MATH 5510 — Mathematical Models of Financial Derivatives

Topic 4 - Financial derivatives with embedded barrier features

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4.1 Product nature of barrier options

A barrier option is either nullified, activated or exercised when the underlying asset price breaches a barrier during the life of the option.

1. An out-barrier option (or knock-out option) is one where the option is nullified prior to expiration if the underlying asset price touches the barrier. The holder of the option may be compensated by a rebate payment for the cancellation of the option. An in-barrier option (or knock-in option) is one where the option only comes in existence if the asset price crosses the in-barrier. The holder has paid the option premium up-front since there can be potential positive payoff with zero chance of negative payoff.

2. When the barrier is upstream with respect to the asset price, the barrier option is called an up-option; otherwise, it is called a down-option.
One can identify eight types of European barrier options, such as down-and-out calls, up-and-out calls, down-and-in puts, down-and-out puts, etc.

\[
\begin{align*}
\text{up} & \quad \text{in} & \quad \text{call} \\
\text{down} & \quad \text{out} & \quad \text{put}
\end{align*}
\]

*How do both buyer and writer benefit from the up-and-out call?*

- With an appropriate rebate paid upon breaching the upside barrier, this type of barrier options provide the upside exposure for option buyer but at a lower cost.

- The option writer is not exposed to unlimited liabilities when the asset price rises significantly since the liability amount is capped at the payoff of the call at the upstream barrier.
Barrier options are attractive since they give investors more flexibility to express their view on the asset price movement in the option contract design.

In general, embedded barrier feature in a derivative refers to the trigger of certain event (say, knock-out with rebate, accumulation of coupons, doubling of purchase, etc.) upon breaching of a barrier level.

_Discontinuity at the barrier (circuit breaker effect upon knock-out)_

- Pitched battles often erupt around popular knock-out barriers in currency barrier options and these add much unwanted volatility to the markets.

- George Soros once said “knock-out options relate to ordinary options the way crack relates to cocaine.”
Accumulators

- Entails the investor entering into a commitment to purchase a fixed number of shares per day at a pre-agreed price (the “Accumulator Price”). This Price is set (typically 10-20%) below the market price of the shares at initiation. This is portrayed as the “discount” to the market price of the shares.

Example

Citic Pacific entered into an Australian dollar accumulator as hedges “with a view to minimizing the currency exposure of the company’s iron ore mining project in Australia”. The company benefits from strengthening in the A$ above the exchange rate of A$1 = US$0.87.
Example of an accumulator on China Life Insurance Company

- **Stock Price Movement of China Life Insurance Company Limited (June 12, 2009 - July 13, 2009)**

![Stock Price Movement Diagram](image-url)
Decomposition of an accumulator under immediate settlement

Under the assumption of continuous monitoring of the upper knock-out barrier and immediate settlement of the accumulated stock, one can decompose an accumulator into a portfolio of up-and-out barrier call and put options. Let $K =$ strike price and $H =$ upper knock-out level, the payoff on the observation date $t_i$ is given by

$$
\begin{cases}
0 & \text{if } \max_{0 \leq \tau \leq t_i} S_\tau \geq H \\
S_{t_i} - K & \text{if } \max_{0 \leq \tau \leq t_i} S_\tau < H \text{ and } S_{t_i} \geq K \\
2(S_{t_i} - K) & \text{if } \max_{0 \leq \tau \leq t_i} S_\tau < H \text{ and } S_{t_i} < K,
\end{cases}
$$

where $\max_{0 \leq t \leq t_i} S_\tau$ signifies continuous monitoring of barrier.

- When $S_{t_i} \geq K$, the $t_i$-maturity put option is out-of-the-money and the $t_i$-maturity call option has the payoff $S_{t_i} - K$.

- When $S_{t_i} < K$, the call option is out-of-the-money and the put option becomes in-the-money with payoff $K - S_{t_i}$. The two put options are in short position, the payoff is $-2(K - S_{t_i}) = 2(S_{t_i} - K)$. 
Pricing formulas

\[ n = \text{total number of observation dates} \]
\[ c_{uo} = \text{up-and-out barrier call option} \]
\[ p_{uo} = \text{up-and-out barrier put option} \]

Fair value of an accumulator (continuous monitoring approximation) = \[ \sum_{i=1}^{n} c_{uo}(t_i; K, H) - 2p_{uo}(t_i; K, H). \]

- For the \( t_i \)-maturity call option, the payoff remains the same, independent of whether the knock-out event occurs on \( t_i \) or otherwise. This is an uncommon type of up-and-out call, where the call payoff is adopted as the rebate upon knock-out.
Delayed settlement

- To take care of the delayed delivery of the stocks, the present value of the purchase cost of each unit of stock on date $t_i$ has to be adjusted by the time value of the strike price $K$ paid on the delivery date (several business days after the ending date of the corresponding accumulation period). How to modify the corresponding barrier option price formula?

