MATH 5510 — Mathematical Models of Financial Derivatives

Topic 2 – Black-Scholes-Merton framework and Martingale Pricing Theory

2.1 Stochastic processes and Ito calculus

2.2 Change of measure: Girsanov’s theorem

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2.1 Stochastic processes and Ito calculus

Brownian motion

The Brownian motion with drift is a stochastic process \( \{X(t); t \geq 0\} \) with the following properties:

(i) Every increment \( X(t + s) - X(s) \) is normally distributed with mean \( \mu t \) and variance \( \sigma^2 t \); \( \mu \) and \( \sigma \) are fixed parameters.

(ii) For every \( t_1 < t_2 < \cdots < t_n \), the increments \( X(t_2) - X(t_1), \cdots, X(t_n) - X(t_{n-1}) \) are independent random variables with distributions given in (i).

(iii) \( X(0) = 0 \) and the sample paths of \( X(t) \) are continuous.

• Note that \( X(t + s) - X(s) \) is independent of the past history of the random path, that is, the knowledge of \( X(\tau) \) for \( \tau < s \) has no effect on the probability distribution for \( X(t + s) - X(s) \). This is precisely the Markovian character of the Brownian motion.
The Markovian property of an asset price process is consistent with the weak form of market efficiency, which assumes that the present value of an asset price already impounds all information in past prices. The particular path taken by the asset price to reach the present value is irrelevant.

For the particular case $\mu = 0$ and $\sigma^2 = 1$, the Brownian motion is called the standard Brownian motion (or standard Wiener process). By virtue of the normal distribution of the Brownian increment $Z(t) - Z(t_0)$, the conditional probability distribution for the standard Wiener process \{ $Z(t); t \geq 0$\} is given by

\[
P[Z(t) \leq z | Z(t_0) = z_0] = P[Z(t) - Z(t_0) \leq z - z_0]
= \frac{1}{\sqrt{2\pi(t-t_0)}} \int_{-\infty}^{z-z_0} \exp \left( -\frac{x^2}{2(t-t_0)} \right) dx
= N \left( \frac{z - z_0}{\sqrt{t-t_0}} \right).
\]
Overlapping Brownian increments

(a) \( E[Z(t)^2] = \text{var}(Z(t)) + E[Z(t)]^2 = t. \)

(b) \( E[Z(t)Z(s)] = \min(t, s). \)

To show the result in (b), we assume \( t > s \) and consider

\[
E[Z(t)Z(s)] = E[\{Z(t) - Z(s)\}Z(s) + Z(s)^2] = E[\{Z(t) - Z(s)\}Z(s)] + E[Z(s)^2].
\]

Since \( Z(t) - Z(s) \) and \( Z(s) \) are independent and both \( Z(t) - Z(s) \) and \( Z(s) \) have zero mean, so

\[
E[Z(t)Z(s)] = E[Z(s)^2] = s = \min(t, s).
\]

When \( t > s \), the correlation coefficient \( \rho \) between the two overlapping Brownian increments \( Z(t) \) and \( Z(s) \) is given by

\[
\rho = \frac{E[Z(t)Z(s)]}{\sqrt{\text{var}(Z(t))\text{var}(Z(s))}} = \frac{s}{\sqrt{st}} = \sqrt{s}t.
\]
Joint distribution of \(Z(s)\) and \(Z(t)\)

Since both \(Z(t)\) and \(Z(s)\) are normally distributed with zero mean and variance \(t\) and \(s\), respectively, the probability distribution of the overlapping Brownian increments is given by the bivariate normal distribution function.

If we define \(X_1 = Z(t)/\sqrt{t}\) and \(X_2 = Z(s)/\sqrt{s}\), then \(X_1\) and \(X_2\) become standard normal random variables. We then have

\[
P[Z(t) \leq z_t, Z(s) \leq z_s] = P[X_1 \leq z_t/\sqrt{t}, X_2 \leq z_s/\sqrt{s}] = N_2(z_t/\sqrt{t}, z_s/\sqrt{s}; \sqrt{s/t})
\]

where the bivariate normal distribution function is given by

\[
N_2(x_1, x_2; \rho) = \int_{-\infty}^{x_2} \int_{-\infty}^{x_1} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{\xi_1^2 - 2\rho\xi_1\xi_2 + \xi_2^2}{2(1-\rho^2)}\right) d\xi_1 d\xi_2.
\]
**Geometric Brownian motion**

Let $X(t)$ denote the Brownian motion with drift parameter $\mu$ and variance parameter $\sigma^2$. The stochastic process defined by

$$Y(t) = e^{X(t)} , \quad t \geq 0,$$

is called the *Geometric Brownian motion*. The value taken by $Y(t)$ is non-negative.

Since $X(t) = \ln Y(t)$ is a Brownian motion, by properties (i) and (ii), we deduce that $\ln Y(t) - \ln Y(0)$ is normally distributed with mean $\mu t$ and variance $\sigma^2 t$. For common usage, $\frac{Y(t)}{Y(0)}$ is said to be lognormally distributed.
The density function of $\frac{Y(t)}{Y(0)}$ is deduced to be

$$f_Y(y, t) = \frac{1}{y\sqrt{2\pi\sigma^2 t}} \exp\left(-\frac{(\ln y - \mu t)^2}{2\sigma^2 t}\right).$$

The mean of $Y(t)$ conditional on $Y(0) = y_0$ is found to be

$$E[Y(t)|Y(0) = y_0] = y_0 \int_0^\infty y f_Y(y, t) \, dy = y_0 \int_{-\infty}^\infty \frac{e^x}{\sqrt{2\pi\sigma^2 t}} \exp\left(-\frac{(x - \mu t)^2}{2\sigma^2 t}\right) \, dx, \quad x = \ln y,$$

$$= y_0 \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi\sigma^2 t}} \exp\left(-\frac{[x - (\mu t + \sigma^2 t)]^2 - 2\mu t\sigma^2 t - \sigma^4 t^2}{2\sigma^2 t}\right) \, dx$$

$$= y_0 \exp\left(\mu t + \frac{\sigma^2 t}{2}\right).$$
The variance of $Y(t)$ conditional on $Y(0) = y_0$ is found to be

$$\text{var}(Y(t) | Y(0) = y_0)$$

$$= y_0^2 \int_0^\infty y^2 f_Y(y, t) \, dy - \left[ y_0 \exp \left( \mu t + \frac{\sigma^2 t}{2} \right) \right]^2$$

$$= y_0^2 \left\{ \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2 t}} \exp \left( -\frac{[x - (\mu t + 2\sigma^2 t)]^2 - 4\mu t \sigma^2 t - 4\sigma^4 t^2}{2\sigma^2 t} \right) \, dx \right\}$$

$$= y_0^2 \exp(2\mu t + \sigma^2 t)[\exp(\sigma^2 t) - 1].$$
Brownian paths are seen to be non-differentiable. The non-differentiability property can be shown easily by proving the finiteness of the quadratic variation of a Brownian motion.

**Quadratic variation of Brownian motions**

Suppose we form a partition $\pi$ of the time interval $[0, T]$ by the discrete points

$$0 = t_0 < t_1 < \cdots < t_n = T,$$

and let $\delta t_{\max} = \max_{k} (t_k - t_{k-1})$. We write $\Delta t_k = t_k - t_{k-1}$, and define the corresponding quadratic variation of the standard Brownian motion $Z(t)$ by

$$Q_\pi = \sum_{k=1}^{n} [Z(t_k) - Z(t_{k-1})]^2.$$

The quadratic variation of $Z(t)$ over $[0, T]$ is given by

$$Q[0,T] = \lim_{\delta t_{\max} \to 0} Q_\pi = T.$$
First, we consider

\[
E[Q_\pi] = \sum_{k=1}^{n} E[\{Z(t_k) - Z(t_{k-1})\}^2] = \sum_{k=1}^{n} \text{var}(Z(t_k) - Z(t_{k-1}))
\]

since \(Z(t_k) - Z(t_{k-1})\) has zero mean

\[
= \text{var}(Z(t_n) - Z(t_0))
\]

since \(Z(t_k) - Z(t_{k-1}), k = 1, \ldots, n\) are independent

\[
= t_n - t_0 = T
\]

so that

\[
\lim_{\delta t_{max} \to 0} E[Q_\pi] = T.
\]
Consider
\[
\text{var}(Q_{\pi} - T) = E \left[ \sum_{k=1}^{n} \sum_{\ell=1}^{n} \{ [Z(t_k) - Z(t_{k-1})]^2 - \Delta t_k \} \{ [Z(t_{\ell}) - Z(t_{\ell-1})]^2 - \Delta t_{\ell} \} \right].
\]

Since the increments \([Z(t_k) - Z(t_{k-1})], k = 1, \cdots, n\) are independent, only those terms corresponding to \(k = \ell\) in the above series survive, so we have
\[
\text{var}(Q_{\pi} - T) = E \left[ \sum_{k=1}^{n} \{ [Z(t_k) - Z(t_{k-1})]^2 - \Delta t_k \}^2 \right] = \sum_{k=1}^{n} E \left[ \{Z(t_k) - Z(t_{k-1})\}^4 \right] - 2\Delta t_k \sum_{k=1}^{n} E \left[ \{Z(t_k) - Z(t_{k-1})\}^2 \right] + \Delta t_k^2.
\]
Since $Z(t_k) - Z(t_{k-1})$ is normally distributed with zero mean and variance $\Delta t_k$, its fourth order moment is known to be

$$E[\{Z(t_k) - Z(t_{k-1})\}^4] = 3\Delta t_k^2,$$

so

$$\text{var}(Q_\pi - T) = \sum_{k=1}^{n} [3\Delta t_k^2 - 2\Delta t_k^2 + \Delta t_k^2] = 2 \sum_{k=1}^{n} \Delta t_k^2.$$ 

In taking the limit $\delta t_{\text{max}} \to 0$, we observe that $\text{var}(Q_\pi - T) \to 0$.

