], \]

where

\[ d_1^{(4)} = \frac{\ln \frac{F}{X} + \left( r_d - r_f + \rho \sigma_F \sigma_S + \frac{\sigma_F^2}{2} \right) \tau}{\sigma_F \sqrt{\tau}}, \quad d_2^{(4)} = d_1^{(4)} - \sigma_F \sqrt{\tau}. \]
Digital quanto option relating 3 currency worlds

\( F_{S\textbackslash U} = \) SGD currency price of one unit of USD currency

\( F_{H\textbackslash S} = \) HKD currency price of one unit of SGD currency

- Digital quanto option payoff: pay one HKD if \( F_{S\textbackslash U} \) is above some strike level \( K \).
- The dynamics of \( F_{S\textbackslash U} \) under \( Q^S \) is governed by

\[
\frac{dF_{S\textbackslash U}}{F_{S\textbackslash U}} = (r_{SGD} - r_{USD}) dt + \sigma_{F_{S\textbackslash U}} dZ^S_{F_{S\textbackslash U}}.
\]
• Given \( \delta^S_{FS\setminus U} = r_{SGD} - r_{USD} \), how to find \( \delta^H_{FS\setminus U} \), which is the risk neutral drift rate of the SGD asset denominated in Hong Kong dollar?

• By the quanto-prewashing technique

\[
\delta^H_{FS\setminus U} = \delta^S_{FS\setminus U} - \rho \sigma_{FS\setminus U} \sigma_{FH\setminus S}.
\]

• Digital option value = \( e^{-r_{HKD}\tau} E^t_{QH} \left[ 1\{FS\setminus U > K\} \right] = e^{-r_{HKD}\tau} N(d) \)

where

\[
d = \frac{\ln \frac{FS\setminus U}{K} + \left( \delta^H_{FS\setminus U} - \frac{\sigma^2_{FS\setminus U} K}{2} \right) \tau}{\sigma_{FS\setminus U} \sqrt{\tau}}.
\]
Example 1

The quanto option pays \( F_{H \backslash S} \) Hong Kong dollars when \( F_{S \backslash U} > K \). This is equivalent to pay one Singaporean dollar. Value of the quanto option in Singaporean dollar is

\[
e^{-r_{SGD} \tau} E^{t \left[ \mathbf{1}_{\{F_{S \backslash U} > K\}} \right]} = e^{-r_{SGD} \tau} N(\hat{d})
\]

where

\[
\hat{d} = \frac{\ln \frac{F_{S \backslash U}}{K} + \left( \delta_{F_{S \backslash U}}^S - \frac{\sigma_{F_{S \backslash U}}^2}{2} \right) \sigma_{F_{S \backslash U}}}{\sigma_{F_{S \backslash U}} \sqrt{\tau}}, \quad \delta_{F_{S \backslash U}}^S = r_{SGD} - r_{USD}.
\]

This option model is similar to \( c_1(S, F, \tau) \), where the option payoff in foreign currency is converted into domestic currency using the prevailing exchange rate at maturity. The most efficient approach is to perform valuation of the option under the foreign currency world. The value of the quanto option in Hong Kong dollar is

\[
F_{H \backslash S} e^{-r_{SGD} \tau} N(\hat{d}).
\]
Example 2

The quanto option pays $F_{H\setminus U}$ Hong Kong dollars when $F_{S\setminus U} > K$. This is equivalent to pay one US dollars.

Method One

Observe that $F_{H\setminus U} = F_{H\setminus S}F_{S\setminus U}$ so that it is like paying $F_{S\setminus U}$ Singaporean dollars when $F_{S\setminus U} > K$.

Value of the quanto option in Hong Kong dollars is

$$F_{H\setminus U}e^{-r_{SGD}\tau}E_{Q_S}\left[F_{S\setminus U}1_{\{F_{S\setminus U}>K\}}\right] = F_{H\setminus U}e^{-r_{SGD}\tau}e^{(r_{SGD}\tau-r_{USD}\tau)}F_{S\setminus U}N(d_1)$$

$$= F_{H\setminus U}e^{-r_{USD}\tau}N(d_1)$$

where

$$d_1 = \frac{\ln \frac{F_{S\setminus U}}{K} + \left(r_{SGD} - r_{USD} + \frac{\sigma_{F_{S\setminus U}}^2}{2}\right)\tau}{\sigma_{F_{S\setminus U}}\sqrt{\tau}}.$$
Method Two

The quanto option pays one US dollars when \( F_{S\backslash U} > K \Leftrightarrow \frac{1}{K} > \frac{1}{F_{S\backslash U}} = F_{U\backslash S} \). Later, we multiply the option value in US currency by the exchange rate \( F_{H\backslash U} \) to convert into Hong Kong dollars.

Value of the quanto option in US dollars is

\[
e^{-r_{USD} \tau} E_{QU}^{t} \left[ 1 \{ F_{U\backslash S} < \frac{1}{K} \} \right] = e^{-r_{USD} \tau} N(-d_2),
\]

where

\[
d_2 = \frac{\ln \frac{F_{U\backslash S}}{1/K} + \left( (r_{USD} - r_{SGD}) - \frac{\sigma_{F_{U\backslash S}}^2}{2} \right) \tau}{\sigma_{F_{U\backslash S}} \sqrt{\tau}} = -d_1.
\]

Remark The quanto option value in Hong Kong dollars using the two approaches agree with each other.
3.3 Transaction costs models

How to construct the hedging strategy that best replicates the payoff of a derivative security in the presence of transaction costs?

Recall that one can create a portfolio containing $\Delta$ units of the underlying asset and money market account which replicates the payoff of the option. By the portfolio replication argument, the value of an option is equal to the initial cost of setting up the replicating portfolio which mimics the payoff of the option.

Leland proposes a modification to the Black-Scholes model where the portfolio is adjusted at regular time intervals. His model assumes proportional transaction costs where the costs in buying and selling the asset are proportional to the monetary value of the transaction.
Let $k$ denote the round trip transaction cost per unit dollar of transaction. Suppose $\alpha$ units of assets are bought ($\alpha > 0$) or sold ($\alpha < 0$) at the price $S$, then the transaction cost is given by $\frac{k}{2} |\alpha|S$ in either buying or selling.

We consider a hedged portfolio of the writer of the option, where he is shorting one unit of option and long holding $\Delta$ units of the underlying asset. The value of this hedged portfolio at time $t$ is given by

$$\Pi(t) = -V(S,t) + \Delta S,$$

where $V(S,t)$ is the value of the option and $S$ is the asset price at time $t$. Let $\delta t$ denote the fixed and small finite time interval between successive rebalancing of the portfolio.
After the small time interval $\delta t$, the change in value of the portfolio is

$$\delta \Pi = -\delta V + \Delta \delta S - \frac{k}{2} |\delta \Delta| S,$$

where $\delta S$ is the change in asset price and $\delta \Delta$ is the change in the number of units of asset held in the portfolio.

