Let $Q$ be the martingale measure with the money market account as the numeraire and $Q^*$ denote the equivalent martingale measure where the asset price $S_t$ is used as the numeraire. Suppose $S_t$ follows the Geometric Brownian process with drift rate $r$ and volatility $\sigma$ under $Q$, where $r$ is the riskless interest rate. By using Eq. (3.2.11), show that
\[
\frac{dQ^*}{dQ} \bigg|_{\mathcal{F}_T} = \frac{S_T}{S_0} e^{-rT} = e^{-\frac{\sigma^2}{2} T + \sigma Z_T},
\]
where $Z_T$ is $Q$-Brownian. Using the Girsanov Theorem, show that
\[
Z^*_T = Z_T - \sigma T
\]
is $Q^*$-Brownian. Explain why
\[
E_{Q^*}[1_{\{S_T > X\}}] = N \left( \frac{\ln \frac{S_0}{X} + \left( r + \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}} \right),
\]
then deduce that [see Eq. (3.3.12b) in Kwok’s text]
\[
E_{Q}[S_T 1_{\{S_T > X\}}] = e^{rT} S_0 N \left( \frac{\ln \frac{S_0}{X} + \left( r + \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}} \right).
\]
Let $U_t$ be another asset whose price dynamics under $Q$ is governed by
\[
\frac{dU_t}{U_t} = r\, dt + \sigma_U\, dZ_t^U,
\]
where $dZ_t^U\, dZ_t = \rho\, dt$ and $\rho$ is the correlation coefficient. Show that
\[
Z_t^{*U} = Z_t^U - \rho \sigma_U T
\]
is a $Q^*$-Brownian process.

$Hint$: Since $dZ_t^U$ and $dZ_t$ are correlated with correlation coefficient $\rho$, we may write
\[
dZ_t^U = \rho\, dZ_t + \sqrt{1 - \rho^2}\, dZ_t^L,
\]
where $Z_t^L$ is uncorrelated with $Z_t$. 
2. Consider the exchange option which enables the holder the right but not the obligation to exchange risky asset $S_2$ for another risky asset $S_1$. Let the price dynamics of $S_1$ and $S_2$ under the risk neutral measure be governed by

$$\frac{dS_i}{S_i} = (r - q_i) dt + \sigma_i dZ_i, \quad i = 1, 2,$$

where $dZ_1 dZ_2 = \rho \, dt$. Let $V(S_1, S_2, \tau)$ denote the price function of the exchange option, whose terminal payoff takes the form

$$V(S_1, S_2, 0) = \max(S_1 - S_2, 0).$$

Show that the governing equation for $V(S_1, S_2, \tau)$ is given by

$$\frac{\partial V}{\partial \tau} = \frac{\sigma_1^2 S_1}{2} \frac{\partial^2 V}{\partial S_1^2} + \rho \sigma_1 \sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S_2} + \frac{\sigma_2^2 S_2}{2} \frac{\partial^2 V}{\partial S_2^2} + (r - q_2) S_1 \frac{\partial V}{\partial S_1} + (r - q_2) S_2 \frac{\partial V}{\partial S_2} - rV.$$

By taking $S_2$ as the numeraire and defining the similarity variables

$$x = \frac{S_1}{S_2} \text{ and } W(x, \tau) = \frac{V(S_1, S_2, \tau)}{S_2},$$

show that the governing equation for $W(x, \tau)$ becomes

$$\frac{\partial W}{\partial \tau} = \frac{\sigma_1^2}{2} x^2 \frac{\partial^2 W}{\partial x^2} + (q_2 - q_1) x \frac{\partial W}{\partial x} - q_2 W.$$

Verify that the solution to $W(x, \tau)$ is given by

$$W(x, \tau) = e^{-q_2 \tau} x N(d_1) - e^{-q_1 \tau} N(d_2)$$

where

$$d_1 = \frac{\ln \frac{S_1}{S_2} + (q_2 - q_1 + \frac{\sigma_2^2}{2}) \tau}{\sigma \sqrt{\tau}}, \quad d_2 = d_1 - \sigma \sqrt{\tau},$$

$$\sigma^2 = \sigma_1^2 - 2 \rho \sigma_1 \sigma_2 + \sigma_2^2.$$

In summary, the price function $V(S_1, S_2, \tau)$ can be expressed as

$$V(S_1, S_2, \tau) = e^{-r \tau} \left[ e^{(r - q_1) \tau} N(d_1) - e^{(r - q_2) \tau} N(d_2) \right],$$

where

$$d_1 = \frac{\ln \frac{S_1}{S_2} + \left[ (r - q_1) - (r - q_2) + \frac{\sigma_2^2}{2} \right] \tau}{\sigma \sqrt{\tau}}, \quad \sigma^2 = \sigma_1^2 - 2 \rho \sigma_1 \sigma_2 + \sigma_2^2,$$

$$d_2 = -\frac{\ln \frac{S_2}{S_1} + \left[ (r - q_2) - (r - q_1) + \frac{\sigma_1^2}{2} \right] \tau}{\sigma \sqrt{\tau}} = d_1 - \sigma \sqrt{\tau}.$$
3. Suppose the terminal payoff of an exchange rate option is \( F_T 1_{\{F_T > X\}} \). Let \( V_d(F, t) \) denote the value of the option in the domestic currency world, show that

\[
V_d(F, t) = F e^{-r_f(T-t)} E_{Q_f} [1_{\{F_T > X\}} | F_t = F] = F e^{-r_f T} N(d)
\]

where

\[
d = \frac{\ln \frac{F}{K} + (r_d - r_f + \frac{\sigma^2}{2}) \tau}{\sigma_F \sqrt{\tau}}, \quad \tau = T - t.
\]