By virtue of $\lim_{n \to \infty} \text{var}(Q_\pi - T) = 0$, we say that $T$ is the mean square limit of $Q_\pi$. 
Remarks

1. In general, the quadratic variation of the Brownian motion with variance rate $\sigma^2$ over the time interval $[t_1, t_2]$ is given by

$$Q_{[t_1, t_2]} = \sigma^2(t_2 - t_1).$$

2. If we write $dZ(t) = Z(t) - Z(t - dt)$, where $dt \to 0$, then we can deduce from the above calculations that

$$E[dZ(t)^2] = dt \quad \text{and} \quad \text{var}(dZ(t)^2) = 2\, dt^2.$$ 

Since $dt^2$ is a higher order infinitesimally small quantity, we may claim that the random quantity $dZ(t)^2$ converges in the mean square sense to the deterministic quantity $dt$. 
**Definition of stochastic integration**

Let $f(t)$ be an arbitrary function of $t$ and $Z(t)$ be the standard Brownian motion. First, we consider the definition of the stochastic integral $\int_0^T f(t) \, dZ(t)$ as a limit of the following partial sums (defined in the usual Riemann-Stieltjes sense):

$$\int_0^T f(t) \, dZ(t) = \lim_{n \to \infty} \sum_{k=1}^n f(\xi_k)[Z(t_k) - Z(t_{k-1})]$$

where the discrete points $0 < t_0 < t_1 < \cdots < t_n = T$ form a partition of the interval $[0, T]$ and $\xi_k$ is some immediate point between $t_{k-1}$ and $t_k$. The limit is taken in the mean square sense.
Unfortunately, the limit depends on how the immediate points are chosen. For example, suppose we take \( f(t) = Z(t) \) and choose \( \xi_k = \alpha t_k + (1 - \alpha)t_{k-1}, 0 \leq \alpha \leq 1 \), for all \( k \). We consider

\[
E \left[ \sum_{k=1}^{n} Z(\xi_k)[Z(t_k) - Z(t_{k-1})] \right]
\]

\[
= \sum_{k=1}^{n} E[Z(\xi_k)Z(t_k) - Z(\xi_k)Z(t_{k-1})]
\]

\[
= \sum_{k=1}^{n} [\min(\xi_k, t_k) - \min(\xi_k, t_{k-1})]
\]

\[
= \sum_{k=1}^{n} (\xi_k - t_{k-1}) = \alpha \sum_{k=1}^{n} (t_k - t_{k-1}) = \alpha T,
\]

so that the expected value of the stochastic integral depends on the choice of the immediate points.
• A function \( f(t) \) is said to be non-anticipative (非預見) with respect to the Brownian motion \( Z(t) \) if the value of the function at time \( t \) is determined by the path history of \( Z(t) \) up to time \( t \). We may write \( f(t) \in \mathcal{F}_t^Z \), where \( \mathcal{F}_t^Z \) is the filtration generated by \( Z(t) \).

**Examples**

1. \( f_1(t) = \begin{cases} 0 & \text{if } \max_{0 \leq s \leq t} Z(s) < 5 \\ 1 & \text{if } \max_{0 \leq s \leq t} Z(s) \geq 5 \end{cases} \)
   is non-anticipative.

2. \( f_2(t) = \begin{cases} 0 & \text{if } \max_{0 \leq s \leq 1} Z(s) < 5 \\ 1 & \text{if } \max_{0 \leq s \leq 1} Z(s) \geq 5 \end{cases} \)
   is not non-anticipative.

For \( t < 1 \), the value of \( f_2(t) \) is determined by the realization of the path of \( Z \) over \([0, 1]\).

• In finance, the investor’s action is non-anticipative in nature since he makes the investment decision before the asset prices move.
Define the stochastic integration by taking $\xi_k = t_{k-1}$ (left-hand point in each sub-interval) so that integration is taken to be non-anticipatory. The Ito definition of stochastic integral is given by

$$\int_0^T f(t) \, dZ(t) = \lim_{n \to \infty} \sum_{k=1}^n f(t_{k-1})[Z(t_k) - Z(t_{k-1})],$$

where $f(t)$ is non-anticipative with respect to $Z(t)$. A path is “sliced” into consecutive Gaussian increments, each increment is multiplied by a random variable, and these numbers are added together to give the stochastic integral.

Consider the $k^{th}$ term: $f(t_{k-1})\Delta Z_k = f(t_{k-1})[Z(t_k) - Z(t_{k-1})]$, once the history of the path up to time $t_{k-1}$ is revealed, the value of $f(t_{k-1})$ is known. The increment of the stochastic integral over $(t_{k-1}, t_k)$ conditional on the path history up to $t_{k-1}$ is Gaussian with mean zero and variance $f(t_{k-1})^2(t_k - t_{k-1})$. Since $Z(t_k) - Z(t_{k-1})$ is the forward Brownian increment beyond $t_{k-1}$, so it is independent of $f(t_{k-1})$, hence

$$E[f(t_{k-1})[Z(t_k) - Z(t_{k-1})]] = E[f(t_{k-1})]E[Z(t_k) - Z(t_{k-1})] = 0.$$
Consider the evaluation of the Ito stochastic integral \( \int_0^T Z(t) \, dZ(t) \).

A naive evaluation according to the usual integration rule gives

\[
\int_0^T Z(t) \, dZ(t) = \frac{1}{2} \int_0^T \frac{d}{dt}[Z(t)]^2 \, dt = \frac{Z(T)^2 - Z(0)^2}{2},
\]

which gives a wrong result. The correct approach is given by

\[
\int_0^T Z(t) \, dZ(t) = \lim_{n \to \infty} \sum_{k=1}^{n} Z(t_{k-1})[Z(t_k) - Z(t_{k-1})]
= \lim_{n \to \infty} \frac{1}{2} \sum_{k=1}^{n} (\{Z(t_{k-1}) + [Z(t_k) - Z(t_{k-1})]\}^2
- Z(t_{k-1})^2 - [Z(t_k) - Z(t_{k-1})]^2)
= \frac{1}{2} \lim_{n \to \infty} [Z(t_n)^2 - Z(t_0)^2]
- \frac{1}{2} \lim_{n \to \infty} \sum_{k=1}^{n} [Z(t_k) - Z(t_{k-1})]^2
= \frac{Z(T)^2 - Z(0)^2}{2} - \frac{T}{2}.
\]
Rearranging the terms,

\[ 2 \int_0^T Z(t) \, dZ(t) + \int_0^T dt = \int_0^T \frac{d}{dt} [Z(t)]^2 \, dt, \]

or in differential form,

\[ 2Z(t) \, dZ(t) + dt = d[Z(t)]^2. \]

Unlike the usual differential rule, we have the extra term \( dt \).

This comes from the finiteness of the quadratic variation of the Brownian motion. This is because \(|Z(t_k) - Z(t_{k-1})|^2\) is of order \( \Delta t_k \) and

\[ \lim_{n \to \infty} \sum_{k=1}^{n} [Z(t_k) - Z(t_{k-1})]^2 \]

remains finite on taking the limit.
Stochastic differentials

Let $\mathcal{F}_t$ be the natural filtration generated by the standard Brownian motion $Z(t)$ through the observation of the trajectory of $Z(t)$. Let $\mu(t)$ and $\sigma(t)$ be non-anticipative with respect to $Z(t)$ with
\[
\int_0^T |\mu(t)| \, dt < \infty \quad \text{and} \quad \int_0^T \sigma^2(t) \, dt < \infty \quad \text{(almost surely)}
\]
for all $T$, then the process $X(t)$ defined by
\[
X(t) = X(0) + \int_0^t \mu(s) \, ds + \int_0^t \sigma(s) \, dZ(s),
\]
is called an Ito process. The differential form of the above equation is given as
\[
dX(t) = \mu(t) \, dt + \sigma(t) \, dZ(t).
\]
Ito’s Lemma

Suppose \( f(x,t) \) is a twice continuously differentiable function and the stochastic process \( Y \) is defined by \( Y = f(X,t) \). Since \( dZ(t)^2 \) converges in the mean square sense to \( dt \), the second order term \( dX^2 \) also contributes to the differential \( dY \). The Ito formula of computing the differential of the stochastic function \( f(X,t) \) is given by

\[
dY = \left[ \frac{\partial f}{\partial t}(X,t) + \mu(t) \frac{\partial f}{\partial x}(X,t) + \frac{\sigma^2(t)}{2} \frac{\partial^2 f}{\partial x^2}(X,t) \right] dt + \sigma(t) \frac{\partial f}{\partial x}(X,t) \ dZ.
\]
Expand $\Delta Y$ by the Taylor series up to the second order terms:

$$\Delta Y = \frac{\partial f}{\partial t} \Delta t + \frac{\partial f}{\partial x} \Delta X \left( \frac{1}{2} \left( \frac{\partial^2 f}{\partial t^2} \Delta t^2 + 2 \frac{\partial^2 f}{\partial x \partial t} \Delta X \Delta t + \frac{\partial^2 f}{\partial x^2} \Delta X^2 \right) \right) + O(\Delta X^3, \Delta t^3).$$

In the limit $\Delta X \to 0$ and $\Delta t \to 0$, we apply the multiplication rules where $dZ^2 = dt, dZ dt = 0$ and $dt^2 = 0$ so that

$$dY = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dX + \sigma^2(t) \frac{\partial^2 f}{\partial x^2} dt.$$

Writing out in full in terms of $dZ$ and $dt$, we obtain the Ito formula.
Example – Governing SDE of exponential Brownian

Consider the exponential Brownian

\[ S(t) = S_0 e^{\left(r - \frac{\sigma^2}{2}\right)t + \sigma Z(t)}. \]

Suppose we write

\[ X(t) = \left(r - \frac{\sigma^2}{2}\right)t + \sigma Z(t) \]

so that

\[ dX(t) = \left(r - \frac{\sigma^2}{2}\right) dt + \sigma dZ(t) \]

\[ S(t) = S_0 e^{X(t)}. \]

Treating \( S \) as a function of the state variable \( X \) and time variable \( t \), the respective partial derivatives of \( S \) are

\[ \frac{\partial S}{\partial t} = 0, \quad \frac{\partial S}{\partial X} = S \quad \text{and} \quad \frac{\partial^2 S}{\partial X^2} = S. \]
Note that $\frac{\partial S}{\partial t} = 0$ since $S = S_0 e^X$, where $S$ contains no explicit dependence on the time variable $t$.