A cautious reader may doubt why the proportional transaction cost term $-\frac{k}{2} |\delta \Delta| S$ appears in $\delta \Pi$ while the term $S \delta \Delta$ is missing.

- The transaction cost term represents the single trip transaction cost paid due to rebalancing of the position in the underlying asset.

- By following the “pragmatic” approach used by Black and Scholes (1973), the number of units $\Delta$ is taken to be instantaneously constant.
By Ito’s lemma, the change in option value in time $\delta t$ to leading orders is given by

$$
\delta V \approx \frac{\partial V}{\partial S} \delta S + \left( \frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} \right) \delta t.
$$

In order to cancel the stochastic terms, one chooses $\Delta = \frac{\partial V}{\partial S}$. The change in the number of units of asset in time $\delta t$ is given by

$$
\delta \Delta = \frac{\partial V}{\partial S} (S + \delta S, t + \delta t) - \frac{\partial V}{\partial S} (S, t).
$$

By Ito’s lemma, the leading order of $|\delta \Delta|$ is found to be

$$
|\delta \Delta| \approx \sigma S \left| \frac{\partial^2 V}{\partial S^2} \right| |\delta Z|.
$$

Formally, we may treat $\delta Z$ as $\tilde{x} \sqrt{\delta t}$, where $\tilde{x}$ is the standard normal variable. The expectation of the reflected Brownian motion $|\delta \Delta|$ is given by

$$
E(|\delta Z|) = \sqrt{\frac{2}{\pi}} \sqrt{\delta t}.
$$
This hedged portfolio should be expected to earn a return same as that of a riskless asset. This gives

\[ E[\delta \Pi] = r \left( -V + \frac{\partial V}{\partial S} S \right) \delta t. \]

By putting all the above results together, the above equation can be rewritten as

\[
\left( -\frac{\partial V}{\partial t} - \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} - k \frac{\sigma^2}{2} S \sqrt{\frac{2}{\pi \delta t}} \left| \frac{\partial^2 V}{\partial S^2} \right| \right) \delta t = r \left( -V + \frac{\partial V}{\partial S} S \right) \delta t.
\]

If we define the Leland number to be \( Le = \sqrt{\frac{2}{\pi}} \left( \frac{k}{\sigma \sqrt{\delta t}} \right) \), we obtain

\[
\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\sigma^2}{2} Le S^2 \left| \frac{\partial^2 V}{\partial S^2} \right| + rS \frac{\partial V}{\partial S} - rV = 0.
\]
In the proportional transaction costs model, the term $\frac{\sigma^2}{2} Le S^2 \left| \frac{\partial^2 V}{\partial S^2} \right|$ is in general non-linear, except when the comparative static $\Gamma = \frac{\partial^2 V}{\partial S^2}$ does not change sign for all $S$. The transaction cost term is dependent on $\Gamma$, where $\Gamma$ measures the sensitivity of the hedge ratio $\Delta$ to underlying asset price $S$.

One may rewrite the equation into the form that resembles the Black-Scholes equation

$$\frac{\partial V}{\partial t} + \frac{\tilde{\sigma}^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0,$$

where the modified volatility is given by

$$\tilde{\sigma}^2 = \sigma^2 [1 + Le \text{ sign}(\Gamma)].$$

The governing equation becomes mathematically ill-posed when $\tilde{\sigma}^2$ becomes negative. This occurs when $\Gamma < 0$ and $Le > 1$. 
It is known that $\Gamma$ is always positive for the vanilla European call and put options in the absence of transaction costs. If we postulate the same sign behavior for $\Gamma$ in the presence of transaction costs, then $\tilde{\sigma}^2 = \sigma^2 (1 + Le) > \sigma^2$.

The governing equation then becomes linear under the above assumption so that the Black-Scholes formulas become applicable except that the modified volatility $\tilde{\sigma}$ is now used as the volatility parameter.

We can deduce $V(S,t)$ to be an increasing function of $Le$ since we expect a higher option value for a high value of modified volatility. Financially speaking, the more frequent the rebalancing (smaller $\delta t$) the higher the transaction costs and so the writer of an option should charge higher for the price of the option.
Let $V(S, t; \tilde{\sigma})$ and $V(S, t; \sigma)$ denote the option values obtained from the Black-Scholes formula with volatility values $\tilde{\sigma}$ and $\sigma$, respectively.

The total transaction costs associated with the replicating strategy is then given by

$$\mathcal{T} = V(S, t; \tilde{\sigma}) - V(S, t; \sigma).$$

When $Le$ is small, $\mathcal{T}$ can be approximated by

$$\mathcal{T} \approx \frac{\partial V}{\partial \sigma} (\tilde{\sigma} - \sigma),$$

Since $\tilde{\sigma} = \sigma [1 + Le \text{sign}(\Gamma)]^{1/2} \approx \sigma \left[1 + \frac{Le}{2} \text{sign}(\Gamma)\right]$, so $\tilde{\sigma} - \sigma \approx \frac{k}{\sqrt{2\pi \delta t}}$.

Note that $\frac{\partial V}{\partial \sigma}$ is the same for both call and put options. For $Le \ll 1$, the total transaction costs for either a call or a put is approximately given by

$$\mathcal{T} \approx \frac{kSe^{-d_1^2/2}}{2\pi} \sqrt{\frac{T - t}{\delta t}}.$$
3.4 Implied volatilities and volatility smiles

The difficulties of setting volatility value in the option price formulas lie in the fact that the input value should be the forecast volatility value over the remaining life of the option rather than an estimated volatility value (*historical volatility*) from the past market data of the asset price.

The Black-Scholes implied volatility $\sigma_{imp}(X, T)$ is the unique solution to

$$V_{market}(X, T) = V^{BS}(S, t; K, T, \sigma_{imp}(X, T)).$$

The above equation is an answer to: what volatility is implied in observed option prices, if the Black-Scholes model is a valid description of market conditions?
Implied volatility surface

- In financial markets, it becomes a common practice for traders to quote an option’s market price in terms of implied volatility $\sigma_{imp}$.

- In particular, several implied volatility values obtained simultaneously from different options with varying maturities and strike prices on the same underlying asset provide an extensive market view about the volatility at varying strikes and maturities.

- The Black-Scholes (BS) implied volatility computed from the market option price by inverting the BS price formula varies with strike price and time to expiration – volatility smile (skew) and volatility term structure, respectively. The plot of implied volatilities against moneyness ($X/S$) and time to expiration $T-t$ generates the implied volatility surface.
DAX option implied volatilities (as black dots) on 2000/05/02. The lower left axis is moneyness and left right axis is time to expiration measured in years.

- $\sigma_{imp}(X,T)$ is non-linear in strikes and time to expiration; and if observed over in calendar time, it is also time-dependent.
Volatility smiles

- The Black-Scholes model assumes a lognormal probability distribution of the asset price at all future times. Since volatility is the only unobservable parameter in the Black-Scholes model, the model gives the option price as a function of volatility.