- More precisely, the underlying asset of the $t_i$-maturity knock-out option should be the forward contract with delivery price $K$ and maturity date $T_i$ ($T_i$ is a few business days after $t_i$), $i = 1, 2, \ldots, n$. 
4.2 Partial differential equation approach and method of images

Pricing formulation of a European single-asset down-and-out call (continuous monitoring of barrier)

\[
\frac{\partial c}{\partial \tau} = \frac{\sigma^2 S^2}{2} \frac{\partial^2 c}{\partial S^2} + rS \frac{\partial c}{\partial S} - rc, \quad S > B \text{ and } \tau \in (0, T],
\]

subject to

- knock-out condition: \( c(B, \tau) = R(\tau) \)
- terminal payoff: \( c(S, 0) = \max(S - X, 0) \),

Here, \( B \) is a down-barrier and \( R(\tau) \) is the time-dependent rebate. Normally, \( B \) is set to be less than \( X \); otherwise, the barrier is breached even when it is in-the-money. The rebate is set so as to avoid jump discontinuity in the payoff structure.
After applying the transformation of variable: \( y = \ln S \), the barrier becomes the vertical line \( y = \ln B \) in the \((y, \tau)\)-plane. The governing equation becomes

\[
\frac{\partial c}{\partial \tau} = \frac{\sigma^2}{2} \frac{\partial^2 c}{\partial y^2} + \left( r - \frac{\sigma^2}{2} \right) \frac{\partial c}{\partial y} - rc,
\]

defined in the semi-infinite domain: \( y > \ln B \) and \( \tau \in (0, T] \).

The boundary condition and initial condition, respectively, become

\[
c(\ln B, \tau) = R(\tau) \quad \text{and} \quad c(y, 0) = \max(e^y - X, 0) \]

Since the down-and-out barrier call option becomes a forward contract at \( S \to \infty \), the far field boundary condition is

\[
\lim_{S \to \infty} c(S, \tau) = S - X e^{-r\tau}.
\]
Recall that the density function
\[ u(x, t) = \frac{1}{\sigma \sqrt{2\pi t}} \exp\left( -\frac{(x - \mu t)^2}{2\sigma^2 t} \right) \]
satisfies
\[ \frac{\partial u}{\partial t} = \frac{\sigma^2}{2} \frac{\partial^2 u}{\partial x^2} - \mu \frac{\partial u}{\partial x} \]
with \( u(x, 0^+) = \delta(x) \).

**Green function**

Setting \( \mu = -\left(r - \frac{\sigma^2}{2}\right) \), the Green function in the infinite domain: \(-\infty < y < \infty\) is given by
\[ G_0(y, \tau; \xi) = \frac{e^{-r\tau}}{\sigma \sqrt{2\pi \tau}} \exp\left( -\frac{(y + \mu \tau - \xi)^2}{2\sigma^2 \tau} \right), \]
where \( G_0(y, \tau; \xi) \) satisfies the initial condition:
\[ \lim_{\tau \to 0^+} G_0(y, \tau; \xi) = \delta(y - \xi). \]
Method of images

Assuming that the Green function in the semi-infinite domain takes the form

\[ G(y, \tau; \xi) = G_0(y, \tau; \xi) - H(\xi)G_0(y, \tau; \eta), \quad y > \ln B, \]

we are required to determine \( H(\xi) \) and \( \eta \) (in terms of \( \xi \)) such that the zero Dirichlet boundary condition \( G(\ln B, \tau; \xi) = 0 \) is satisfied.