By the Ito lemma, we obtain

$$dS = \left(r - \frac{\sigma^2}{2} + \frac{\sigma^2}{2}\right) S \, dt + \sigma S \, dZ$$

or

$$\frac{dS}{S} = r \, dt + \sigma \, dZ.$$

Since $E[X(t)] = \left(r - \frac{\sigma^2}{2}\right)t$ and $\text{var}(X(t)) = \sigma^2 t$, the mean and variance of $\ln \frac{S(t)}{S_0}$ are found to be $\left(r - \frac{\sigma^2}{2}\right)t$ and $\sigma^2 t$, respectively.
Multi-dimensional version of Ito’s lemma

Suppose \( f(x_1, \cdots, x_n, t) \) is a multi-dimensional twice continuously differentiable function and the stochastic process \( Y_n \) is defined by

\[
Y_n = f(X_1, \cdots, X_n, t),
\]

where the process \( X_j(t) \) follows the Ito process

\[
dX_j(t) = \mu_j(t) \, dt + \sigma_j(t) \, dZ_j(t), \quad j = 1, 2, \cdots, n.
\]

The Brownian motions \( Z_j(t) \) and \( Z_k(t) \) are assumed to be correlated with correlation coefficient \( \rho_{jk} \) so that \( dZ_j \, dZ_k = \rho_{jk} \, dt \).
In a similar manner, we expand $\Delta Y_n$ up to the second order term in $\Delta X_j$:

$$\Delta Y_n = \frac{\partial f}{\partial t}(X_1, \cdots, X_n, t) \Delta t + \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(X_1, \cdots, X_n, t) \Delta X_j$$

$$+ \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial^2 f}{\partial x_j \partial x_k}(X_1, \cdots, X_n, t) \Delta X_j \Delta X_k$$

$$+ O(\Delta t \Delta X_j) + O(\Delta t^2).$$
In the limits $\Delta X_j \to 0, j = 1, 2, \cdots, n$, and $\Delta t \to 0$, we neglect the higher order terms in $O(\Delta t \Delta X_j)$ and $O(\Delta t^2)$ and observe $dX_j \, dX_k = \sigma_j(t)\sigma_k(t)\rho_{jk} \, dt$. We then obtain the following multi-dimensional version of the Ito lemma:

$$dY_n = \left[ \frac{\partial f}{\partial t}(X_1, \cdots, X_n, t) + \sum_{j=1}^{n} \mu_j(t) \frac{\partial f}{\partial x_j}(X_1, \cdots, X_n, t) \\
+ \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} \sigma_j(t)\sigma_k(t)\rho_{jk} \frac{\partial^2 f}{\partial x_j \partial x_k}(X_1, \cdots, X_n, t) \right] dt \\
+ \sum_{j=1}^{n} \sigma_j(t) \frac{\partial f}{\partial x_j}(X_1, \cdots, X_n, t) \, dZ_j.$$
Martingale property of a zero-drift Ito process

Consider an Ito process defined in integral form

\[ X(t) = X(0) + \int_0^t \mu(s) \, ds + \int_0^t \sigma(s) \, dZ(s) \]

with non-zero drift term \( \mu(t) \). We write \( \mathcal{M}(t) = \int_0^t \sigma(s) \, dZ(s) \).

Note that

\[ \mathcal{M}(T) = \mathcal{M}(t) + \int_t^T \sigma(s) \, dZ(s), \quad T > t. \]

Suppose we take the conditional expectation of \( \mathcal{M}(T) \) given the history of the Brownian path up to the time \( t \) (denoted by the operator \( E_t \)), we obtain

\[ E_t[\mathcal{M}(T)] = \mathcal{M}(t) \]

since the stochastic integral has zero conditional expectation. Hence, \( \mathcal{M}(t) \) is a martingale. However, \( X(t) \) is not a martingale if \( \mu(t) \) is non-zero.
Transition density function

Let \( X_t \) be the unrestricted zero-drift Brownian motion with variance rate \( \sigma^2 \). Write \( u(x, t) \) as the density function such that \( X_t \) falls within the interval \( \left( x - \frac{dx}{2}, x + \frac{dx}{2} \right) \) with probability \( u(x, t) \, dx \).

Assume that \( X_0 = \xi \), that is, the Brownian path starts at the position \( \xi \) at \( t = 0 \). The governing equation for \( u(x, t) \) is given by

\[
\frac{\partial u}{\partial t} = \frac{\sigma^2}{2} \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \ t > 0,
\]

with the initial condition: \( u(x, 0) = \delta(x - \xi) \).

*When a discrete random variable \( X \) assumes discrete values \( x_1, x_2, \ldots, x_n \), its probability mass function is

\[
f_X(x) = \sum_{i=1}^{n} P[X = x_i] \delta(x - x_i).
\]

Recall \( \delta(x - x_i) = \frac{d}{dx} H(x - x_i) \), where \( H(x - x_i) = \begin{cases} 1 & \text{if } x \geq x_i \\ 0 & \text{otherwise} \end{cases} \).
The solution to $u(x, t)$ is known to be

$$u(x, t) = \frac{1}{\sigma \sqrt{2\pi t}} \exp \left( -\frac{(x - \xi)^2}{2\sigma^2 t} \right).$$

This is the same as the density function of a normal random variable with mean $\xi$ and variance $\sigma^2 t$.

For a Brownian motion with variance rate $\sigma^2$ and drift rate $\mu$, the density function is

$$u(x, t) = \frac{1}{\sigma \sqrt{2\pi t}} \exp \left( -\frac{(x - \mu t - \xi)^2}{2\sigma^2 t} \right)$$

so that the mean position at time $t$ is $\xi + \mu t$. The corresponding governing equation becomes

$$\frac{\partial u}{\partial t} = -\mu \frac{\partial u}{\partial x} + \frac{\sigma^2}{2} \frac{\partial^2 u}{\partial x^2}.$$
If we let \( x = y + \mu t \), then \( y \) gives the spatial position when the frame of reference is moving at the rate \( \mu \). Say, a position at \( \xi + \mu t \) in the \( x \)-frame becomes \( \xi \) in the \( y \)-frame.

In terms of \( y \), the density function becomes

\[
    u(y, t) = \frac{1}{\sigma \sqrt{2\pi t}} \exp \left( -\frac{(y - \xi)^2}{2\sigma^2 t} \right),
\]

which gives the density function of a zero-drift Brownian motion with variance rate \( \sigma^2 \) and starting position \( \xi \) under the \( y \)-frame.

In our subsequent discussion, for simplicity of presentation, we consider Brownian motion with unit variance rate so that \( \sigma^2 = 1 \). Also, the starting position \( \xi \) is taken to be zero.
2.2 Change of measure: Girsanov theorem

Consider the standard $P$-Brownian motion $Z_P(t)$, which is known to have zero drift and unit variance rate under the measure $P$. Adding the drift term $\mu t$ to $Z_P(t)$ (here $\mu$ is taken to be constant) and writing

$$ Z_\tilde{P}(t) = Z_P(t) + \mu t, $$

then $Z_\tilde{P}(t)$ is a Brownian motion with drift rate $\mu$ under the measure $P$.

Suppose we set the position of the Brownian motion $Z_P(t)$ be $y$, so the position of $Z_\tilde{P}(t)$ is $y + \mu t = x$. The density function of $Z_P(t)$ is $\frac{1}{\sqrt{2\pi t}} \exp \left( -\frac{y^2}{2t} \right)$. If we multiply the above density function by $\exp \left( -\mu y - \frac{\mu^2 t}{2} \right)$, then the density function of $Z_\tilde{P}(t)$ in terms of $x$ is given by $\frac{1}{\sqrt{2\pi t}} \exp \left( -\frac{x^2}{2t} \right)$. 

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Radon-Nikodym derivatives

Can we modify the probability density through multiplication of \( dP \) by a factor such that \( Z_P(t) \) becomes a Brownian motion (zero drift) under the modified measure \( \tilde{P} \)?

The factor is called the Radon-Nikodym derivative \( \frac{d\tilde{P}}{dP} \). This procedure is called the change of measure from the original measure \( P \) to the new measure \( \tilde{P} \). The expectation formulas of the random variable \( X \) under \( P \) and \( \tilde{P} \) are related by

\[
E_{\tilde{P}}[X] = \int X \, d\tilde{P} = \int X \frac{d\tilde{P}}{dP} \, dP = E_P \left[ X \frac{d\tilde{P}}{dP} \right].
\]

For some fixed value of \( T \), the corresponding Radon-Nikodym derivative can be found to be

\[
\frac{d\tilde{P}}{dP} = \exp \left( -\mu Z_P(T) - \frac{\mu^2}{2T} \right).
\]
To verify the claim, it suffices to show that $Z_{\tilde{P}}(T)$ as a random variable is normal with zero mean and variance $T$ under the measure $\tilde{P}$ by looking at the corresponding moment generating function.

A random variable $X$ is normal with mean $m$ and variance $\sigma^2$ under a measure $P$ if and only if

$$E_P[\exp(\alpha X)] = \exp \left( \alpha m + \frac{\alpha^2}{2}\sigma^2 \right), \quad \text{for any real } \alpha.$$ 

Now, we consider

$$E_{\tilde{P}} \left[ \exp(\alpha Z_{\tilde{P}}(T)) \right]$$

$$= E_P \left[ \frac{d\tilde{P}}{dP} \exp(\alpha Z_P(T) + \alpha \mu T) \right]$$

$$= E_P \left[ \exp \left( (\alpha - \mu) Z_P(T) \right) \exp \left( \alpha \mu T - \frac{\mu^2}{2} T \right) \right]$$

$$= \exp \left( \frac{(\alpha - \mu)^2}{2} T + \alpha \mu T - \frac{\mu^2}{2} T \right) = \exp \left( \frac{\alpha^2}{2} T \right), \quad \text{for any real } \alpha,$$

hence $Z_{\tilde{P}}(T)$ is normal with zero mean and variance $T$ under $\tilde{P}$. 

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Is the Radon-Nikodym derivative process $\exp\left(-\mu Z_P(t) - \frac{\mu^2 t}{2}\right)$ a martingale under $P$?

For $s < t$, consider

$$E_P\left[ \exp\left(-\mu Z_P(t) - \frac{\mu^2 t}{2}\right) \mid \mathcal{F}_s \right]$$

$$= E_P\left[ \exp\left(-\mu Z_P(s) - \frac{\mu^2 s}{2}\right) \exp\left(-\mu \left( Z_P(t) - Z_P(s) \right) - \frac{\mu^2}{2} (t - s) \right) \mid \mathcal{F}_s \right]$$

$$= \exp\left(-\mu Z_P(s) - \frac{\mu^2 s}{2}\right) \exp\left(\frac{\mu^2}{2} (t - s)\right) \exp\left(-\frac{\mu^2}{2} (t - s)\right)$$

$$= \exp\left(-\mu Z_P(s) - \frac{\mu^2 s}{2}\right).$$

Referring to the solution to the SDE: $\frac{dS^*_t}{S^*_t} = \sigma \, dZ_t$, which is given by

$$S_t^* = S_0^* e^{-\frac{\sigma^2}{2} t + \sigma Z_t},$$

we can deduce that $S_t^*$ is a martingale.
Consider a stochastic process $\gamma(t)$ which satisfies the Novikov condition:

$$E[e^{\int_0^t \frac{1}{2} \gamma(s)^2 \, ds}] < \infty,$$

and consider the Radon-Nikodym derivative:

$$\frac{d\tilde{P}}{dP} = \rho(t)$$

where

$$\rho(t) = \exp \left( \int_0^t -\gamma(s) \, dZ(s) - \frac{1}{2} \int_0^t \gamma(s)^2 \, ds \right).$$

Here, $Z_P(t)$ is a Brownian motion under the measure $P$ (called $P$-Brownian process). Under the measure $\tilde{P}$, the stochastic process

$$Z_{\tilde{P}}(t) = Z_P(t) + \int_0^t \gamma(s) \, ds$$

is $\tilde{P}$-Brownian.
Feynman-Kac representation formula

Suppose the Ito process $X(t)$ is governed by the stochastic differential equation

$$dX(s) = \mu(X(s), s) \, ds + \sigma(X(s), s) \, dZ(s), \quad t \leq s \leq T,$$

with initial condition: $X(t) = x$.