- If we plot the implied volatility of the exchange-traded options, like index options, against their strike price for a fixed maturity, the curve is typically convex in shape, rather than a straight horizontal line as suggested by the simple Black-Scholes model. This phenomenon is commonly called the volatility smile by market practitioners.

- These smiles exhibit widely differing properties, depending on whether the market data were taken before or after the October, 1987 market crash.
The implied volatility values are obtained by averaging over exchange-traded European index options of different maturities.

A typical pattern of pre-crash smile. The implied volatility curve is convex with a dip.
A typical pattern of post-crash smile. The implied volatility drops against $X/S$, indicating that out-of-the-money puts ($X/S < 1$) are traded at higher implied volatility than out-of-the-money calls ($X/S > 1$). The market price of the out-of-the-money puts became more expensive than the Black-Scholes price after the 1987 crash (investors are generally worried about market clashes and buy puts for protection).
Comparison of the risk neutral probability density of asset price (solid curve) implied from market data and the theoretical lognormal distribution (dotted curve). The risk neutral probability density is thicker at the left tail and thinner at the right tail, indicating that there is a higher chance of more acute drop when $S$ is low and a lower chance of further increase when $S$ is high.
Negative correlation between stock price process and volatility process

In real market situation, it is a common occurrence that when the asset price is high, volatility tends to decrease, making it less probable for a higher asset price to be realized. When the asset price is low, volatility tends to increase, that is, it is more probable that the asset price plummets further down. In other words, stock price process and volatility process are in general negatively correlated.
Extreme events in stock price movements

Probability distributions of stock market returns have typically been estimated from historical time series. Unfortunately, common hypotheses may not capture the probability of extreme events. The clash events are rare and may not be present in the historical record.

Examples

1. On October 19, 1987, the two-month S&P 500 futures price fell 29%. Under the lognormal hypothesis of annualized volatility of 20%, this is a $-27$ standard deviation event with probability $10^{-160}$ (virtually impossible).

2. On October 13, 1989, the S&P 500 index fell about 6%, a $-5$ standard deviation event. Under the maintained hypothesis, this should occur only once in 14,756 years.
Different volatilities across time

Supply and demand
When markets are very quiet, the implied volatilities of the near month options are generally lower than those of the far month. When markets are very volatile, the reverse is generally true.

• In very volatile markets, everyone wants or needs to load with gamma. Near-dated options provide the most gamma and the resultant buying pressure will have the effect of pushing prices up.

• In quiet markets, no one wants a portfolio long of near dated options.
Different volatilities for different strike prices

1. *Stock options* – higher volatilities at lower strike and lower volatilities at higher strikes

- In a falling market, everyone needs out-of-the-money puts for insurance and will pay a higher price for the lower strike options.

- Equity fund managers are long billions of dollars worth of stock and writing out-of-the-money call options against their holdings as a way of generating extra income. This pushes the value of out-of-the-money call options down.
2. *Commodity options* – higher volatilities at higher strike and lower volatilities at lower strikes

- Government intervention – no worry about a large price fall. Speculators are tempted to sell puts aggressively.

- Risk of shortages – no upper limit on the price. Demand for higher strike price options.
Term structure of volatility

The Black-Scholes formulas remain valid under time dependent volatility except that \( \sqrt{\frac{1}{T-t} \int_t^T \sigma(\tau)^2 \, d\tau} \) is used to replace \( \sigma \).

How to obtain the term structure of volatility \( \sigma(t) \) given the implied volatility measured at time \( t^* \) of a European option expiring at time \( t \)? For an option with time to expiry \( t - t^* \), the substitution of the implied volatility \( \sigma_{imp}(t^*, t) \) into the standard Black-Scholes formula under constant volatility gives the option price. The equivalence of giving the same observed option price by adopting the two different forms of volatility in the two separate option price formulas leads to

\[
\int_{t^*}^{t} \sigma(u)^2 \, du = \sigma_{imp}^2(t^*, t)(t - t^*).
\]

Differentiating with respect to \( t \), we obtain the term structure of volatility in terms of the term structure of implied volatility

\[
\sigma(t) = \sqrt{\sigma_{imp}(t^*, t)^2 + 2(t - t^*) \sigma_{imp}(t^*, t) \frac{\partial \sigma_{imp}(t^*, t)}{\partial t}}.
\]
**Approximation of $\sigma(t)$ as a piecewise constant function**

Practically, we do not have a continuous differentiable implied volatility function $\sigma_{imp}(t^*, t)$, but rather implied volatilities are available at discrete instants $t_i, \ i = 1, 2, \ldots, n$. Suppose we assume $\sigma(t)$ to be piecewise constant over $(t_{i-1}, t_i)$, where $\sigma(t) = \sigma_i, \ t_{i-1} < t < t_i, \ i = 1, 2, \ldots, n$. We then have

$$\int_{t^*}^{t_i} \sigma^2(\tau) \, d\tau - \int_{t^*}^{t_{i-1}} \sigma^2(\tau) \, d\tau = (t_i - t^*)\sigma_{imp}^2(t^*, t_i) - (t_{i-1} - t^*)\sigma_{imp}^2(t^*, t_{i-1})$$

$$= \int_{t_{i-1}}^{t_i} \sigma^2(\tau) \, d\tau = \sigma_i^2(t_i - t_{i-1}), \quad t_{i-1} < t < t_i,$$

giving

$$\sigma_i = \sqrt{\frac{(t_i - t^*)\sigma_{imp}^2(t^*, t_i) - (t_{i-1} - t^*)\sigma_{imp}^2(t^*, t_{i-1})}{t_i - t_{i-1}}}, \quad t_{i-1} < t < t_i.$$
Risk neutral density function

- Let $\psi(S_T, T; S_t, t)$ denote the transition density function of the asset price. The time-$t$ price of a European call with maturity date $T$ and strike price $X$ is given by

$$
c(S_t, t; X, T) = e^{-r(T-t)} \int_X^\infty (S_T - X) \psi(S_T, T; S_t, t) \, dS_T.
$$

- If we differentiate $c$ with respect to $X$, we obtain

$$
\frac{\partial c}{\partial X} = -e^{-r(T-t)} \int_X^\infty \psi(S_T, T; S_t, t) \, dS_T;
$$

and differentiate once more, we have

$$
\psi(X, T; S_t, t) = e^{r(T-t)} \frac{\partial^2 c}{\partial X^2}.
$$

- Suppose that market European option prices at all strikes are available, the risk neutral density function can be inferred completely from the market prices of options with the same maturity and different strikes, without knowing the volatility function.
Dupire equation and local volatility function

Assuming that the asset price dynamics under the risk neutral measure is governed by

$$\frac{dS_t}{S_t} = (r - q)dt + \sigma(S_t, t)dZ_t,$$

where the local volatility function is assumed to have both state and time dependence. Write $c = c(X, T)$, the Dupire equation takes the form

$$\frac{\partial c}{\partial T} = -qc - (r - q)X \frac{\partial c}{\partial X} + \frac{\sigma^2(X, T)}{2}X^2 \frac{\partial^2 c}{\partial X^2}.$$
Proof