Note that both \( G_0(y, \tau; \xi) \) and \( H(\xi)G_0(y, \tau; \eta) \) satisfy the differential equation. Also, provided that \( \eta \notin (\ln B, \infty) \), then

\[ \lim_{\tau \to 0^+} G_0(y, \tau; \eta) = 0 \text{ for all } y > \ln B. \]
By imposing the boundary condition at \( y = \ln B \), one observes

\[
H(\xi) = \frac{G_0(\ln B, \tau; \xi)}{G_0(\ln B, \tau; \eta)} = \exp \left( \frac{(\xi - \eta)[2(\ln B + \mu \tau) - (\xi + \eta)]}{2\sigma^2 \tau} \right).
\]

The assumed form of \( G(y, \tau; \xi) \) is feasible only if the right hand side becomes a function of \( \xi \) only. This can be achieved by the judicious choice of

\[
\eta = 2 \ln B - \xi,
\]

so that

\[
H(\xi) = \exp \left( \frac{2\mu}{\sigma^2}(\xi - \ln B) \right).
\]
• This method works only if \( \mu/\sigma^2 \) is a constant, independent of \( \tau \). In other words, the method fails when the model parameters are time dependent.

• The parameter \( \eta \) lies outside \((\ln B, \infty)\). Actually, it can be visualized as the mirror image of \( \xi \) with respect to the barrier \( y = \ln B \). In engineering perspective, an image sink of magnitude \( H(\xi) \) is placed at the image point \( \eta = 2 \ln B - \xi \) so that the combination of the source of unit strength at \( \xi \) and image sink of strength \( H(\xi) \) at \( \eta \) give zero value at the barrier \( y = \ln B \).
Pictorial representation of the method of images. The mirror is placed along $y = \ln B$. 

$\eta = 2\ln B - \xi$
Once $\eta$ and $H(\xi)$ are determined, we have

$$H(\xi)G_0(y, \tau; \eta) = \exp\left(\frac{2\mu}{\sigma^2}(\xi - \ln B)\right) \frac{e^{-r\tau}}{\sigma \sqrt{2\pi\tau}} \exp\left( - \frac{[y + \mu \tau - (2 \ln B - \xi)]^2}{2\sigma^2\tau} \right)$$

$$= \left(\frac{B}{S}\right)^{2\mu/\sigma^2} \frac{e^{-r\tau}}{\sigma \sqrt{2\pi\tau}} \exp\left( - \frac{[(y - \xi) + \mu \tau - 2(y - \ln B)]^2}{2\sigma^2\tau} \right).$$

In the last expression, the scalar multiple of the Gaussian term is now independent of $\xi$ so that integration with respect to $\xi$ can be performed more effectively.

The Green function in the specified semi-infinite domain: $\ln B < y < \infty$ becomes

$$G(y, \tau; \xi) = \frac{e^{-r\tau}}{\sigma \sqrt{2\pi\tau}} \left\{ \exp\left( - \frac{(u - \mu \tau)^2}{2\sigma^2\tau} \right) - \left(\frac{B}{S}\right)^\lambda \exp\left( - \frac{(u - 2\beta - \mu \tau)^2}{2\sigma^2\tau} \right) \right\},$$

where $u = \xi - y$ and $\beta = \ln B - y = \ln \frac{B}{S}$. Also, $\lambda = \frac{2\mu}{\sigma^2} = \frac{2r}{\sigma^2} - 1 = \delta - 1$ with $\delta = \frac{2r}{\sigma^2}$. 17
Zero-rebate case

We consider the down-and-out barrier call option with zero rebate, where \( R(\tau) = 0 \), and let \( K = \max(B, X) \), so \( e^\xi - X > 0 \) when \( \xi \in (\ln K, \infty) \). The price of the zero-rebate European down-and-out call can be expressed as

\[
c_{do}(y, \tau) = \int_{\ln B}^{\infty} \max(e^\xi - X, 0) G(y, \tau; \xi) \, d\xi \\
= \int_{\ln K}^{\infty} (e^\xi - X) G(y, \tau; \xi) \, d\xi \\
= \frac{e^{-r\tau}}{\sigma \sqrt{2\pi\tau}} \int_{\ln K/S}^{\infty} (Se^u - X) \left[ \exp \left( -\frac{(u - \mu \tau)^2}{2\sigma^2\tau} \right) \right. \\
\left. - \left( \frac{B}{S} \right)^{2\mu/\sigma^2} \exp \left( -\frac{(u - 2\beta - \mu \tau)^2}{2\sigma^2\tau} \right) \right] \, du, \\
\ln B < y < \infty, \quad \tau > 0.
\]
The direct evaluation of the integral gives