Consider a twice differentiable function $F(X(t), t)$, by virtue of the Ito lemma, the differential of which is given by

$$dF = \left[ \frac{\partial F}{\partial t} + \mu(X, t) \frac{\partial F}{\partial X} + \frac{\sigma^2(X, t)}{2} \frac{\partial^2 F}{\partial X^2} \right] dt + \sigma \frac{\partial F}{\partial X} dZ.$$

Suppose $F$ satisfies the parabolic partial differential equation

$$\frac{\partial F}{\partial t} + \mu(X, t) \frac{\partial F}{\partial X} + \frac{\sigma^2(X, t)}{2} \frac{\partial^2 F}{\partial X^2} = 0$$

with terminal condition: $F(X(T), T) = h(X(T))$, then $dF$ becomes

$$dF = \sigma \frac{\partial F}{\partial X} \, dZ.$$
Supposing that \( \sigma \frac{\partial F}{\partial X} \) is non-anticipative with the Brownian motion \( Z(t) \), we can express the above stochastic differential form into the following integral form

\[
F(X(T), T) = F(X(t), t) + \int_t^T \sigma(X(u), u) \frac{\partial F}{\partial X}(X(u), u) \, dZ(u).
\]

The stochastic integral can be viewed as a sum of inhomogeneous consecutive Gaussian increments with mean zero, hence it has zero conditional expectation based on \( \mathcal{F}_t \).

By taking the conditional expectation and setting \( F(X(T), T) = h(X(T)) \), we then obtain the following Feynman-Kac representation formula

\[
F(x, t) = E_{x,t}[h(X(T))], \quad t < T,
\]

where \( F(x, t) \) satisfies the partial differential equation and \( E_{x,t} \) refers to expectation taken conditional on \( X(t) = x \).
Example

Let the dynamics of \( S_t \) under a martingale pricing measure \( Q \) be governed by

\[
dS_t = rS_t \, dt + \sigma S_t \, dZ^Q_t.
\]

Let \( V(S_t, t) \) be the price function of a financial derivative with the underlying asset price \( S_t \) and \( M_t \) be the money market account process. Given that \( V^*(S_t, t) = V(S_t, t)/M_t \) is a martingale under \( Q \), we have

\[
V^*(S, t) = E_t^S[V^*(S_T, T)].
\]

From the Feynman-kac representation theorem, we deduce that \( V^*(S, t) \) is governed by

\[
\frac{\partial V^*}{\partial t} + rS \frac{\partial V^*}{\partial S} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V^*}{\partial S^2} = 0.
\]

Incorporating discounting effect, the equation for \( V(S, t) \) becomes

\[
\frac{\partial V}{\partial t} + rS \frac{\partial V}{\partial S} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0.
\]
2.3 Riskless hedging principle and dynamic replicating strategy

Riskless hedging principle

Writer of a call option – hedges his exposure by holding certain units of the underlying asset in order to create a riskless portfolio.

In an efficient market with no riskless arbitrage opportunity, a riskless portfolio must earn its rate of return equals the riskless interest rate.

Let \( \Pi(t) \) be the value of a riskless hedged portfolio. By invoking no-arbitrage argument, we must have

\[
d\Pi(t) = r \Pi(t) dt,
\]

where \( r \) is the riskfree interest rate.
Dynamic replication strategy

How to replicate an option dynamically by a portfolio of the riskless asset in the form of money market account and the risky underlying asset?

The cost of constructing the replicating portfolio gives the fair price of an option.

Equality of market price of risk

Hedgeable securities should have the same market price of risk. Recall

\[ \lambda_S = \frac{\rho_S - r}{\sigma_S} \quad \text{and} \quad \lambda_V = \frac{\rho_V - r}{\sigma_V} \]

and \( \lambda_S = \lambda_V \) if the stock and option (both tradeable) are hedgeable with each other.
Black-Scholes’ assumptions on the financial market

(i) Trading takes place continuously in time.
(ii) The riskless interest rate $r$ is known and constant over time.
(iii) The asset pays no dividend.
(iv) There are no transaction costs in buying or selling the asset or the option, and no taxes.
(v) The assets are perfectly divisible.
(vi) There are no penalties to short selling and the full use of proceeds is permitted.
(vii) There are no arbitrage opportunities.

- The ability to construct a perfectly hedged portfolio relies on the assumption of continuous trading and continuous asset price process.
• The stochastic process of the asset price \( S_t \) is assumed to follow the Geometric Brownian process

\[
\frac{dS_t}{S_t} = \rho \, dt + \sigma \, dZ_t.
\]

• Consider a portfolio which involves short selling of one unit of a European call option and long holding of \( \Delta_t \) units of the underlying asset. The portfolio value \( \Pi(S_t, t) \) at time \( t \) is given by

\[
\Pi = -c + \Delta_t S_t,
\]

where \( c = c(S_t, t) \) denotes the call price as a function of the state variable \( S_t \) and time \( t \).

• Note that \( \Delta_t \) changes with time \( t \), reflecting the dynamic nature of hedging. Since both \( c \) and \( \Pi \) are random variables, we apply the Ito Lemma to give

\[
dc = \frac{\partial c}{\partial t} \, dt + \frac{\partial c}{\partial S_t} \, dS_t + \frac{\sigma^2}{2} S_t^2 \frac{\partial^2 c}{\partial S_t^2} \, dt.
\]
Black and Scholes assume that $\Delta_t$ is held fixed from $t$ to $t + dt$, so that the differential change in the portfolio value $\Pi$ is given by

$$
-dc + \Delta_t \, dS_t = \left( -\frac{\partial c}{\partial t} - \frac{\sigma^2}{2} S_t^2 \frac{\partial^2 c}{\partial S_t^2} \right) dt + \left( \Delta_t - \frac{\partial c}{\partial S_t} \right) dS_t
$$

$$
= \left[ -\frac{\partial c}{\partial t} - \frac{\sigma^2}{2} S_t^2 \frac{\partial^2 c}{\partial S_t^2} + \left( \Delta_t - \frac{\partial c}{\partial S_t} \right) \rho S_t \right] dt + \left( \Delta_t - \frac{\partial c}{\partial S_t} \right) \sigma S_t \, dZ_t.
$$

By taking $\Delta_t = \frac{\partial c}{\partial S_t}$, the stochastic term associated with $dZ_t$ vanishes. Also, the term involving $\rho$ also vanishes. The riskless hedged portfolio should earn the riskless rate of return. We then have

$$
d\Pi_t = r\Pi_t \, dt
$$

so that

$$
-\frac{\partial c}{\partial t} - \frac{\sigma^2}{2} S_t^2 \frac{\partial^2 c}{\partial S_t^2} = r \left( -c + S_t \frac{\partial c}{\partial S_t} \right)
$$

$$
\Leftrightarrow \frac{\partial c}{\partial t} + rS \frac{\partial c}{\partial S} + \frac{\sigma^2}{2} S_t^2 \frac{\partial^2 c}{\partial S_t^2} - rV = 0, \text{ where } c = c(S,t).
$$
Integral formulation

The financial gain on the portfolio from zero time to time $t$ is given by the following

$$G_t^\Pi = \int_0^t -dc + \int_0^t \Delta_u dS_u$$

$$= \int_0^t \left[ -\frac{\partial c}{\partial u} - \frac{\sigma^2}{2} S_u^2 \frac{\partial^2 c}{\partial S_u^2} + \left( \Delta_u - \frac{\partial c}{\partial S_u} \right) \rho S_u \right] du$$

$$+ \int_0^t \left( \Delta_u - \frac{\partial c}{\partial S_u} \right) \sigma S_u dZ_u.$$

Recall that $\Delta_u$ should observe the non-anticipative property in the stochastic integral in order that the integral is well defined.

This fits well with the financial scenario where $\Delta_u$ is held fixed over $(u, u + du)$ and the differential change on the asset position is attributed to the change in the asset price $dS_u$. 
- The stochastic component of the portfolio gain stems from the last term: \( \int_0^t \left( \Delta_u - \frac{\partial c}{\partial S_u} \right) \sigma_s \, du \). Suppose we adopt the dynamic hedging strategy by choosing \( \Delta_u = \frac{\partial c}{\partial S_u} \), at all earlier times \( u < t \), then the financial gain becomes deterministic at all times.
- Interestingly, by setting \( \Delta_u = \frac{\partial c}{\partial S_u} \), both the stochastic term and drift term disappear (one stone for two birds). The dependence of gain on \( \rho \) disappears together with the disappearance of randomness.
- By virtue of no arbitrage, the financial gain should be the same as the gain from investing on the riskfree asset (in the form of money market account) with dynamic position whose value equals \(-c + S_u \frac{\partial c}{\partial S_u}\) at time \( u \).

The deterministic gain from this dynamic position of riskless asset is given by

\[
G_t^M = \int_0^t r \left( -c + S_u \frac{\partial c}{\partial S_u} \right) \, du.
\]
By equating these two deterministic gains: \( G^\Pi_t = G^M_t \), we have

\[
G^\Pi_t = G^M_t \\
\Leftrightarrow 0 = \int_0^t \left[ \frac{\partial c}{\partial u} + \frac{\sigma^2}{2} S_u^2 \frac{\partial^2 c}{\partial S_u^2} + r \left( -c + S_u \frac{\partial c}{\partial S_u} \right) \right] du, \quad 0 < u < t,
\]

which is satisfied for any asset price \( S_u \) provided that the call price function \( c(S, t) \) always satisfies the relation

\[
\frac{\partial c}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 c}{\partial S^2} + rS \frac{\partial c}{\partial S} - rc = 0.
\]

Since the time-\( u \) value of the hedged portfolio \(-c(S_u, u) + \Delta_u S_u\) is riskless; so it is equivalent to a money market account \( M_u \) (riskfree asset). We may deduce the replication of the call option by a portfolio of underlying asset and money market account as follows:

\[
c(S_u, u) = \Delta_u S_u - M_u.
\]

The hedge ratio \( \Delta_u \) is equal to \( \frac{\partial c}{\partial S_u} \).
• The above parabolic partial differential equation is called the *Black-Scholes equation*. Note that the parameter \( \rho \), which is the expected rate of return of the asset, does not appear in the equation.

• The terminal payoff at time \( T \) of the European call with strike price \( X \) is translated into the following terminal condition:

\[
c(S, T) = \max(S - X, 0).
\]

• The option pricing model involves five parameters: \( S, T, X, r \) and \( \sigma \), all except the volatility \( \sigma \) are directly observable parameters.

• The independence of the pricing model on \( \rho \) is related to the concept of *risk neutrality*. 
Deficiencies in the model

1. Geometric Brownian process assumption of the asset price process? Actual asset price dynamics is much more complicated. Later models allow the asset price process to follow the jump-diffusion process and exhibit stochastic volatility.