We differentiate $\psi(X,T; S_t, t)$ with respect to $T$ to obtain

$$\frac{\partial \psi}{\partial T} = e^{r(T-t)} \left( r \frac{\partial^2 c}{\partial X^2} + \frac{\partial^2}{\partial X^2} \frac{\partial c}{\partial T} \right),$$

and $\psi(X, T; S, t)$ satisfies the forward Fokker-Planck equation, where

$$e^{-r(T-t)} \frac{\partial \psi}{\partial T} = \frac{\partial^2}{\partial X^2} \left[ \frac{\sigma^2(X, T)}{2} X^2 \psi \right] - \frac{\partial}{\partial X} [(r - q) X \psi].$$
Combining the above equations and eliminating the common factor $e^{r(T-t)}$, we have

$$
\frac{r}{\partial X^2} \frac{\partial^2 c}{\partial X^2} + \frac{\partial^2}{\partial X^2 \partial T} \frac{\sigma^2(X,T)}{2} X^2 \frac{\partial^2 c}{\partial X^2} - \frac{\partial}{\partial X} \left[ (r - q) X \frac{\partial^2 c}{\partial X^2} \right].
$$

Integrating the above equation with respect to $X$ twice, we obtain

$$
\frac{\partial c}{\partial T} + rc + (r - q) \left( X \frac{\partial c}{\partial X} - c \right)
= \frac{\sigma^2(X,T)}{2} X^2 \frac{\partial^2 c}{\partial X^2} + \alpha(T) X + \beta(T),
$$

where $\alpha(T)$ and $\beta(T)$ are arbitrary functions of $T$.

Since all functions involving $c$ in the above equation vanish as $X$ tends to infinity, hence $\alpha(T)$ and $\beta(T)$ must be zero. Grouping the remaining terms in the equation, we obtain the Dupire equation.
From the Dupire equation, we may express the local volatility $\sigma(X, T)$ explicitly in terms of the call price function and its derivatives, where

$$\sigma^2(X, T) = \frac{2 \left[ \frac{\partial c}{\partial T} + qc + (r - q)X \frac{\partial c}{\partial X} \right]}{X^2 \frac{\partial^2 c}{\partial X^2}}.$$ 

• Suppose a sufficiently large number of market option prices are available at many maturities and strikes, we can estimate the local volatility from the above equation by approximating the derivatives of $c$ with respect to $X$ and $T$ using the market data.

• In real market conditions, market prices of options are available only at limited number of maturities and strikes.
Relationship between local volatility and implied volatility

Dupire’s equation shows how to compute $\sigma_{loc}(X,T)$ from market prices of European options. On the other hand, the market quote prices for European options are in terms of their implied volatilities. One may want to relate $\sigma_{loc}(X,T)$ with $\sigma_{imp}(X,T)$. We have

$$\sigma_{loc}^2(X,T) = \frac{\sigma_{imp}^2 + 2T\sigma_{imp}\frac{\partial\sigma_{imp}}{\partial T} + 2(r-q)XT\sigma_{imp}\frac{\partial\sigma_{imp}}{\partial X}}{(1 + Xd_1T\frac{\partial\sigma_{imp}}{\partial X})^2 + X^2T\sigma_{imp}^2} - d_1T \left(\frac{\partial\sigma_{imp}}{\partial X}\right)^2,$$

where

$$d_1 = \frac{\ln \frac{S}{X} + \left[r - q + \frac{\sigma_{imp}^2(X,T)}{2}\right]T}{\sigma_{imp}(X,T)\sqrt{T}}.$$
3.5 Volatility trading: variance and volatility derivatives, VIX

*Characteristics of volatility (hidden stochastic process*)

- Likely to grow when uncertainty and risk increase. May serve as a proxy for market confidence – fear gauge.

- Volatilities appear to revert to the mean (non-linear drift).
  - After a large volatility spike, the volatility can potentially decrease rapidly.
  - After a low volatility period, it can be slow to increase.

- Volatility is often negatively correlated with stock or index level, and tends to stay high after large downward moves.

- Stock options are impure: they provide exposure to both direction of the stock price and its volatility. If one hedges the options according to Black-Scholes prescription, then she can remove the exposure to the stock price.
Businesses that are implicitly short volatility (lose when volatility increases)

- Investors following active benchmarking strategies may require more frequent rebalancing and incur higher transaction expenses during volatile periods.

- Equity funds are probably short volatility due to the negative correlation between index level and volatility.

- Hedge funds that take positions on the spread between stocks of companies planning mergers will narrow. If volatility increases, the merger may become less likely and spread may widen.

- Volatility swaps are forward contracts on future realized stock volatility; and similarly, variance swaps on future variance (square of future volatility). They provide pure exposure to volatility and variance, respectively.
Replication of variance swaps - continuous model

The fair strike of a variance swap (continuously monitored) is given by

\[ K_{\text{var}} = E_0[V_R] = E_0 \left( \frac{1}{T} \int_0^T \sigma_t^2 \, dt \right) \]

Suppose the asset price process \( S_t \) follows the following Brownian motion:

\[ \frac{dS_t}{S_t} = r \, dt + \sigma_t \, dW_t, \]

where \( W_t \) is the standard Brownian motion and \( \sigma_t \) is non-stochastic (though may be state dependent). We may rewrite the dynamics equation as follows:

\[ d \ln S_t = \left( r - \frac{\sigma_t^2}{2} \right) dt + \sigma_t \, dW_t. \]
Subtracting the two, we obtain

\[
\frac{dS_t}{S_t} - d \ln S_t = \frac{\sigma_t^2}{2} \, dt
\]

The measure of the continuous realized variance is then given by

\[
V_R = \frac{1}{T} \int_0^T \sigma_t^2 \, dt = \frac{2}{T} \left( \int_0^T \frac{dS_t}{S_t} - \ln \frac{S_t}{S_0} \right).
\]

The formula dictates the strategy that can be adopted to replicate the realized variance.

We take \( \frac{1}{S_t} \) units of stock at time \( t \) paying $1, and enter a “static” short position at time 0 in a forward contract which at maturity has a payoff equals to the logarithm of the total return of on the stock \( \ln \frac{S_T}{S_0} \), where \( \frac{S_T}{S_0} \) is the total return over \([0, T]\).
This is a self-financing strategy

Suppose the stock price goes up, the investor sells \( \frac{1}{S_t} \) units of stock and buys \( \frac{1}{S_{t+dt}} \) units paying $1. The net amount \( \frac{S_{t+dt}}{S_t} - 1 \) is invested in the riskfree asset. Over the same period, the forward value \( F_t = E_t \left[ \ln \frac{S_T}{S_0} \right] \) of the log contract increases in value. The short position in the log contract offsets the gain on the long stock position. Recall

\[
\ln \frac{S_{t+dt}}{S_t} \approx \frac{S_{t+dt}}{S_t} - 1
\]

since \( \ln(1 + x) \approx x \).
The pricing issue is to find the fair strike of the variance swap.

\[
K_{\text{var}} = E_0[V_R] = \frac{2}{T} E_0 \left[ \int_0^T \frac{dS_t}{S_t} - \ln \frac{S_T}{S_0} \right]
\]

\[
= \frac{2}{T} \left\{ E_0 \left[ \int_0^T r \ dt \right] + E_0 \left[ \int_0^T \sigma_t \ dW_t \right] - E_0 \left[ \ln \frac{S_T}{S_0} \right] \right\}.
\]

The expectation of the long stock position gives \( rT \) since the dollar value of the stock position is always $1. How to replicate the log contract using basic instruments of forward contracts, calls and puts?