\[
\begin{align*}
    c_{do}(S, \tau) &= S \left[ N(d_1) - \left( \frac{B}{S} \right)^{\delta+1} N(d_3) \right] \\
    &\quad - X e^{-r\tau} \left[ N(d_2) - \left( \frac{B}{S} \right)^{\delta-1} N(d_4) \right],
\end{align*}
\]

where

\[
\begin{align*}
    d_1 &= \frac{\ln \frac{S}{K} + \left( r + \frac{\sigma^2}{2} \right) \tau}{\sigma \sqrt{\tau}}, \\
    d_2 &= d_1 - \sigma \sqrt{\tau}, \\
    d_3 &= d_1 + \frac{2}{\sigma \sqrt{\tau}} \ln \frac{B}{S}, \\
    d_4 &= d_2 + \frac{2}{\sigma \sqrt{\tau}} \ln \frac{B}{S}, \\
    \delta &= \frac{2r}{\sigma^2}.
\end{align*}
\]
Suppose we define the modified European call price formula

\[ \tilde{c}_E(S, \tau; X, B) = SN(d_1) - Xe^{-r\tau}N(d_2), \]

then \( c_{do}(S, \tau; X, B) \) can be expressed in the following succinct form

\[ c_{do}(S, \tau; X, B) = \tilde{c}_E(S, \tau; X, B) - \left( \frac{B}{S} \right)^{\delta-1} \tilde{c}_E \left( \frac{B^2}{S}, \tau; X, B \right). \]

One can show by direct calculation that the function \( \left( \frac{B}{S} \right)^{\delta-1} \tilde{c}_E \left( \frac{B^2}{S}, \tau \right) \) satisfies the Black-Scholes equation identically. Also, we observe

\[ \tilde{c}_E \left( \frac{B^2}{S}, 0^+ \right) = 0, \quad \ln B < S < \infty. \]

The above form allows us to observe readily the satisfaction of the boundary condition: \( c_{do}(B, \tau) = 0 \), and the terminal payoff condition.
Remarks

1. Closed form analytic price formulas for barrier options with exponential time dependent barrier, \( B(\tau) = Be^{-\gamma\tau} \), can also be derived. However, when the barrier level has an arbitrary time dependence, the search for an analytic price formula for the barrier option fails.

2. Closed form price formulas for barrier options can also be obtained for other types of diffusion process followed by the underlying asset price. The types of processes include the square root constant elasticity of variance process (volatility is a power function of the stock price) and the double exponential jump diffusion process.

3. The monitoring period for breaching of the barrier may be limited to only part of the life of the option. The pricing of this type of partial barrier option is related to pricing a compound option.
4. Since the nullification of the out-option is compensated by the activation of the in-option counterpart, it is obvious that

\[ c_{di}(S, \tau; X, B) + c_{do}(S, \tau; X, B) = c_E(S, \tau; X), \]
valid for either \( B < X \) or \( B \geq X \). Assuming \( B < X \), so that \( K = \max(B, X) = X \), the price of a down-and-in call option can be deduced to take the following simple form:

\[ c_{di}(S, \tau; X, B) = \left( \frac{B}{S} \right)^{\delta-1} c_E \left( \frac{B^2}{S}, \tau; X \right). \]

5. With a rebate \( B(\tau) \) paid upon knock-out at \( S = B \), the value of the rebate provision is given by

\[
\int_0^\tau e^{-ru} \frac{\ln \frac{S}{B}}{\sqrt{2\pi \sigma}} \frac{u^{3/2}}{u^{3/2}} \exp \left( -\frac{\ln \frac{S}{B} + \left( r - \frac{\sigma^2}{2} \right) u}{2\sigma^2 u} \right) R(\tau - u) \, du,
\]
where \( u \) is the time lapsed from the current time.
4.3 Probabilistic approach: density functions of restricted Brownian motions and first passage time density functions

Realized extremum value of the asset price process

The realized maximum and minimum value of the asset price process from time zero to time $t$ (under continuous monitoring) are defined by

\[ m^t_0 = \min_{0 \leq u \leq t} S_u \]
\[ M^t_0 = \max_{0 \leq u \leq t} S_u, \]

respectively. The terminal payoffs of the various types of barrier options can be expressed in terms of $m^T_0$ and $M^T_0$. For example, consider the down-and-out call and up-and-out put with barrier $B$ (downstream or upstream), their respective terminal payoff can be expressed as

\[ c_{do}(S_T, T; X, B) = \max(S_T - X, 0) \mathbf{1}_{\{m^T_0 > B\}} \]
\[ p_{uo}(S_T, T; X, B) = \max(X - S_T, 0) \mathbf{1}_{\{M^T_0 < B\}}. \]
First passage time