4. Let \( F_{SU} \) denote the Singaporean currency price of one unit of US currency and \( F_{HS} \) denote the Hong Kong currency price of one unit of Singaporean currency. Suppose we assume \( F_{SU} \) to be governed by the following dynamics under the risk neutral measure \( Q_S \) in the Singaporean currency world:

\[
\frac{dF_{SU}}{F_{SU}} = (r_{SGD} - r_{USD}) dt + \sigma_{F_{SU}} dZ_{F_{SU}}^S,
\]

where \( r_{SGD} \) and \( r_{USD} \) are the Singaporean and US riskless interest rates, respectively. Similar Geometric Brownian process assumption is made for other exchange rate processes. The digital quanto option pays one US dollar at maturity if \( F_{SU} \) is above \( \alpha F_{HS} \) for some constant value \( \alpha \). Find the value of the digital quanto option in Hong Kong dollar in terms of the exchange rates, the riskless interest rates of the different currency worlds and volatility values.

5. Show that the total transaction costs in Leland’s model (Leland, 1985) increases (decreases) with the strike price \( X \) when \( X < X^* \) (\( X > X^* \)), where

\[
X^* = S e^{(r + \frac{\sigma^2}{2})(T-t)}.
\]

**Hint:** Use the result

\[
\frac{\partial}{\partial X} \left( \frac{\partial V}{\partial \sigma} \right) = \frac{S}{\sqrt{2\pi(T-t)}} \frac{d_1}{\sigma} \exp \left( -\frac{d_1^2}{2} \right).
\]

6. Suppose the transaction costs are proportional to the number of units of asset traded rather than the dollar value of the asset traded as in the original Leland’s model. Find the corresponding governing equation for the price of a derivative based on this new transaction costs assumption.

7. Let the asset price process under the physical measure \( P \) follows

\[
\frac{dS_t}{S_t} = \mu_t dt + \sigma_t d\tilde{Z}_t^1 \\
\frac{d\sigma_t}{\sigma_t} = \alpha(S_t, \sigma_t, t) dt + \beta(S_t, \sigma_t, t)(\rho d\tilde{Z}_t^1 + \sqrt{1-\rho^2} d\tilde{Z}_t^2),
\]

where \( \tilde{Z}_t^1 \) and \( \tilde{Z}_t^2 \) are uncorrelated. Let \( \lambda_S \) and \( \lambda_\beta \) be the market price of risk for the Brownian motions \( \tilde{Z}_t^1 \) and \( \tilde{Z}_t^2 \), respectively, where

\[
\lambda_S = \frac{\mu_t - r}{\sigma_t}
\]
and $\lambda_\beta$ has yet to be determined. Recall that the Radon-Nikodym derivative of the risk neutral measure $Q$ with respect to the physical measure $P$ is given by

$$\left. \frac{dQ}{dP} \right|_{\mathcal{F}_t} = \exp \left( - \int_0^t (\lambda_S^2 + \lambda_\beta^2) du + \lambda_S d\tilde{Z}_u^1 + \lambda_\beta d\tilde{Z}_u^2 \right).$$

Recall that the $P$-Brownian motions $\tilde{Z}_t^1$ and $\tilde{Z}_t^2$ are related to the $Q$-Brownian motions $Z_t^1$ and $Z_t^2$ by

$$dZ_t^1 = d\tilde{Z}_t^1 + \lambda_S \, dt \quad \text{and} \quad dZ_t^2 = d\tilde{Z}_t^2 + \lambda_\beta \, dt.$$

Show that $\lambda_\sigma = \rho \lambda_S + \sqrt{1 - \rho^2} \lambda_\beta$.

Under the physical measure $P$, suppose the price dynamics of a derivative is governed by

$$\frac{dF_t}{F_t} = \mu_F \, dt + \sigma_{FS} \, d\tilde{Z}_t^1 + \sigma_{F\sigma} \, d\tilde{Z}_t^2,$$

where

$$\sigma_{FS} = \frac{S}{F} \frac{\partial F}{\partial S} \sigma \quad \text{and} \quad \sigma_{F\sigma} = \frac{1}{F} \frac{\partial F}{\partial \sigma} \beta.$$

Show that the Sharpe ratio of the derivative is given by

$$\frac{\mu_F - r}{\sigma_F} = \frac{\sigma_{FS}}{\sigma_F} \lambda_S + \frac{\sigma_{F\sigma}}{\sigma_F} \lambda_\sigma,$$

where

$$\sigma_F^2 = \sigma_{FS}^2 + \sigma_{F\sigma}^2 + 2 \rho \sigma_{FS} \sigma_{F\sigma}.$$