**Technical result** For any twice-differentiable function \( f: \mathbb{R} \to \mathbb{R} \), and any \( S_* \geq 0 \), we have

\[
f(S_T) = f(S_*) + f'(S_*)(S_T - S_*) + \int_0^{S_*} f''(K)(K - S_T)^+ \ dK
\]

\[
+ \int_{S_*}^{\infty} f''(K)(S_T - K)^+ \ dK.
\]
The log payoff $\ln \frac{S_T}{S_0}$ can be rewritten as

$$\ln \frac{S_T}{S_0} = \ln \frac{S_T}{S_*} + \ln \frac{S_*}{S_0}, \ \text{where} \ S_* \ \text{is an arbitrage non-negative number.}$$

Applying the technical formula for $f(S_T) = \ln S_T$, we have

$$\ln S_T - \ln S_* = \frac{S_T - S_*}{S_*} - \int_0^{S_*} \frac{1}{K^2} (K - S_T)^+ \, dK - \int_{S_*}^{\infty} \frac{1}{K^2} (S_T - K)^+ \, dK.$$

- Hold a long position in $\frac{1}{S_*}$ forward contracts with forward price $S_*;$
- Short positions in $\frac{1}{K^2}$ put options with strike $K$, $K$ from 0 to $S_*;$ short positions in $\frac{1}{K^2}$ call options with strike $K$, $K$ from $S_*$ to $\infty$.

All contracts have the same maturity $T.$
Valuation of fair strike

\[
K_{\text{var}} = \frac{2}{T} \left\{ rT - E_0 \left[ \ln \frac{S}{S_0} + \frac{S_T - S_\ast}{S_\ast} - \int_0^{S_\ast} \frac{1}{K^2} (K - S_T)^+ \, dK - \int_{S_\ast}^{\infty} \frac{1}{K^2} (S_T - K)^+ \, dK \right] \right\}.
\]

Note that

\[
S_0 = e^{-rT} E_0[S_T], \quad C_0(K) = e^{-rT} E_0[(S_T - K)^+],
\]
\[
P_0(K) = e^{-rT} E_0[(K - S_T)^+].
\]

We then have

\[
K_{\text{var}} = \frac{2}{T} \left[ rT - \left( \frac{S_0}{S_\ast} e^{rT} - 1 \right) - \ln \frac{S_\ast}{S_0} \right.
\]
\[
\left. + \, e^{rT} \int_0^{S_\ast} \frac{1}{K^2} P_0(K) \, dK + e^{rT} \int_{S_\ast}^{\infty} \frac{1}{K^2} C_0(K) \, dK \right].
\]

The formula requires an infinite number of strikes in order to be exact, while the market provides only a finite number of options.
Volatility swaps

Even though variance swaps can be priced and replicated easily, they are still less actively traded compared to volatility swaps.

First order approximation:

\[
(\sqrt{V_R})^2 - (K_{s/d})^2 \approx 2K_{s/d}(\sqrt{V_R} - K_{s/d})
\]

or

\[
\sqrt{V_R} - K_{s/d} \approx \frac{1}{2K_{s/d}} [V_R - (K_{s/d})^2], \quad K_{s/d} = \sqrt{K_{\text{var}}}
\]

To make \( K_{s/d} < \sqrt{K_{\text{var}}} \), consider the second order Taylor expansion of \( g(V_R) = \sqrt{V_R} \) around \( K_{\text{var}} = E_0[V_R] \), we have

\[
\sqrt{V_R} \approx \sqrt{K_{\text{var}}} + \frac{1}{2\sqrt{K_{\text{var}}}}(V_R - K_{\text{var}}) - \frac{1}{8(K_{\text{var}})^{3/2}}(V_R - K_{\text{var}})^2.
\]
Variance and Standard deviation Payoffs
Taking the expected values on both sides, we obtain

\[ K_{s/d} = E_0[\sqrt{V_R}] \approx \sqrt{K_{\text{var}}} - \frac{1}{8(K_{\text{var}})^{3/2}} \frac{E_0[(V_R - K_{\text{var}})^2]}{\text{var}_0(V_R)} \]

The convexity correction represents the mismatch between \( K_{s/d} \) and \( \sqrt{K_{\text{var}}} \). Under this approximation, we achieve \( K_{s/d} < \sqrt{K_{\text{var}}} \).

- The above formula does not give a straightforward formula for \( K_{s/d} \) since the conditional variance of the realized variance has to be estimated.

- Broadie and Jain (2008) show that this convexity correction formula to approximate fair volatility strikes may not provide good estimates in jump-diffusion models.
The tale of VIX

In response to the emergence of the over-the-counter volatility derivatives market, the Chicago Board of Options Exchange (CBOE) introduced the CBOE volatility index (VXO) that was designed to reveal the market forecast of the future realized volatility of S&P 100 index based on the traded prices of options.

- Compute an average of the Black-Scholes (BS) option implied volatility with strike prices close to the current spot index level and maturities interpolated at about one month.

- Implied volatility measure (so does VXO) is used as an indicator of market stress. However, this is based on the BS model (not model free).
Steps

1. Consider 2 maturities $T_1$ and $T_2$ that bracket maturity date on one month later.

At maturity $T_i$, $i = 1, 2$, the near-the-money Black-Scholes (BS) implied volatility for each $K_j^{(i)}$ is obtained by averaging the BS implied volatilities of one call and one put at $T_i$ and $K_j^{(i)}$. Taking two strikes $K_1^{(i)}$ and $K_2^{(i)}$, the CBOE interpolates linearly the implied volatilities to obtain the at-the-money BS implied volatility [as denoted by $\text{ATMV}(t, T_i)$].
2. Since the BS implied volatility is based on “actual calendar days/365”, the CBOE converts it into “trading days/252” convention:

\[ TV(t, T_i) = \text{ATMV}(t, T_i) \frac{\sqrt{\text{NC}(t, T_i)}}{\sqrt{\text{NT}(t, T_i)}}. \]

\[ \text{NC}(t, T_i) = \text{number of actual calendar days between } t \text{ and } T_i \]
\[ \text{NT}(t, T_i) = \text{number of trading days between } t \text{ and } T_i \]
\[ = \text{NC}(t, T_i) - 2 \times \text{int}\left(\frac{\text{NC}(t, T_i)}{T}\right) \]