Suppose $B$ is the down-barrier, we define $\tau_B$ to be the stopping time at which the underlying asset price crosses the barrier and enters into the down-region (stopping event) for the first time:

$$\tau_B = \inf\{t | S_t \leq B\}, \quad S_0 = S.$$ 

Assume $S > B$ and asset price path continuity, we may express $\tau_B$ (commonly called the first passage time) as

$$\tau_B = \inf\{t | S_t = B\}.$$ 

In a similar manner, if $B$ is the up-barrier and $S < B$, we have

$$\tau_B = \inf\{t | S_t \geq B\} = \inf\{t | S_t = B\}.$$ 

• A random variable $\tau : \Omega \to [0, \infty)$ is called a $\mathcal{F}_t$-stopping time if $\{\tau \leq t\} \in \mathcal{F}_t$ for all $t \in [0, \infty)$. That is, it is possible to decide whether $\{\tau \leq t\}$ has occurred on the basis of knowledge of $\mathcal{F}_t$. 
Expectation representation of a European down-and-out call

Assuming $S > B$, it is easily seen that $\{\tau_B > T\}$ and $\{m_T^0 > B\}$ are equivalent events if $B$ is a down-barrier. By virtue of the risk neutral valuation principle, the price of a down-and-out call at time zero is given by

$$c_{do}(S, 0; X, B) = e^{-rT}E_Q[\max(S_T - X, 0)1_{\{m_T^0 > B\}}]$$

$$= e^{-rT}E_Q[\max(S_T - X, 0)\mathbf{1}_{S_T > X} \mathbf{1}_{\tau_B > T}].$$

The determination of the price function $c_{do}(S, 0; X, B)$ requires the determination of the joint density function of $S_T$ and $m_T^0$. 
Reflection principle

Let $W^0_t (W^\mu_t)$ denote the Brownian motion that starts at zero, with constant volatility $\sigma$ and zero drift rate (constant drift rate $\mu$). We would like to find $P[m^T_0 < m, W^\mu_T > x]$, where $x \geq m$ and $m \leq 0$.

Zero-drift Brownian motion $W^0_t$

Given that the minimum value $m^T_0$ falls below $m$, then there exists some time instant $\xi, 0 < \xi < T$, such that $\xi$ is the first time that $W^0_\xi$ equals $m$. Here, $\xi$ is seen to be the first passage time to the down-barrier $m$. As Brownian paths are continuous, there exist some times during which $W^0_t < m$. In other words, $W^0_t$ decreases at least below $m$ and then increases at least up to level $x$ (higher than or equal to $m$) at time $T$. 
Pictorial representation of the reflection principle of the Brownian motion $W_t^0$. The dotted path after the stopping time $\xi$ is the mirror reflection of the Brownian path at the level $m$. Suppose $W_T^0$ ends up at a value higher than $x$, then the reflected path $\tilde{W}_T^0$ at time $T$ has a value lower than $2m - x$. 
Suppose we define the random process

$$\widetilde{W}_t^0 = \begin{cases} W_t^0 & \text{for } t < \xi \\ 2m - W_t^0 & \text{for } \xi \leq t \leq T, \end{cases}$$

that is, $\widetilde{W}_t^0$ is the mirror reflection of $W_t^0$ at the level $m$ within the time interval between $\xi$ and $T$.

- Note that $W_t^0$ is $\mathcal{F}_t$-Brownian and the first passage time $\xi$ is a $\mathcal{F}_t$-stopping time. The strong Markov property of a Brownian motion states that for each stopping time $\xi$, the increment $W_{\xi+u}^0 - W_\xi^0$, $u \geq 0$, is a Brownian motion that is independent of the path history from time zero up to $\xi$.

