1. Consider a portfolio containing $\Delta$ units of asset and $M$ dollars of riskless asset in the form of money market account. The portfolio is dynamically adjusted so as to replicate an option. Let $S$ and $V(S, t)$ denote the value of the underlying asset and the option, respectively. Let $r$ denote the riskless interest rate and $\Pi$ denote the value of the self-financing replicating portfolio. When the self-financing trading strategy is adopted, explain why

$$\Pi = \Delta S + M$$

and

$$d\Pi = \Delta dS + rM dt,$$

where $r$ is the riskless interest rate. Here, the differential term $S d\Delta$ does not enter into $d\Pi$. Assume that the asset price dynamics follows the Geometric Brownian process:

$$\frac{dS}{S} = \rho \, dt + \sigma \, dZ.$$

Using the condition that the option value and the value of the replicating portfolio should match at all times, show that the number of units of asset held must be given by

$$\Delta = \frac{\partial V}{\partial S}.$$

How to proceed further in order to obtain the Black-Scholes equation for $V$?

2. Let the dynamics of the stochastic state variable $S_t$ be governed by the Ito process

$$dS_t = \mu(S_t, t) \, dt + \sigma(S_t, t) \, dZ_t.$$

For a twice differentiable function $f(S_t)$, the differential of $f(S_t)$ is given by

$$df = \left[ \mu(S_t, t) \frac{\partial f}{\partial S_t} + \frac{\sigma^2(S_t, t)}{2} \frac{\partial^2 f}{\partial S_t^2} \right] dt + \sigma(S_t, t) \frac{\partial f}{\partial S_t} dZ_t.$$

We let $\psi(S_t; S_0, t_0)$ denote the transition density function of $S_t$ at the future time $t$, conditional on the value $S_0$ at an earlier time $t_0$. By considering the time-derivative of the expected value of $f(S_t)$ and equating $d dt E[f(S_t)]$ and $E \left[ \frac{df(S_t)}{dt} \right]$, where

$$E \left[ \frac{df(S_t)}{dt} \right] = \int_{-\infty}^{\infty} \psi(\xi; S_0, t_0) \left[ \mu(\xi, t) \frac{\partial f}{\partial \xi} + \frac{\sigma^2(\xi, t)}{2} \frac{\partial^2 f}{\partial \xi^2} \right] \psi(\xi, t; S_0, t_0) \, d\xi,$$

we have

$$\frac{d}{dt} E[f(S_t)] = \int_{-\infty}^{\infty} f(\xi) \frac{\partial \psi}{\partial t}(\xi; S_0, t_0) \, d\xi \quad \text{(i)}$$

and

$$E \left[ \frac{df(S_t)}{dt} \right] = \int_{-\infty}^{\infty} \left[ \mu(\xi, t) \frac{\partial f}{\partial \xi} + \frac{\sigma^2(\xi, t)}{2} \frac{\partial^2 f}{\partial \xi^2} \right] \psi(\xi, t; S_0, t_0) \, d\xi, \quad \text{(ii)}$$
show that $\psi(S_t, t; S_0, t_0)$ is governed by the following forward Fokker-Planck equation:

$$\frac{\partial \psi}{\partial t} + \frac{\partial}{\partial S_t} [\mu(S_t, t)] \psi - \frac{\partial^2}{\partial S_t^2} \left[ \frac{\sigma^2(S_t, t)}{2} \psi \right] = 0.$$ 

**Hint:** Perform parts integration of the integral in Eq. (i).

3. From the Black-Scholes price function $c(S, \tau)$ for a European vanilla call, show that the limiting values of the call price at vanishing volatility and infinite volatility are the lower and upper bounds of the European call price respectively, namely,

$$\lim_{\sigma \to 0^+} c(S, \tau) = \max(S - X e^{-r\tau}, 0),$$

and

$$\lim_{\sigma \to \infty} c(S, \tau) = S.$$

Give an appropriate financial interpretation of the above results. Apparently, $X$ does not appear in $c(S, \tau)$ when $\sigma \to \infty$. Is it justifiable from financial intuition?

4. Suppose the dividends and interest incomes are taxed at the rate $R$ but capital gains taxes are zero. Find the price formulas for the European put and call on an asset which pays a continuous dividend yield at the constant rate $q$, assuming that the riskless interest rate $r$ is also constant.

**Hint:** Explain why the riskless interest rate $r$ and dividend yield $q$ should be replaced by $r(1 - R)$ and $q(1 - R)$, respectively, in the Black-Scholes formulas.

5. Consider a futures on an underlying asset which pays $N$ discrete dividends between $t$ and $T$ and let $D_i$ denote the amount of the $i$th dividend paid on the ex-dividend date $t_i$. Show that the futures price is given by

$$F(S, t) = S e^{r(T-t)} - \sum_{i=1}^{N} D_i e^{r(T-t_i)},$$

where $S$ is the current asset price and $r$ is the riskless interest rate. Consider a European call option on the above futures. Show that the governing differential equation for the price of the call, $c_F(F, t)$, is given by (Brenner et al., 1985)

$$\frac{\partial c_F}{\partial t} + \frac{\sigma^2}{2} \left[ F + \sum_{i=1}^{N} D_i e^{r(T-t_i)} \right]^2 \frac{\partial^2 c_F}{\partial F^2} - r c_F = 0.$$
6. A forward start option is an option which comes into existence at some future time $T_1$ and expires at $T_2$ ($T_2 > T_1$). The strike price is set equal the asset price at $T_1$ such that the option is at-the-money at the future option’s initiation time $T_1$. Consider a forward start call option whose underlying asset has value $S$ at current time $t$ and constant dividend yield $q$, show that the value of the forward start call is given by

$$e^{-qt_1}c(S, T_2 - T_1; S)$$

where $c(S, T_2 - T_1; S)$ is the value of an at-the-money call (strike price same as asset price) with time to expiry $T_2 - T_1$.

*Hint:* The value of an at-the-money call option is proportional to the asset price.

7. Consider a European capped call option whose terminal payoff function is given by

$$c_M(S, 0; X, M) = \min(\max(S - X, 0), M),$$

where $X$ is the strike price and $M$ is the cap. Show that the value of the European capped call is given by

$$c_M(S, \tau; X, M) = c(S, \tau; X) - c(S, \tau; X + M),$$

where $c(S, \tau; X + M)$ is the value of a European vanilla call with strike price $X + M$.

8. Consider the value of a European call option written by an issuer whose only asset is $\alpha (< 1)$ units of the underlying asset. At expiration, the terminal payoff of this call is then given by

$$S_T - X \quad \text{if} \quad \alpha S_T \geq S_T - X \geq 0$$
$$\alpha S_T \quad \text{if} \quad S_T - X > \alpha S_T$$

and zero otherwise. Show that the value of this European call option is given by (Johnson and Stulz, 1987)

$$c_L(S, \tau; X, \alpha) = c(S, \tau; X) - (1 - \alpha)c \left( S, \tau; \frac{X}{1 - \alpha} \right), \quad \alpha < 1,$$

where $c \left( S, \tau; \frac{X}{1 - \alpha} \right)$ is the value of a European vanilla call with strike price $\frac{X}{1 - \alpha}$.