2. Continuous hedging at all times
   — trading usually involves transaction costs.

3. Interest rate should be stochastic instead of deterministic.

Black and Scholes use the differential formulation of $d\Pi$ and follow the “pragmatic” approach of keeping the hedge ratio $\Delta_t$ to be instantaneously “frozen” in the next differential time interval $dt$. Mathematicians may be puzzled since the product rule in elementary calculus is not observed, where $d(\Delta_t S_t) = \Delta_t dS_t + S_t d\Delta_t$. 
Merton’s formulation – Dynamic replication strategy

\[ Q_S(t) = \text{number of units of asset} \]

\[ Q_V(t) = \text{number of units of option} \]

\[ M_S(t) = \text{dollar value of } Q_S(t) \text{ units of asset} \]

\[ M_V(t) = \text{dollar value of } Q_V(t) \text{ units of option} \]

\[ M(t) = \text{value of riskless asset invested in money market account} \]

- Construction of a self-financing and dynamically hedged portfolio containing risky asset, option and riskless asset (in the form of money market account).
• Dynamic replication: Composition is allowed to change at all times in the replication process.

• The self-financing portfolio is set up with zero initial net investment cost and no additional funds added or withdrawn afterwards.

The zero net investment condition at time \( t \) is

\[
\Pi(t) = M_S(t) + M_V(t) + M(t) = Q_S(t)S + Q_V(t)V + M(t) = 0.
\]

Using Ito’s lemma, we compute the differential of option value \( V \) as follows:

\[
dV = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} dt + \sigma S \frac{\partial V}{\partial S} dZ.
\]
Formally, we write the stochastic dynamics of $V$ as

$$\frac{dV}{V} = \rho_V \, dt + \sigma_V \, dZ$$

where

$$\rho_V = \frac{\frac{\partial V}{\partial t} + \rho_S \frac{\partial V}{\partial S} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2}}{V} \quad \text{and} \quad \sigma_V = \frac{\sigma S \frac{\partial V}{\partial S}}{V}.$$ 

$$d\Pi(t) = [Q_S(t) \, dS + Q_V(t) \, dV + rM(t) \, dt]$$

\[ + \left[ S \, dQ_S(t) + V \, dQ_V(t) + dM(t) \right] \]

zero due to self-financing trading strategy

- The additional term $rM(t) \, dt$ gives the interest amount earned from the money market account over $dt$.

- $dM(t)$ represents the change in the money market account due to the net dollar gained/lost from the sale of the underlying asset and option in the portfolio.
The instantaneous portfolio return \( d\Pi(t) \) can be expressed in terms of \( M_S(t) \) and \( M_V(t) \) as follows:

\[
\begin{align*}
    d\Pi(t) &= Q_S(t) \, dS + Q_V(t) \, dV + r \, M(t) \, dt \\
    &= M_S(t) \frac{dS}{S} + M_V(t) \frac{dV}{V} + r \, M(t) \, dt \\
    &= [(\rho - r) \, M_S(t) + (\rho_V - r) \, M_V(t)] \, dt \\
    &\quad + [\sigma M_S(t) + \sigma_V M_V(t)] \, dZ.
\end{align*}
\]

We make the self-financing portfolio to be instantaneously riskless by choosing \( M_S(t) \) and \( M_V(t) \) such that the stochastic term becomes zero.

From the relation:

\[
\sigma M_S(t) + \sigma_V M_V(t) = \sigma S Q_S(t) + \frac{\sigma S \partial V}{V} V Q_V(t) = 0,
\]

we obtain the following ratio of the units of asset and derivative to be held

\[
\frac{Q_S(t)}{Q_V(t)} = -\frac{\partial V}{\partial S}.
\]
Taking $Q_V(t) = -1$, and knowing

$$0 = \Pi(t) = -V + \Delta S + M(t)$$

we obtain

$$V = \Delta S + M(t), \text{ where } \Delta = \frac{\partial V}{\partial S}.$$

- This corresponds to the case of shorting one unit of the option. The above equation implies that the position of one unit of option can be replicated by a self-financing trading strategy using $S$ and $M(t)$, where $\Delta = \frac{\partial V}{\partial S}$.

**Numerical example**

Suppose the call option value increases by $0.3$ when the underlying asset increases $1$ in value, then $\frac{\partial V}{\partial S} \approx 0.3$. To hedge the sale of one unit of the call, the hedger holds $0.3$ units of the underlying asset so that

$$1 \times 0.3 + 0.3 \times (-1) = 0.$$
The dynamic replicating portfolio is riskless and requires no net investment, so \( d\Pi(t) = 0 \). Putting all these relations together, we obtain

\[
0 = [ (\rho - r)M_S(t) + (\rho_V - r)M_V(t) ] \, dt.
\]

Putting \( \frac{Q_S(t)}{Q_V(T)} = -\frac{\partial V}{\partial S} \), we obtain

\[
(\rho - r)S\frac{\partial V}{\partial S} = (\rho_V - r)V.
\]

Substituting \( \rho_V \) by

\[
\left[ \frac{\partial V}{\partial t} + \rho S \frac{\partial V}{\partial S} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} \right] / V,
\]

we obtain the Black-Scholes equation

\[
\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0.
\]
Alternative perspective on risk neutral valuation

From \( \rho_V = \frac{\partial V}{\partial t} + \rho S \frac{\partial V}{\partial S} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} S \frac{\partial^2 V}{\partial S^2} \), we obtain

\[
\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + \rho S \frac{\partial V}{\partial S} - \rho_V V = 0.
\]

We need to calibrate the parameters \( \rho \) and \( \rho_V \), or find some other means to avoid such nuisance.

From \((\rho - r)S \frac{\partial V}{\partial S} = (\rho_V - r)V\), by setting \( S \frac{\partial V}{\partial S} = \frac{\sigma_V}{\sigma} V \), we obtain

\[
\frac{\rho_V - r}{\sigma} = \frac{\rho - r}{\sigma_V} \Rightarrow \text{Black-Scholes equation.}
\]

\( \lambda_V \) and \( \lambda_S \) are the market price of risk of \( V \) and \( S \), respectively. For risk aversion (risk neutral) investors, they demand positive (zero) market price of risk.
Arguments of risk neutrality

• The market price of risk is the rate of excess return above $r$ per unit risk. The two hedgeable securities (option and asset) should have the same market price of risk. Apparently, the Black-Scholes equation can be obtained by setting $\rho = \rho_V = r$ (implying zero market price of risk).

• We find the price of a derivative relative to that of the underlying asset $\Rightarrow$ mathematical relationship between the prices is invariant to the risk preference.

• Be careful that the actual rate of return of the underlying asset would affect the asset price and thus indirectly affects the absolute derivative price. We simply use the convenience of risk neutrality to arrive at the mathematical relationship.
“How we came up with the option formula?” — Black (1989)

- It started with tinkering and ended with delayed recognition.
- The expected return on a warrant should depend on the risk of the warrant in the same way that a common stock’s expected return depends on its risk.
- I spent many, many days trying to find the solution to that (differential) equation. I have a PhD in applied mathematics, but had never spent much time on differential equations, so I didn’t know the standard methods used to solve problems like that. I have an A.B. in physics, but I didn’t recognize the equation as a version of the heat equation, which has well-known solutions.
Expectation representation of derivative price

Under the actual probability measure $P$, the governing pde is

$$
\frac{\partial V}{\partial t} + \rho S \frac{\partial V}{\partial S} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} - \rho V V = 0, \quad V(S,T) = h(S).
$$

By the Feynman-Kac representation, $V(S, t)$ admits the expectation representation

$$
V(S, t) = e^{-\rho V(T-t)} E_P^t [h(S_T)],
$$

when $E_P^t$ denotes the expectation under $P$ conditional on filtration $\mathcal{F}_t$.

- Option valuation can be performed in the risk neutral world by *artificially* taking the expected rate of returns of the asset and option to be $r$. We choose a pricing measure (called risk neutral measure or martingale measure) such that the expected rate of return of any risky instrument is $r$ or the discounted value has zero expected rate of return.
Suppose the governing pde is the Black-Scholes equation, where

\[
\frac{\partial V}{\partial t} + rS \frac{\partial V}{\partial S} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0,
\]

then the derivative price function admits the expectation representation

\[
V(S, t) = e^{-r(T-t)} E^t_Q[ h(S_T) ].
\]

This is simply the risk neutral valuation principle. Under the pricing (risk neutral) measure \( Q \), the dynamics of \( S_t \) is governed by

\[
\frac{dS_t}{S_t} = r \, dt + \sigma \, dZ_t^Q, \quad Z_t^Q \text{ is } Q\text{-Brownian}.
\]

For the call option, the hedge ratio \( \Delta_t \) is either given by \( e^{-\rho V(T-t)} E^t_P [ S_T 1\{S_T > X \} ] \) or \( e^{-r(T-t)} E^t_Q [ S_T 1\{S_T > X \} ] \). It can be verified that both give the same value, by virtue of the equality of market price of risk of asset and option. Suppose the investor is risk averse, a higher expected rate of asset \( \rho \) is counterbalanced by a higher value of \( \rho V \) in the discount factor.
Derivative whose underlying is a non-tradable index

What happens when the underlying is not a tradeable security?

Suppose the derivative price \( V(Q, t; T) \) is dependent on some price index \( Q \) whose dynamics is

\[
dQ_t = \mu(Q_t, t) \, dt + \sigma_Q(Q_t, t) \, dZ_t.
\]

Now, \( Q \) is not a traded security. We can only hedge two derivatives with respective maturity \( T_1 \) and \( T_2 \), whose values are dependent on \( Q \).