- Though the stopping time $\xi$ depends on the path history $\{W_t^0 : 0 \leq t \leq \xi\}$, it will not affect the Brownian motion at later times. The reflection of the Brownian path dictates that

$$\widetilde{W}_{\xi+u}^0 - \widetilde{W}_\xi^0 = -(W_{\xi+u}^0 - W_\xi^0), \quad u > 0.$$
By the strong Markov property of Brownian motions, the two Brownian increments have the same distribution, and the distribution has zero mean and variance $\sigma^2 u$. In other words, for every Brownian path that starts at 0, travels at least $m$ units (downward, $m \leq 0$) before $T$ and later travels at least $x - m$ units (upward, $x \geq m$), there is an equally likely path that starts at 0, travels $m$ units (downward, $m \leq 0$) some time before $T$ and travels at least $m - x$ units (further downward, $m \leq x$).

Hence, $\{W_0^T > x\} \cap \{m_0^T < m\}$ is equivalent to $\{\tilde{W}_0^T < 2m - x\}$. Equivalently, we claim that the two events $\{W_0^T > x\} \cap \{m_0^T < m\}$ and $\{2m - W_0^T > x\}$ are equal in probability. We then have

$$P[W_0^T > x, m_0^T < m] = P[2m - W_0^T > x]$$
$$= P[W_0^T < 2m - x] \quad \text{since } W_0^T \text{ has zero drift}$$
$$= N\left(\frac{2m - x}{\sigma \sqrt{T}}\right), \quad m \leq \min(x, 0).$$
Non-zero drift Brownian motion $W_t^\mu$

We apply the Girsanov Theorem to effect the change of measure for finding the above joint distribution when the Brownian motion has non-zero drift.

Suppose under the measure $Q$, $W_t^\mu$ is a Brownian motion with variance rate $\sigma^2$ and drift rate $\mu$. We change the measure from $Q$ to $\tilde{Q}$ such that $W_t^\mu$ becomes a Brownian motion with variance rate $\sigma^2$ and zero drift under $\tilde{Q}$. As an illustration, we consider

$$P[W_T^\mu < y] = E_Q[1_{\{W_T^\mu < y\}}] = E_{\tilde{Q}} \left[ 1_{\{W_T^\mu < y\}} \exp \left( \frac{\mu W_T^\mu}{\sigma^2} - \frac{\mu^2 T}{2\sigma^2} \right) \right]$$

$$= \int_{-\infty}^y \frac{1}{\sqrt{2\pi \sigma^2 T}} e^{-\frac{z^2}{2\sigma^2 T}} e^{\frac{\mu z}{\sigma^2} e^{-\frac{\mu^2 T}{2\sigma^2}}} dz$$

$$= \int_{-\infty}^y \frac{1}{\sqrt{2\pi \sigma^2 T}} e^{-\frac{(z-\mu T)^2}{2\sigma^2 T}} dz = N \left( \frac{y - \mu T}{\sigma \sqrt{T}} \right).$$
• Note that the Radon-Nikodym derivative: \( \exp \left( \frac{\mu W_T^\mu}{\sigma^2} - \frac{\mu^2 T}{2\sigma^2} \right) \) is appended in transforming from \( Q \) to \( \tilde{Q} \). Also, the density function of \( W_T^\mu \) under \( \tilde{Q} \) is given by
\[
\tilde{Q}[W_T^\mu \in dz] = \frac{1}{\sqrt{2\pi \sigma^2 T}} e^{-z^2 / 2\sigma^2 T} dz.
\]

• When the \( \mu \)-drift Brownian motion \( W_T^\mu \) does not go beyond \( y \), the zero-drift Brownian motion \( W_t^0 \) does not go beyond \( y - \mu T \). This intuition gives
\[
P[W_T^\mu < y] = N \left( \frac{y - \mu T}{\sigma \sqrt{T}} \right).
\]

• In order that we can apply the reflection principle that is applicable under the zero-drift case, we perform all expectation calculations under \( \tilde{Q} \) whereby \( W_t^\mu \) becomes a zero-drift Brownian motion.
Recall that the two events \( \{ W^0_T > x \} \cap \{ m_0^T < m \} \) and \( \{ 2m - W^0_T > x \} \) are equal in distribution. We transform from \( Q \) to \( \tilde{Q} \) by appending 
\[
\exp \left( \frac{\mu}{\sigma^2} (2m - W^\mu_T) - \frac{\mu^2 T}{2\sigma^2} \right)
\]
under which \( 2m - W^\mu_T \) becomes zero-drift Brownian motion. Also, \( W^\mu_T \) is a Brownian motion with zero-drift under \( \tilde{Q} \). For \( m \leq \min \{ x, 0 \} \), we then have