3. The CBOE interpolates linearly \( TV(t, T_1) \) and \( TV(t, T_2) \) to estimate the 22 trading days (one month) at-the-money implied volatility

\[ \text{VXO}_t = \frac{TV(t, T_1)(NT_2 - 22) + TV(t, T_2)(22 - NT_1)}{NT_2 - NT_1}. \]
• $VXO_t$ over-estimates the one-month realized volatility consistently due to
(i) linear interpolation error
(ii) trading day conversion.
Mathematical derivation of VIX

VIX expresses volatility in percentage points. It is calculated as 100 times the square root of the expected 30-day variance (var) of the S&P 500 rate of return.

\[
\text{VIX} = 100\sqrt{\text{forward price of expected realized cumulative variance}}
\]

Suppose the forward price \( F_t \) of the index \textit{without jump} follows

\[
\frac{dF_t}{F_t} = \sigma_t \, dW_t \text{ so that } d\ln F_t = -\frac{\sigma_t^2}{2} \, dt + \sigma_t \, dW_t.
\]

Subtracting the two equations, we obtain the cumulative variance over \([0, T]\)

\[
\frac{dF_t}{F_t} - d\ln F_t = \frac{\sigma_t^2}{2} dt, \quad \text{so} \quad \int_0^T \sigma_t^2 dt = 2 \left[ \int_0^T \frac{dF_t}{F_t} - \ln \frac{F_T}{F_0} \right].
\]

From a well known result in Taylor expansion, we have

\[
f(F_T) - f(F_0) = f'(F_0)(F_T - F_0) + \int_0^{F_0} f''(K)(K - F_T)^+ dK + \int_{F_0}^{\infty} f''(K)(F_T - K)^+ dK; \quad F_0 = \text{time-0 forward price}
\]
\textbf{Proof}

\[ f(S_T) = \int_0^{S_*} f(K)\delta(S_T - K) \, dK + \int_{S_*}^{\infty} f(K)\delta(S_T - K) \, dK \]

\[ = f(K)1_{\{S_T<K\}}[0]^{S_*} - \int_0^{S_*} f'(K)1_{\{S_T<K\}} \, dK \]

\[ + f(K)1_{\{S_T\geq K\}}[\infty]^{S_*} - \int_{S_*}^{\infty} f'(K)1_{\{S_T\geq K\}} \, dK \]

\[ = f(S_*)1_{\{S_T<S_*\}} - \left[ f'(K)(K - S_T)^+ \right]_{0}^{S_*} + \int_{0}^{S_*} f''(K)(K - S_T)^+ \, dK \]

\[ + f(S_*)1_{\{S_T\geq S_*\}} - \left[ f'(K)(S_T - K)^+ \right]_{S_*}^{\infty} + \int_{S_*}^{\infty} f''(K)(S_T - K)^+ \, dK \]

\[ = f(S_*) + f'(S_*)(S_T - S_*)^+ - (S_* - S_T)^+ \]

\[ + \int_{0}^{S_*} f''(K)(K - S_T)^+ \, dK + \int_{S_*}^{\infty} f''(K)(S_T - K)^+ \, dK \]

\[ f(S_T) - f(S_*) = f'(S_*)(S_T - S_*) + \int_{0}^{S_*} f''(K)(K - S_T)^+ \, dK \]

\[ + \int_{S_*}^{\infty} f''(K)(S_T - K)^+ \, dK. \]
Taking $f(F_T) = \ln F_T$ and take $S^*$ to be $F_0$, we have

$$\ln \frac{F_T}{F_0} = \frac{F_T - F_0}{F_0} - \int_0^{F_0} \frac{(K - F_T)^+}{K^2} dK - \int_{F_0}^{\infty} \frac{(F_T - K)^+}{K^2} dK.$$ 

Combining the results, the total variance is

$$\text{var}_T = 2 \left[ \int_0^T \frac{dF_t}{F_t} - \frac{F_T - F_0}{F_0} + \int_0^{F_0} \frac{(K - F_T)^+}{K^2} dK + \int_{F_0}^{\infty} \frac{(F_T - K)^+}{K^2} dK \right].$$

How does CBOE construct the VIX based on traded option prices?

- Taking expectation of the second term gives $e^{rT} - 1$. As an approximation, the first two terms cancel for small values of $rT$. 
• Note that the dynamic rebalancing of \( \frac{1}{F_t} \) units of time-\( t \) forward represent the constant one-dollar exposure. The replicating cost of this dynamic position over \([0, T]\) is \( rT \), where \( r \) is the riskfree interest rate.

• The last two terms represent continuum of puts whose strikes are below \( F_0 \) and calls whose strikes are above \( F_0 \). They represent out-of-the-money options with respect to \( F_0 \). The CBOE’s choice is more natural since out-of-the-money options tend to be more liquid contracts.
In reality, we can only approximate the continuum of options by strips of out-of-the-money puts and calls, whose strikes have finite distance \( \Delta K \) apart. Let \( K_0 \) be the closest listed strike below \( F_0 \). An additional term \( \left( \frac{F_0}{K_0} - 1 \right)^2 \) is subtracted due to the adjustment compensating for the strips of options that are not centered around a strike exactly at \( F_0 \). This adjustment term becomes zero when \( F_0 = K_0 \).

The adjustment can be visualized as the correction required when the limits of integration in the two integrals of the put and call options are changed from \( F_0 \) to \( K_0 \).
The forward price of the expected realized cumulative variance is approximated by

\[ P = 2e^{rT} \left[ \sum_{0}^{K_0} \frac{\Delta K}{K^2} \text{put}_K + \sum_{K_0}^{\infty} \frac{\Delta K}{K^2} \text{call}_K \right] - \left( \frac{F_0}{K_0} - 1 \right)^2, \]

where \( e^{rT} \text{put}_K \) and \( e^{rT} \text{call}_K \) are forward prices of out-of-the-money put and call, respectively.

\[ VIX^2_t = \left\{ \frac{2}{30/365} \sum_i \frac{\Delta K_i}{K_i^2} e^{r(30/365)} Q(K_i) - \frac{1}{30/365} \left( \frac{F}{F_0} - 1 \right)^2 \right\} \times 100^2, \]

\( K_0 \) is the first strike below the forward index level \( F_0 \), \( Q(K_i) \) is the time-\( t \) out-of-the-money option with strike \( K_i \).
Plot of the VIX index (02/01/1990–02/01/2009).
3.6 Stochastic volatility models

- The daily fluctuations of the return of asset prices typically exhibit volatility clustering where large moves follow large moves and small moves follow small moves.

- Also, the distribution of asset price returns is highly peaked and fat-tailed, indicating mixtures of distribution with different variances.

- It is natural to model volatility as a random variable.