The portfolio value \( \Pi \) is given by

\[
\Pi = V_1(Q, t; T_1) - V_2(Q, t; T_2),
\]

where

\[
\frac{dV_i}{V_i} = \mu_V(Q, t; T_i) \, dt + \sigma_V(Q, t; T_i) \, dZ_t, \quad i = 1, 2.
\]
By Ito’s lemma:

\[
\begin{align*}
\mu_V(Q, t; T_i) &= \frac{1}{V_i} \left( \frac{\partial V_i}{\partial t} + \mu \frac{\partial V_i}{\partial Q} + \frac{\sigma_Q^2}{2} \frac{\partial^2 V_i}{\partial Q^2} \right) \\
\sigma_V(Q, t; T_i) &= \frac{\sigma_Q V_i}{V_i} \frac{\partial V_i}{\partial Q}, \quad i = 1, 2.
\end{align*}
\]

The change in portfolio value is

\[
\begin{align*}
d\Pi &= [V_1 \mu_V(Q, t; T_1) - V_2 \mu_V(Q, t; T_2)] \, dt \\
&\quad + [V_1 \sigma_V(Q, t; T_1) - V_2 \sigma_V(Q, t; T_2)] \, dZ_t.
\end{align*}
\]

Suppose \( V_1 \) and \( V_2 \) are chosen such that

\[
V_1 = \frac{\sigma_V(T_2)}{\sigma_V(T_2) - \sigma_V(T_1)} \Pi \quad \text{and} \quad V_2 = \frac{\sigma_V(T_1)}{\sigma_V(T_2) - \sigma_V(T_1)} \Pi,
\]

then the stochastic term vanishes and \( \Pi = V_1 - V_2 \) is satisfied.
Once randomness is eliminated, the riskless hedged portfolio value observes
\[
\frac{d\Pi}{\Pi} = \frac{\mu_V(T_1)\sigma_V(T_2) - \mu_V(T_2)\sigma_V(T_1)}{\sigma_V(T_2) - \sigma_V(T_1)} \sigma_V(T_1) \sigma_V(T_2) - \sigma_V(T_1) \sigma_V(T_2) dt = r dt.
\]
Rearranging, we obtain
\[
\frac{\mu_V(T_1) - r}{\sigma_V(T_1)} = \frac{\mu_V(T_2) - r}{\sigma_V(T_2)}.
\]
The relation is valid for arbitrary maturity dates $T_1$ and $T_2$. Hence,
\[
\frac{\mu_V(Q,t) - r}{\sigma_V(Q,t)} = \lambda(r,t) = \text{market price of risk of } Q.
\]
We obtain
\[
\frac{\partial V}{\partial t} + \mu \frac{\partial V}{\partial Q} + \frac{\sigma_Q^2}{2} \frac{\partial^2 V}{\partial Q^2} - rV = \lambda \sigma_Q \frac{\partial V}{\partial Q}.
\]
The governing equation for the derivative value $V = V(Q, t; T)$ becomes

$$\frac{\partial V}{\partial t} + (\mu - \lambda \sigma_Q) \frac{\partial V}{\partial Q} + \frac{\sigma_Q^2}{2} \frac{\partial^2 V}{\partial Q^2} - rV = 0,$$

where the market price of risk is involved. When the index $Q$ is non-tradeable, the drift rate in the option pricing equation is reduced by $\lambda \sigma_Q$ with respect to the actual drift rate $\mu$.

What happens when $Q$ becomes the price of a tradeable security? In this case, $V = Q$ should satisfy the above equation. This gives

$$\mu - \lambda \sigma_Q = rQ.$$

Furthermore, we set $\sigma_Q = \sigma Q$, where $\sigma$ is a constant. We recover the Black-Scholes equation

$$\frac{\partial V}{\partial t} + rQ \frac{\partial V}{\partial Q} + \frac{\sigma^2}{2} Q^2 \frac{\partial^2 V}{\partial Q^2} - rV = 0.$$
2.4 Martingale pricing theory

Continuous time securities model

- Uncertainty in the financial market is modeled by the filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, P)\), where \(\Omega\) is a sample space, \(\mathcal{F}\) is a \(\sigma\)-algebra on \(\Omega\), \(P\) is a probability measure on \((\Omega, \mathcal{F})\), \(\mathcal{F}_t\) is the filtration and \(\mathcal{F}_T = \mathcal{F}\).

- There are \(M\) risky securities whose price processes are modeled by the adapted stochastic processes \(S_m(t), m = 1, \ldots, M\). Let \(S_0(t)\) denote the money market account process.

- We define \(h_m(t)\) to be the number of units of the \(m\)th security held in the portfolio at time \(t\); \(m = 0, 1, \ldots, M\).

- The trading strategy \(H(t)\) is the vector stochastic process \((h_0(t) h_1(t) \cdots h_M(t))^T\), where \(H(t)\) is a \((M+1)\)-dimensional predictable process since the portfolio composition is determined by the investor based on the information available before time \(t\).
• The value process associated with the trading strategy $H(t)$ is defined by

$$V(t) = \sum_{m=0}^{M} h_m(t)S_m(t), \quad 0 \leq t \leq T,$$

and the gain process $G(t)$ is given by

$$G(t) = \sum_{m=0}^{M} \int_{0}^{t} h_m(u) \, dS_m(u), \quad 0 \leq t \leq T.$$

• Similar to discrete models, $H(t)$ is self-financing if and only if

$$V(t) = V(0) + G(t).$$
**Discounted price processes and gain process**

- The money market account process $S_0(t)$ grows at the riskless interest rate $r(t)$, that is,
  
  $$dS_0(t) = r(t)S_0(t) \, dt.$$  
  
- The discounted security price process $S_m^*(t)$ is defined as
  
  $$S_m^*(t) = S_m(t)/S_0(t), \quad m = 1, 2, \ldots, M.$$  
  
- The discounted value process $V^*(t)$ is defined by dividing $V(t)$ by $S_0(t)$. The discounted gain process $G^*(t)$ is defined by
  
  $$G^*(t) = V^*(t) - V^*(0).$$
Arbitrage and equivalent martingale measure

- A self-financing trading strategy $H$ represents an arbitrage opportunity if and only if (i) $G^*(T) \geq 0$ and (ii) $E_P G^*(T) > 0$ where $P$ is the actual probability measure of the states of occurrence associated with the securities model.

- A probability measure $Q$ on the space $(\Omega, \mathcal{F})$ is said to be an equivalent martingale measure if it satisfies

  (i) $Q$ is equivalent to $P$, that is, both $P$ and $Q$ have the same null set;

  (ii) the discounted security price processes $S^*_m(t), m = 1, 2, \cdots, M$ are martingales under $Q$, that is,

  \[ E_Q[S^*_m(u)|\mathcal{F}_t] = S^*_m(t), \quad \text{for all } 0 \leq t \leq u \leq T. \]

**Remark** The restriction on trading strategies based on “no arbitrage” is not sufficient for the existence of an equivalent martingale measure. One common choice of a sufficient condition is “no free lunch with vanishing risk”.

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existence of an equivalent martingale measure $\Rightarrow$ absence of arbitrage

- Assume that an equivalent martingale measure exists and $H$ is a self-financing strategy under $P$ so it is also self-financing under $Q$.

- The time-$t$ discounted value $V^*(t)$ of the portfolio generated by $H$ is $Q$-martingale so that $V^*(0) = E_Q[V^*(T)]$.

- We start with $V(0) = V^*(0) = 0$, and suppose we claim that $V^*(T) \geq 0$ with strict inequality for some states of the world. Since $Q(\omega) > 0$, we have $E_Q[V^*(T)] > 0$. However, by virtue of the martingale property, we have $E_Q[V^*(T)] = V^*(0) = 0$, a contradiction.

In conclusion, starting with $V^*(0) = 0$, it is impossible to have "$V^*(T) \geq 0$ and $V^*(T)$ is strictly positive for some states". Hence, there cannot exist any arbitrage opportunities.
Contingent claims are modeled as $\mathcal{F}_T$-measurable random variables. A contingent claim is said to be attainable if there exists at least an admissible trading strategy $H$ such that $V(T) = Y$. A self-financing trading strategy is said to be $Q$-admissible if the discounted gain process $G^*(t)$ is a $Q$-martingale.

**Theorem**

Assume that an equivalent martingale measure $Q$ exists. Let $Y$ be an attainable contingent claim generated by a $Q$-admissible self-financing trading strategy $H$. Then for each time $t, 0 \leq t \leq T$, the arbitrage price of $Y$ is given by

$$V(t; H) = S_0(t)E_Q \left[ \frac{Y}{S_0(T)} \right| \mathcal{F}_t] .$$
Risk neutral valuation formula

The validity of the Theorem is readily seen if we consider the discounted value process $V^*(t; H)$ to be a martingale under $Q$. This leads to

$$V(t; H) = S_0(t)V^*(t; H) = S_0(t)E_Q[V^*(T; H)|\mathcal{F}_t].$$

Furthermore, by observing that the time-$T$ value of the discounted replicating portfolio $V^*(T; H)$ is equal to $Y/S_0(T)$, the risk neutral valuation formula then follows.
Black-Scholes model revisited

The price processes of $S(t)$ and $M(t)$ in the physical measure $P$ are

$$
\frac{dS(t)}{S(t)} = \rho \, dt + \sigma \, dZ(t)$$
$$dM(t) = rM(t) \, dt.
$$

The discounted price process $S^*(t) = S(t)/M(t)$ becomes

$$
\frac{dS^*(t)}{S^*(t)} = (\rho - r)dt + \sigma \, dZ(t).
$$

Compared to $S(t)$, the expected rate of return of $S^*(t)$ is reduced by $r$ while the level of randomness remains unchanged. We would like to find the equivalent martingale measure $Q$ such that the discounted asset price $S^*$ is $Q$-martingale. Let $\tilde{Z}(t)$ denote the standard Brownian motion under the probability measure $Q$. To achieve the martingale property, it is appropriate to define

$$
d\tilde{Z}(t) = dZ(t) + \frac{\rho - r}{\sigma} \, dt.
$$
The corresponding Radon-Nikodym derivative is given by

\[ \frac{dQ}{dP} = \exp \left( -\frac{\rho - r}{\sigma} Z(t) - \left( \frac{\rho - r}{\sigma} \right)^2 \frac{t}{2} \right). \]

Under the \( Q \)-measure, the process of \( S^*(t) \) now becomes

\[ \frac{dS^*(t)}{S^*(t)} = \sigma \ d\tilde{Z}(t), \]

hence \( S^*(t) \) is \( Q \)-martingale. The asset price \( S(t) \) under the \( Q \)-measure is governed by

\[ \frac{dS(t)}{S(t)} = r \ dt + \sigma \ d\tilde{Z}(t). \]

When the money market account is used as the numeraire, the corresponding equivalent martingale measure is called the \textit{risk neutral measure} and the drift rate of \( S \) under the \( Q \)-measure is called the \textit{risk neutral drift rate}. 
Summary

The arbitrage price of a derivative is given by

$$V(S, t) = e^{-r(T-t)}E^t,S_Q[h(S_T)]$$

where $E^t,S_Q$ is the expectation under the risk neutral measure $Q$ conditional on the filtration $\mathcal{F}_t$ with $S_t = S$. By the Feynman-Kac representation formula, the fact that $V(S, t)$ satisfies the pde

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

is equivalent to $V(S, t)$ admits the above expectation representation.

For the European call option whose terminal payoff is $\max(S_T - X, 0)$, the call price $c(S, t)$ is given by

$$c(S, t) = e^{-r(T-t)}E^t,S_Q[\max(S_T - X, 0)]$$

$$= e^{-r(T-t)}\{E^t,S_Q[S_T \mathbf{1}_{\{S_T \geq X\}}] - XE^t,S_Q[\mathbf{1}_{\{S_T \geq X\}}]\}.$$
Exchange rate process under domestic risk neutral measure

Consider a foreign currency option whose payoff function depends on the exchange rate $F$, which is defined to be the domestic currency price of one unit of foreign currency.