\[
P[W^\mu_T > x, m_0^T < m] = E_{\tilde{Q}} \left[ 1_{\{ 2m - W^\mu_T > x \}} \exp \left( \frac{\mu}{\sigma^2} (2m - W^\mu_T) - \frac{\mu^2 T}{2\sigma^2} \right) \right]
\]

\[
= e^{\frac{2\mu m}{\sigma^2}} E_{\tilde{Q}} \left[ 1_{\{ W^\mu_T < 2m - x \}} \exp \left( -\frac{\mu}{\sigma^2} W^\mu_T - \frac{\mu^2 T}{2\sigma^2} \right) \right]
\]

\[
= e^{\frac{2\mu m}{\sigma^2}} \int_{-\infty}^{2m-x} \frac{1}{\sqrt{2\pi\sigma^2 T}} \exp \left( -\frac{z^2}{2\sigma^2 T} - \frac{\mu z}{\sigma^2} - \frac{\mu^2 T}{2\sigma^2} \right) dz
\]

\[
= e^{\frac{2\mu m}{\sigma^2}} \int_{-\infty}^{2m-x} \frac{1}{\sqrt{2\pi\sigma^2 T}} \exp \left( -\frac{(z + \mu T)^2}{2\sigma^2 T} \right) dz
\]

\[
= e^{\frac{2\mu m}{\sigma^2}} N \left( \frac{2m - x + \mu T}{\sigma\sqrt{T}} \right).
\]
Consider the restricted Brownian motion $W_t^\mu$ that has a downstream barrier $m$ over the period $[0, T]$ so that $m^T_0 > m$. Given that $W_t^\mu$ does not breach the barrier $m$, we would like to derive the joint distribution

$$P[W_T^\mu > x, m^T_0 > m], \quad \text{and} \quad m \leq \min(x, 0).$$

By applying the law of total probabilities, we obtain

$$P[W_T^\mu > x, m^T_0 > m] = P[W_T^\mu > x] - P[W_T^\mu > x, m^T_0 < m] = N \left( \frac{-x + \mu T}{\sigma \sqrt{T}} \right) - e^{\frac{2\mu m}{\sigma^2}} N \left( \frac{2m - x + \mu T}{\sigma \sqrt{T}} \right), \quad m \leq \min(x, 0). \quad (A)$$

By setting $m = x$, and since $W_T^\mu > m$ is implicitly implied from $m^T_0 > m$, we obtain the following distribution function for $m^T_0$:

$$P[m^T_0 > m] = N \left( \frac{-m + \mu T}{\sigma \sqrt{T}} \right) - e^{\frac{2\mu m}{\sigma^2}} N \left( \frac{m + \mu T}{\sigma \sqrt{T}} \right).$$
Extension to upstream barrier

Consider the restricted Brownian motion $W^\mu_t$ that has an upstream barrier $M$ over the period $[0,T]$ so that $M_0^T < M$, the joint distribution function of $W^\mu_T$ and $M_0^T$ can be deduced using the following relation between $M_0^T$ and $m_0^T$:

$$M_0^T = \max_{0 \leq t \leq T} (\sigma Z_t + \mu t) = -\min_{0 \leq t \leq T} (-\sigma Z_t - \mu t),$$

where $Z_t$ is the standard Brownian motion. Since $-Z_t$ has the same distribution as $Z_t$, the distribution of the maximum value of $W^\mu_t$ is the same as that of the negative of the minimum value of $W^{-\mu}_t$. 
By swapping $-\mu$ for $\mu$, $-M$ for $m$ and $-y$ for $x$, we obtain

$$P[-W_T^\mu > -y, -M_0^T < -M] = P[W_T^\mu < y, M_0^T > M] = e^{\frac{2\mu M}{\sigma^2}} N \left( \frac{y - 2M - \mu T}{\sigma \sqrt{T}} \right), \quad M \geq \max(y, 0).$$

In a similar manner, we obtain

$$P[W_T^\mu < y, M_0^T < M] = \text{P}[W_T^\mu < y] - P[W_T^\mu < y, M_0^T > M] = N \left( \frac{y - \mu T}{\sigma \sqrt{T}} \right) - e^{\frac{2\mu M}{\sigma^2}} N \left( \frac{y - 2M - \mu T}{\sigma \sqrt{T}} \right), \quad M \geq \max(y, 0). \quad \text{(B)}$$