- The modeling of the stochastic behavior of volatility is more difficult since volatility is a hidden process. Although volatility is driving asset prices, it is not directly observable.
Heston’s stochastic volatility model

The asset price $S_t$ and the variance of asset price $v_t$ are assumed to follow the joint stochastic processes

$$
\begin{align*}
dS_t &= \mu S_t \, dt + \sqrt{v_t} S_t \, dZ_S \\
dv_t &= k(\bar{v} - v_t) \, dt + \eta \sqrt{v_t} \, dZ_v.
\end{align*}
$$

where the Brownian motions are correlated with $dZ_S \, dZ_v = \rho \, dt$. The variance process is seen to have a mean reversion level $\bar{v}$ and reversion speed $k$, and $\eta$ is the volatility of variance. The asset price process has the drift rate $\mu$ under the physical measure. All model parameters are assumed to be constant.

The price of an option on the underlying asset should be a function of $S, v, t$. Let $V(S, v, t; T)$ denote the price of an option with maturity date $T$. Applying Ito’s lemma, the differential $dV$ is given by

$$
\begin{align*}
dV &= \left( \frac{\partial V}{\partial t} + \frac{v}{2} S^2 \frac{\partial^2 V}{\partial S^2} + \rho \eta v S \frac{\partial^2 V}{\partial S \partial v} + \frac{\eta^2 v}{2} \frac{\partial^2 V}{\partial v^2} \right) \, dt \\
&\quad + \frac{\partial V}{\partial S} \, dS + \frac{\partial V}{\partial v} \, dv.
\end{align*}
$$
Since the price variance $v$ is not a traded security, it is necessary to include options of different maturity dates $T_1$ and $T_2$ and the underlying asset in order to construct a riskless hedged portfolio.

Let the portfolio contain $\Delta_1$ units of the option with maturity date $T_1$, $\Delta_2$ units of the option with maturity date $T_2$ and $\Delta_S$ units of the underlying asset. The value of the portfolio is given by

$$\Pi = \Delta_1 V(S, v, t; T_1) + \Delta_2 V(S, v, t; T_2) + \Delta_S S.$$  

Henceforth, we suppress the dependence of $S, v$ and $t$ when options of different maturities are referred. Suppose we write formally

$$\frac{dV(T_i)}{V(T_i)} = \mu_i dt + \sigma_i^S dZ_S + \sigma_i^v dZ_v, \quad i = 1, 2.$$
We obtain

\[
\mu_i = \frac{1}{V(T_i)} \left[ \frac{\partial V(T_i)}{\partial t} + \frac{v S^2 \partial^2 V(T_i)}{2 \partial S^2} + \rho \eta v S \frac{\partial^2 V(T_i)}{\partial S \partial v} + \frac{\eta^2 v \partial V(T_i)}{2 \partial v^2} + \mu S \frac{\partial V(T_i)}{\partial S} + k(\bar{v} - v) \frac{\partial V(T_i)}{\partial v} \right],
\]

\[
\sigma^S_i = \frac{1}{V(T_i)} \sqrt{v S \frac{\partial V(T_i)}{\partial S}}, \quad \sigma^v_i = \frac{1}{V(T_i)} \eta \sqrt{v \frac{\partial V(T_i)}{\partial v}}, \quad i = 1, 2.
\]

Since there are only two risk factors (as modeled by the two Brownian motions \(Z_S\) and \(Z_v\)) and three traded securities are available in the portfolio, it is always possible to form an instantaneously riskless portfolio. Assuming the trading strategy to be self-financing so that the change in portfolio value arises only from changes in the prices of the traded securities.
By following the “pragmatic” Black-Scholes approach of taking the units of securities held to be instantaneously constant, the differential change in the portfolio value is then given by

\[
d\Pi = \Delta_1 dV(T_1) + \Delta_2 dV(T_2) + \Delta_S dS \\
= [\Delta_1 \mu_1 V(T_1) + \Delta_2 \mu_2 V(T_2) + \Delta_S \mu S] \, dt \\
+ [\Delta_1 \sigma_1^S V(T_1) + \Delta_2 \sigma_2^S V(T_2) + \Delta_S \sqrt{v} S] \, dZ_S \\
+ [\Delta_1 \sigma_1^v V(T_1) + \Delta_2 \sigma_2^v V(T_2)] \, dZ_v.
\]

In order to cancel the stochastic terms in \(d\Pi\), we must choose \(\Delta_1\), \(\Delta_2\) and \(\Delta_S\) such that they satisfy the following pair of equations

\[
\begin{align*}
\Delta_1 \sigma_1^S V(T_1) + \Delta_2 \sigma_2^S V(T_2) + \Delta_S \sqrt{v} S &= 0 \\
\Delta_1 \sigma_1^v V(T_1) + \Delta_2 \sigma_2^v V(T_2) &= 0.
\end{align*}
\]
The portfolio now becomes instantaneously riskless. Using no-arbitrage principle, the instantaneously riskless portfolio must earn the riskless interest rate \( r \), that is,

\[
d\Pi = \left[ \Delta_1 \mu_1 V(T_1) + \Delta_2 \mu_2 V(T_2) + \Delta_S \mu S \right] dt
\]

\[
= r[\Delta_1 V(T_1) + \Delta_2 V(T_2) + \Delta_S S] dt
\]

giving the third equation for \( \Delta_1, \Delta_2 \) and \( \Delta_S \):

\[
\Delta_1 (\mu_1 - r)V(T_1) + \Delta_2 (\mu_2 - r)V(T_2) + \Delta_S (\mu - r)S = 0.
\]

We put the three linear equations for \( \Delta_1, \Delta_2 \) and \( \Delta_S \) in the following matrix form:

\[
\begin{pmatrix}
(\mu_1 - r)V(T_1) & (\mu_2 - r)V(T_2) & (\mu - r)S \\
\sigma_1^S V(T_1) & \sigma_2^S V(T_2) & \sqrt{v}S \\
\sigma_1^V V(T_1) & \sigma_2^V V(T_2) & 0
\end{pmatrix}
\begin{pmatrix}
\Delta_1 \\
\Delta_2 \\
\Delta_S
\end{pmatrix}
= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.
\]
The second and third rows are seen to be independent. Nontrivial solution to $\Delta_1, \Delta_2$ and $\Delta_S$ exist for the above homogeneous system of equations only if the first row in the above coefficient matrix can be expressed as a linear combination of the second and third rows. In this case, the coefficient matrix becomes singular. This is equivalent to the existence of a pair of multipliers $\lambda_S(S,v,t)$ and $\lambda_v(S,v,t)$ such that

$$\mu_i - r = \lambda_S \sigma_i^S + \lambda_v \sigma_i^v, \quad i = 1, 2,$$

and

$$\mu - r = \lambda_S \sqrt{v}.$$

In other words, we set the first row to be the sum of $\lambda_S$ times the second row and $\lambda_v$ times the third row. The multipliers $\lambda_S$ and $\lambda_v$ can be interpreted as the market price of risk of the asset price and variance, respectively. In general, they are functions of $S, v$ and $t$. 
Substituting the expression for $\mu_i, \sigma_i^S$ and $\sigma_i^v$, we obtain (dropping the subscript “$i$”)

$$\frac{\partial V}{\partial t} + \frac{v}{2} S^2 \frac{\partial^2 V}{\partial S^2} + \rho \eta v S \frac{\partial^2 V}{\partial S \partial v} + \frac{\eta^2 v}{2} \frac{\partial V}{\partial v^2} + r S \frac{\partial V}{\partial S}$$

$$+ \left[ k(v - \bar{v}) - \lambda_v \eta \sqrt{v} \right] \frac{\partial V}{\partial v} - rV = 0.$$ 