Let $M_d$ and $M_f$ denote the money market account process in the domestic market and foreign market, respectively. The processes of $M_d(t), M_f(t)$ and $F(t)$ are governed by

$$dM_d(t) = r M_d(t) \, dt, \quad dM_f(t) = r_f M_f(t) \, dt, \quad \frac{dF(t)}{F(t)} = \mu dt + \sigma \, dZ,$$

where $r$ and $r_f$ denote the riskless domestic and foreign interest rates, respectively.
We may treat the domestic money market account and the foreign money market account in domestic dollars (whose value is given by $FM_f$) as traded securities in the domestic currency world.

With reference to the domestic equivalent martingale measure, $M_d$ is used as the numeraire.

The relative price process of the foreign money market account $X(t) = F(t)M_f(t)/M_d(t)$ is governed by

$$\frac{dX(t)}{X(t)} = (r_f - r + \mu) \, dt + \sigma \, dZ.$$
With the choice of $\gamma = (r_f - r + \mu)/\sigma$ in the Girsanov Theorem, we define

$$dZ_d = dZ + \gamma \, dt,$$

where $Z_d$ is a Brownian process under $Q_d$.

Under the domestic equivalent martingale measure $Q_d$, the process of $X$ now becomes

$$\frac{dX(t)}{X(t)} = \sigma \, dZ_d$$

so that $X$ is $Q_d$-martingale.

The exchange rate process $F$ under the $Q_d$-measure is given by

$$\frac{dF(t)}{F(t)} = (r - r_f) \, dt + \sigma \, dZ_d.$$

The risk neutral drift rate of $F$ under $Q_d$ is found to be $r - r_f$. 
Summary of risk neutral pricing

Option pricing equation before Black-Scholes-Merton risk neutral pricing framework

\[
\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + \rho S \frac{\partial V}{\partial S} - \rho V V = 0
\]

where

\[dS_t = \rho dt + \sigma dZ_t\]

and

\[\frac{dV_t}{V_t} = \rho_V dt + \sigma_V dZ_t.\]

By the Feynman-Kac formula, we obtain

\[V_t = e^{-\rho V(T-t)}E_t^P[V_T],\] where \(P\) is the physical measure.

One has to estimate \(\rho\) and \(\rho_V\).
What would happen when the riskless hedging procedure is adopted?

1. When the underlying is tradeable, we obtain

\[
\frac{\rho V - r}{\sigma_V} = \frac{\rho - r}{\sigma}, \text{ same market price of risk for asset and option.}
\]

The resulting governing equation for \( V \) is given by

\[
\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0
\]

\[V_t = e^{-r(T-t)} E_t^Q [V_T],\text{ where } Q \text{ is the martingale measure.}\]

Under \( Q \), the dynamics of \( S_t \) is governed by

\[
\frac{dS_t}{S_t} = r \, dt + \sigma \, dZ_t \quad \text{or} \quad \frac{dS^*_t}{S^*_t} = \sigma \, dZ_t.
\]

Apparently, we can set \( \rho = \rho_V = r \). This is equivalent to saying the investor to be risk neutral since she demands zero excess rate of return above the risk free rate on risky instruments.
2. The underlying is non-tradeable so that 

\[
\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + (\rho - \lambda \sigma) \frac{\partial V}{\partial S} - rV = 0,
\]

where 

\[\lambda = \frac{\rho V - r}{\sigma V}.\]

Under hedgeability of two derivatives on \( S \), the drift rate is modified to \( \rho - \lambda \sigma \). When \( S \) becomes tradeable, we have \( \lambda = \frac{\rho V - r}{\rho V} = \frac{\rho - r}{\sigma} \) so \( \rho - \lambda \sigma \) becomes \( r \). This recovers the standard Black-Scholes-Merton equation.
2.5 European option pricing formulas

Recall that the Black-Scholes equation for a European vanilla call option takes the form

$$\frac{\partial c}{\partial \tau} = \frac{\sigma^2}{2} S^2 \frac{\partial^2 c}{\partial S^2} + rS \frac{\partial c}{\partial S} - rc, \quad 0 < S < \infty, \quad \tau > 0, \quad \tau = T - t.$$ 

*Initial condition (payoff at expiry)*

$$c(S, 0) = \max(S - X, 0), \quad X \text{ is the strike price.}$$

Using the transformation: $y = \ln S$ and $c(y, \tau) = e^{-r\tau} w(y, \tau)$, the Black-Scholes equation is transformed into

$$\frac{\partial w}{\partial \tau} = \frac{\sigma^2}{2} \frac{\partial^2 w}{\partial y^2} + \left( r - \frac{\sigma^2}{2} \right) \frac{\partial w}{\partial y}, \quad -\infty < y < \infty, \quad \tau > 0.$$ 

The initial condition for the model now becomes

$$w(y, 0) = \max(e^y - X, 0).$$
Green function approach

The infinite domain Green function is known to be

$$\phi(y, \tau) = \frac{1}{\sigma \sqrt{2\pi \tau}} \exp \left( -\frac{[y + (r - \frac{\sigma^2}{2})\tau]^2}{2\sigma^2\tau} \right), \quad -\infty < y < \infty. $$

Here, $\phi(y, \tau)$ satisfies the initial condition: $\lim_{\tau \to 0^+} \phi(y, \tau) = \delta(y)$, where $\delta(y)$ is the Dirac function representing a unit impulse at the origin.

The Green function can be identified as the density function of the Brownian motion with drift rate $-\left(r - \frac{\sigma^2}{2}\right)$ and variance rate $\sigma^2$. This is not surprising since

$$d \ln S_t = \left(r - \frac{\sigma^2}{2}\right) dt + \sigma dZ^Q_t. $$

The drift rate of $\ln S_t$ is $r - \frac{\sigma^2}{2}$ moving forward in time. However, the drift rate is swapped in sign when we solve the option pricing problem backward in time.
The initial condition can be expressed as

\[ w(y, 0) = \int_{-\infty}^{\infty} w(\xi, 0) \delta(y - \xi) \, d\xi, \]

so that \( w(y, 0) \) can be considered as the superposition of impulses with varying magnitude \( w(\xi, 0) \) ranging from \( \xi \to -\infty \) to \( \xi \to \infty \).

Since the Black-Scholes equation is linear, the response in position \( y \) and at time to expiry \( \tau \) due to an impulse of magnitude \( w(\xi, 0) \) in position \( \xi \) at \( \tau = 0 \) is given by \( w(\xi, 0) \phi(y - \xi, \tau) \). From the principle of superposition for a linear differential equation, the solution is obtained by summing up the responses due to these impulses.

\[
c(y, \tau) = e^{-r \tau} w(y, \tau) \\
= e^{-r \tau} \int_{-\infty}^{\infty} w(\xi, 0) \phi(y - \xi, \tau) \, d\xi \\
= e^{-r \tau} \int_{\ln X}^{\infty} \left( e^\xi - X \right) \frac{1}{\sigma \sqrt{2\pi \tau}} \\
\quad \exp \left( -\frac{[y + (r - \frac{\sigma^2}{2})\tau - \xi]^2}{2\sigma^2 \tau} \right) \, d\xi.
\]

This integral representation resembles the Feynman-Kac representation (risk neutral valuation formula).
It is relatively straightforward to show that

\[
\int_{\ln X}^{\infty} \frac{1}{\sigma \sqrt{2\pi \tau}} \exp \left( -\frac{[y + (r - \frac{\sigma^2}{2})\tau - \xi]^2}{2\sigma^2\tau} \right) d\xi
\]

\[
= N \left( \frac{y + (r - \frac{\sigma^2}{2})\tau - \ln X}{\sigma \sqrt{\tau}} \right) = N \left( \frac{\ln \frac{S}{X} + (r - \frac{\sigma^2}{2})\tau}{\sigma \sqrt{\tau}} \right), \quad y = \ln S.
\]

By performing the procedure of completing square with respect to \( \xi \), we obtain

\[
\int_{\ln X}^{\infty} e^\xi \frac{1}{\sigma \sqrt{2\pi \tau}} \exp \left( -\frac{[y + (r - \frac{\sigma^2}{2})\tau - \xi]^2}{2\sigma^2\tau} \right) d\xi
\]

\[
= \exp(y + r\tau) \int_{\ln X}^{\infty} \frac{1}{\sigma \sqrt{2\pi \tau}} \exp \left( -\frac{[y + (r + \frac{\sigma^2}{2})\tau - \xi]^2}{2\sigma^2\tau} \right) d\xi
\]

\[
= e^{r\tau} SN \left( \frac{\ln \frac{S}{X} + (r + \frac{\sigma^2}{2})\tau}{\sigma \sqrt{\tau}} \right), \quad y = \ln S.
\]
Hence, the price formula of the European call option is found to be

$$c(S, \tau) = SN(d_1) - Xe^{-r\tau}N(d_2),$$

where

$$d_1 = \frac{\ln \frac{S}{X} + (r + \frac{\sigma^2}{2})\tau}{\sigma \sqrt{\tau}}, \quad d_2 = d_1 - \sigma \sqrt{\tau}.$$  

- The initial condition is seen to be satisfied by observing that the limits of $N(d_1)$ and $N(d_2)$ tend to $N(\infty) = 1$ or $N(-\infty) = 0$, depending on $S > X$ or $S < X$.

- The boundary conditions are satisfied by observing

$$\lim_{S \to \infty} N(d_1) = \lim_{S \to \infty} N(d_2) = 1$$

and

$$\lim_{S \to 0^+} N(d_1) = \lim_{S \to 0^+} N(d_2) = 0.$$
The call value lies within the bounds

\[
\max(S - X e^{-r \tau}, 0) \leq c(S, \tau) \leq S, \quad S \geq 0, \tau \geq 0.
\]
Risk neutral transition density function

\[ c(S, \tau) = e^{-r\tau} E_Q[(S_T - X) \mathbf{1}_{\{S_T \geq X\}}] \]
\[ = e^{-r\tau} \int_0^{\infty} \max(S_T - X, 0) \psi(S_T, T; S, t) \, dS_T. \]

Under the risk neutral measure \( Q \), the asset price dynamics is

\[
\frac{dS_t}{S_t} = r \, dt + \sigma \, dZ^Q_t \Leftrightarrow d\ln S_t = \left( r - \frac{\sigma^2}{2} \right) dt + \sigma \, dZ^Q_t,
\]

\[
\ln \frac{S_T}{S} = \left( r - \frac{\sigma^2}{2} \right) \tau + \sigma Z^Q(\tau), \text{ where } \tau = T - t,
\]

so that \( \ln \frac{S_T}{S} \) is normally distributed with mean \( \left( r - \frac{\sigma^2}{2} \right) \tau \) and variance \( \sigma^2 \tau \), and \( S \) is time-\( t \) asset price. From the density function of a normal random variable, the transition density function is

\[
\psi(S_T, T; S, t) = \frac{1}{S_T \sigma \sqrt{2\pi\tau}} \exp \left( -\frac{\left( \ln \frac{S_T}{S} - \left( r - \frac{\sigma^2}{2} \right) \tau \right)^2}{2\sigma^2 \tau} \right).
\]
In terms of \( \ln S_T \) and \( \ln S_t \), the transition density \( \psi(\ln S_T, T; \ln S_t, t) \), where

\[
\psi(\ln S_T, T; \ln S_t, t) \, d \ln S_T = \psi(S_T, T; S_t, t) \, dS_T,
\]

is equivalent to the fundamental solution of \( \phi(y - \xi, \tau) \).