Interestingly, only the market price of variance risk $\lambda_v$ appears in the governing equation while the market price of asset price risk $\lambda_S$ is eliminated by the relation: $\mu - r = \lambda_S \sqrt{v}$. This is because the underlying asset is a tradeable security while the price variance is not directly tradeable, though options whose values dependent on the price variance are tradeable.
Heston made the assumption that $\lambda_v(S, v, t)$ is a constant multiple of $\sqrt{v}$ so that the coefficient of $\frac{\partial V}{\partial v}$ becomes a linear function of $v$. Without loss of generality, we may express the drift term in the form $k'(\bar{v} - v)$ for some constants $k'$ and $\bar{v}'$, where $k'$ and $\bar{v}'$ can be treated as the risk adjusted parameters for the drift of $v$.

**European futures call option**

It may be more convenient to work with the futures call option. Let $f_t$ denote the time-$t$ price of the futures on the underlying asset with expiration date $T$ and define $x_t = \ln \frac{f_t}{X}$. Let $c(x, v, \tau; X)$ denote the futures call price function, $\tau = T - t$, whose governing equation is given by

$$\frac{\partial c}{\partial \tau} = \frac{v}{2} \frac{\partial^2 c}{\partial x^2} - \frac{v}{2} \frac{\partial c}{\partial x} + \eta^2 v \frac{\partial^2 c}{\partial v^2} + \rho \eta v \frac{\partial^2 c}{\partial x \partial v} + k'(\bar{v} - v) \frac{\partial c}{\partial v}$$

with initial condition:

$$c(x, v, 0) = \max(e^x - 1, 0).$$
The futures call price function takes the form:

\[
c(x, v, \tau) = e^x G_1(x, v, \tau) - G_0(x, v, \tau),
\]

where \(G_0(x, v, \tau)\) is the risk neutral probability that the futures call option is in-the-money at expiration and \(G_1(x, v, \tau)\) is related to the risk neutral expectation of the terminal futures price given that the option expires in-the-money. The two functions \(G_j(x, v, \tau), j = 0, 1\), satisfy the following differential equations:

\[
\frac{\partial G_j}{\partial \tau} = \frac{v}{2} \frac{\partial^2 G_j}{\partial x^2} - \left(\frac{1}{2} - j\right) v \frac{\partial G_j}{\partial x} + \frac{\eta^2 v}{2} \frac{\partial^2 G_j}{\partial v^2} + \rho \eta v \frac{\partial^2 G_j}{\partial x \partial v} + k'(\bar{v}' - v) \frac{\partial G_j}{\partial v}, \quad j = 0, 1,
\]

with initial condition:

\[
G_j(x, v, 0) = 1_{\{x \geq 0\}}.
\]
Fourier transform method

Let $\hat{G}_j(m, v, \tau)$ denote the Fourier transform of $G_j(x, v, \tau)$, where

$$\hat{G}_j(m, v, \tau) = \int_{-\infty}^{\infty} e^{-imx} G_j(x, v, \tau) \, dx, \quad j = 0, 1.$$ 

The Fourier transform of the initial condition is

$$\hat{G}_j(m, v, 0) = \int_{-\infty}^{\infty} e^{-imx} G_j(x, v, 0) \, dx \quad \begin{align*}
\quad &= \int_{0}^{\infty} e^{-imx} \, dx = \frac{1}{im}, \quad j = 0, 1. 
\end{align*}$$
Taking the Fourier transform of the differential equation, we obtain

\[
\frac{\partial \hat{G}_j}{\partial \tau} = -\frac{m^2}{2}v\hat{G}_j - imv\left(\frac{1}{2} - j\right)\hat{G}_j
\]

\[
+ \frac{\eta^2}{2}v\frac{\partial^2 \hat{G}_j}{\partial v^2} + im\rho\eta v\frac{\partial \hat{G}_j}{\partial v} + k'(\bar{v}' - v)\frac{\partial \hat{G}_j}{\partial v}
\]

\[
= v\left(\alpha \hat{G}_j + \beta \frac{\partial \hat{G}_j}{\partial v} + \gamma \frac{\partial^2 \hat{G}_j}{\partial v^2}\right) + \delta \frac{\partial \hat{G}_j}{\partial v}, \quad j = 0, 1,
\]

where

\[
\alpha = -\frac{m^2}{2} - im\left(\frac{1}{2} - j\right), \quad \beta = im\rho\eta - k',
\]

\[
\gamma = \frac{\eta^2}{2}, \quad \delta = k'\bar{v}'.
\]
We seek solution of the affine form for $\hat{G}_j$ such that

$$\hat{G}_j(m, v, \tau) = \exp(A(m, \tau) + B(m, \tau)v)G_j(m, v, 0).$$

By substituting the above assumed form into the differential equation, we obtain

$$\frac{\partial B}{\partial \tau} = \alpha + \beta B + \gamma B^2 = \gamma(B - \rho_+)(B - \rho_-)$$
$$\frac{\partial A}{\partial \tau} = \delta B$$

with $B(m, 0) = 0$ and $A(m, 0) = 0$. Here, $\rho_{\pm} = \frac{-\beta \pm \sqrt{\beta^2 - 4\alpha \gamma}}{2\gamma}$.

Writing

$$\rho = \rho_- / \rho_+ \quad \text{and} \quad \xi = \sqrt{\beta^2 - 4\alpha \gamma},$$

the solution to $B(m, \tau)$ and $A(m, \tau)$ are found to be

$$B(m, \tau) = \rho_- \frac{1 - e^{-\xi \tau}}{1 - \rho e^{-\xi \tau}}$$
$$A(m, \tau) = \delta \left(\rho_- \tau - \frac{2}{\eta^2} \ln \frac{1 - \rho e^{-\xi \tau}}{1 - \rho} \right).$$
From the Fourier inversion theorem, we have

\[ F(x) = \frac{1}{2} - \frac{1}{\pi} \int_0^\infty \text{Re} \left[ \frac{e^{-i\phi x} f(\phi)}{i\phi} \right] d\phi, \]

\[ \text{Pr} \left[ X > x \right] = 1 - F(x) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \text{Re} \left[ \frac{e^{-i\phi x} f(\phi)}{i\phi} \right] d\phi. \]

The solution to \( G_j(x, v, \tau) \) is obtained by taking the Fourier inversion of \( \hat{G}_j(m, v, \tau) \), giving

\[ G_j(x, v, \tau) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \text{Re} \left( \frac{\exp(imx + A(m, \tau) + B(m, \tau)v)}{im} \right) dm, \quad j = 0, 1. \]