When we compare the price formula with the expectation representation, we can deduce that

\[
N(d_2) = E_Q[\mathbf{1}_{\{S_T \geq X\}}] = Q[S_T \geq X] \\
SN(d_1) = e^{-r\tau} E_Q[S_T \mathbf{1}_{\{S_T \geq X\}}].
\]

- \( N(d_2) \) is recognized as the probability under the risk neutral measure \( Q \) that the call expires in-the-money, so \( X e^{-r\tau} N(d_2) \) represents the present value of the risk neutral expectation of payment paid by the option holder at expiry.

- \( SN(d_1) \) is the discounted risk neutral expectation of the terminal asset price conditional on the call being in-the-money at expiry.
2.6 Continuous dividend yield models

Let $q$ denote the constant continuous dividend yield, that is, the holder receives dividend of amount equal to $qS \, dt$ within the interval $dt$. The asset price dynamics is assumed to follow the Geometric Brownian Motion

$$\frac{dS}{S} = \rho \, dt + \sigma \, dZ.$$ 

We form a riskless hedging portfolio by short selling one unit of the European call and long holding $\Delta$ units of the underlying asset. The differential of the portfolio value $\Pi$ is given by

$$d\Pi = -dc + \Delta \, dS + q\Delta S \, dt$$

$$= \left( -\frac{\partial c}{\partial t} - \frac{\sigma^2}{2} S^2 \frac{\partial^2 c}{\partial S^2} + q\Delta S \right) dt + \left( \Delta - \frac{\partial c}{\partial S} \right) dS.$$
The last term $q\Delta S \, dt$ is the wealth added to the portfolio due to the dividend payment received. By choosing $\Delta = \frac{\partial c}{\partial S}$, we obtain a riskless hedge for the portfolio. The hedged portfolio should earn the riskless interest rate.

We then have

$$d\Pi = \left(-\frac{\partial c}{\partial t} - \frac{\sigma^2}{2} S^2 \frac{\partial^2 c}{\partial S^2} + qS \frac{\partial c}{\partial S}\right) dt = r \left(-c + S \frac{\partial c}{\partial S}\right) dt,$$

which leads to

$$\frac{\partial c}{\partial \tau} = \frac{\sigma^2}{2} S^2 \frac{\partial^2 c}{\partial S^2} + (r - q)S \frac{\partial c}{\partial S} - rc, \quad \tau = T - t, \quad 0 < S < \infty, \quad \tau > 0.$$
Martingale pricing approach

Suppose all the dividend yields received are used to purchase additional units of asset, then the wealth process of holding one unit of the underlying asset initially is given by

$$\hat{S}_t = e^{qt} S_t,$$

where $e^{qt}$ represents the growth factor in the number of units. The wealth process $\hat{S}_t$ follows

$$\frac{d\hat{S}_t}{\hat{S}_t} = (\rho + q) dt + \sigma dZ.$$

We would like to find the equivalent risk neutral measure $Q$ under which the discounted wealth process $\hat{S}_t^*$ is $Q$-martingale. We choose $\gamma(t)$ in the Radon-Nikodym derivative to be

$$\gamma(t) = \frac{\rho + q - r}{\sigma}.$$
Now $\tilde{Z}$ is a Brownian process under $Q$ and
\[ d\tilde{Z} = dZ + \frac{\rho + q - r}{\sigma} \, dt. \]

Also, $\hat{S}_t^*$ becomes $Q$-martingale since
\[ \frac{d\hat{S}_t^*}{\hat{S}_t^*} = \sigma \, d\tilde{Z}. \]

The asset price $S_t$ under the equivalent risk neutral measure $Q$ becomes
\[ \frac{dS_t}{S_t} = (r - q) \, dt + \sigma \, d\tilde{Z}. \]

Hence, the risk neutral drift rate of $S_t$ is $r - q$.

**Analogy with the foreign currency options**

The continuous yield model is also applicable to options on foreign currencies where the continuous dividend yield can be considered as the yield due to the interest earned by the foreign currency at the foreign interest rate $r_f$. 
Call and put price formulas

The price of a European call option on a continuous dividend paying asset can be obtained by changing $S$ to $Se^{-q\tau}$ in the price formula since the asset price $S$ will be depleted at the rate $q$ due to payment of dividend yield. Suppose we let $\hat{S} = Se^{-q\tau}$, then the option pricing equation becomes

$$\frac{\partial c}{\partial \tau} = \sigma^2 \hat{S}^2 \frac{\partial^2 c}{\partial \hat{S}^2} + r\hat{S} \frac{\partial c}{\partial \hat{S}} - rc.$$

The European call price formula with continuous dividend yield $q$ is

$$c(S, \tau) = Se^{-q\tau}N(\hat{d}_1) - Xe^{-r\tau}N(\hat{d}_2),$$

where

$$\hat{d}_1 = \frac{\ln \frac{S}{X} + (r - q + \frac{\sigma^2}{2})\tau}{\sigma \sqrt{\tau}}, \quad \hat{d}_2 = \hat{d}_1 - \sigma \sqrt{\tau}.$$  

Alternatively, knowing that the expected rate of return of $S_t$ is $r - q$, we can deduce that

$$c(S, \tau) = e^{-r\tau}[Se^{(r-q)\tau}N(\hat{d}_1) - XN(\hat{d}_2)].$$
Similarly, the European put formula with continuous dividend yield $q$ can be deduced from the Black-Scholes put price formula to be

$$p = Xe^{-r\tau}N(-\hat{d}_2) - Se^{-q\tau}N(-\hat{d}_1).$$

The new put and call prices satisfy the *put-call parity relation*

$$p = c - Se^{-q\tau} + Xe^{-r\tau}.$$

Furthermore, the following *put-call symmetry relation* can also be deduced from the above call and put price formulas

$$c(S, \tau; X, r, q) = p(X, \tau; S, q, r).$$

That is, the put price formula can be obtained from the corresponding call price formula by interchanging $S$ with $X$ and $r$ with $q$ in the formula.
• Recall that a call option entitles its holder the right to exchange the riskless asset for the risky asset, and vice versa for a put option. The dividend yield earned from the risky asset is $q$ while that from the riskless asset is $r$.

• If we interchange the roles of the riskless asset and risky asset in a call option, the call becomes a put option, thus giving the justification for the put-call symmetry relation.

As a verification, consider

$$p(X, \tau; S, q, r) = Se^{-q\tau} N\left(-\frac{\ln \frac{X}{S} + \left(q - r - \frac{\sigma^2}{2}\right) \tau}{\sigma \sqrt{\tau}}\right)$$

$$- X e^{-r\tau} N\left(-\frac{\ln \frac{X}{S} + \left(q - r + \frac{\sigma^2}{2}\right) \tau}{\sigma \sqrt{\tau}}\right)$$

$$= Se^{-q\tau} N(\hat{d}_1) - X e^{-r\tau} N(\hat{d}_2)$$

$$= c(S, \tau; X, r, q).$$
Time dependent parameters

Suppose the model parameters become time dependent functions, the Black-Scholes equation has to be modified as follows

\[ \frac{\partial V}{\partial \tau} = \frac{\sigma^2(\tau)}{2} S^2 \frac{\partial^2 V}{\partial S^2} + [r(\tau) - q(\tau)] S \frac{\partial V}{\partial S} - r(\tau)V, \quad 0 < S < \infty, \quad \tau > 0, \]

where \( V \) is the price of the derivative security.

When we apply the following transformations: \( y = \ln S \) and \( w = e^{\int_0^\tau r(u) \, du} V \), then

\[ \frac{\partial w}{\partial \tau} = \frac{\sigma^2(\tau)}{2} \frac{\partial^2 w}{\partial y^2} + \left[ r(\tau) - q(\tau) - \frac{\sigma^2(\tau)}{2} \right] \frac{\partial w}{\partial y}. \]

Consider the following form of the fundamental solution

\[ f(y, \tau) = \frac{1}{\sqrt{2\pi s(\tau)}} \exp \left( -\frac{[y + e(\tau)]^2}{2s(\tau)} \right), \]

which satisfies the initial condition: \( f(y, 0^+) = \delta(y) \).
By direct differentiation, it can be shown that $f(y, \tau)$ satisfies the parabolic equation

$$\frac{\partial f}{\partial \tau} = \frac{1}{2} s'(\tau) \frac{\partial^2 f}{\partial y^2} + e'(\tau) \frac{\partial f}{\partial y}.$$  

Suppose we let

$$s(\tau) = \int_0^\tau \sigma^2(u) \, du$$

and

$$e(\tau) = \int_0^\tau [r(u) - q(u)] \, du - \frac{s(\tau)}{2},$$

then $f$ satisfies the same differential equation as that for $w(y, \tau)$. One can deduce that the fundamental solution is given by

$$\phi(y, \tau) = \frac{1}{\sqrt{2\pi \int_0^\tau \sigma^2(u) \, du}} \exp \left( - \frac{\{y + \int_0^\tau [r(u) - q(u) - \frac{\sigma^2(u)}{2}] \, du\}^2}{2 \int_0^\tau \sigma^2(u) \, du} \right).$$

Given the initial condition $w(y, 0)$, the solution can be expressed as

$$w(y, \tau) = \int_{-\infty}^{\infty} w(\xi, 0) \phi(y - \xi, \tau) \, d\xi.$$
Note that the time dependency of the coefficients $r(\tau), q(\tau)$ and $\sigma^2(\tau)$ will not affect the spatial integration with respect to $\xi$. We may simply make the following substitutions in the option price formulas

- $r$ is replaced by $\frac{1}{\tau} \int_0^\tau r(u) \, du$
- $q$ is replaced by $\frac{1}{\tau} \int_0^\tau q(u) \, du$
- $\sigma^2$ is replaced by $\frac{1}{\tau} \int_0^\tau \sigma^2(u) \, du$.

For example, the European call price formula is modified as follows:

\[
c = S e^{-\int_0^\tau q(u) \, du} N(\tilde{d}_1) - X e^{-\int_0^\tau r(u) \, du} N(\tilde{d}_2)
\]

where

\[
\tilde{d}_1 = \ln \frac{S}{X} + \int_0^\tau [r(u) - q(u) + \frac{\sigma^2(u)}{2}] \, du, \quad \tilde{d}_2 = \tilde{d}_1 - \sqrt{\int_0^\tau \sigma^2(u) \, du}.
\]