Lastly, by setting $y = M$, we obtain the following distribution function for $M_0^T$:

$$P[M_0^T < M] = N \left( \frac{M - \mu T}{\sigma \sqrt{T}} \right) - e^{\frac{2\mu M}{\sigma^2}} N \left( -\frac{M + \mu T}{\sigma \sqrt{T}} \right).$$
Density function of a restricted Brownian motion with one-sided downstream barrier

We define $f_{\text{down}}(x, m, T)$ to be the density function of $W_T^\mu$ with the downstream barrier $m$, where $m \leq \min(x, 0)$, that is,

$$f_{\text{down}}(x, m, T) \, dx = P[W_T^\mu \in dx, m_0^T > m].$$

By differentiating eq. (A) with respect to $x$ and swapping the sign, we obtain

$$f_{\text{down}}(x, m, T) = \frac{1}{\sigma \sqrt{T}} \left[ n \left( \frac{x - \mu T}{\sigma \sqrt{T}} \right) - e^{\frac{2\mu m}{\sigma^2}} n \left( \frac{x - 2m - \mu T}{\sigma \sqrt{T}} \right) \right] 1_{\{m \leq \min(x, 0)\}}.$$
**Extension to upstream barrier**

Similarly, we define $f_{up}(x, M, T)$ to be the density function of $W_T^\mu$ with the upstream barrier $M$, where $M > \max(y, 0)$. By differentiating eq. (B) with respect to $y$, we obtain

$$
P[W_T^\mu \in dy, M_0^T < M] = f_{up}(y, M, T) dy
$$

$$
= \frac{1}{\sigma \sqrt{T}} \left[ n \left( \frac{y - \mu T}{\sigma \sqrt{T}} \right) - e^{\frac{2\mu M}{\sigma^2}} n \left( \frac{y - 2M - \mu T}{\sigma \sqrt{T}} \right) \right] dy \mathbf{1}_{\{M \geq \max(y, 0)\}}.
$$
Transition density function of a restricted Geometric Brownian motion with downstream barrier

Suppose the asset price $S_t$ follows the Geometric Brownian motion under the risk neutral measure such that $\ln \frac{S_t}{S} = W_t^\mu$, where $S$ is the asset price at time zero and the drift rate $\mu = r - \frac{\sigma^2}{2}$. Let $\psi(S_T; S, B)$ denote the transition density of the asset price $S_T$ at time $T$ given the asset price $S$ at time zero and conditional on $S_t > B$ for $0 \leq t \leq T$. Here, $B$ is the downstream barrier. From the density function $f_{\text{down}}(x, m, T)$, we deduce that $\psi(S_T; S, B)$ is given by

$$
\psi(S_T; S, B) = \frac{1}{\sigma \sqrt{TS_T}} \left[ n \left( \ln \frac{S_T}{S} - \left( r - \frac{\sigma^2}{2} \right) T \right) \right]
$$

$$
- \left( \frac{B}{S} \right)^{2r - 1} \left( \ln \frac{S_T}{S} - 2 \ln \frac{B}{S} - \left( r - \frac{\sigma^2}{2} \right) T \right)
$$

$$
\times n \left( \frac{\ln \frac{S_T}{S} - \left( r - \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}} \right).
$$
First passage time density function of a Brownian motion

Let \( Q(u; m) \) denote the density function of the first passage time at which the downstream barrier \( m \) is first hit by the Brownian path \( W^\mu_t \), that is, \( Q(u; m) \, du = P[\tau_m \in du] \).

We determine the distribution function \( P[\tau_m > u] \) by observing that \( \{\tau_m > u\} \) and \( \{m^u_0 > m\} \) are equivalent events. This gives

\[
P[\tau_m > u] = P[m^u_0 > m] = N\left(\frac{-m + \mu u}{\sigma \sqrt{u}}\right) - e^{\frac{2\mu m}{\sigma^2}} N\left(\frac{m + \mu u}{\sigma \sqrt{u}}\right).
\]
The first passage time density function $Q(u; m)$ associated with the downstream barrier is then given by

$$
Q(u; m) \, du = P[\tau_m \in du]
= -\frac{\partial}{\partial u} \left[ N \left( \frac{-m + \mu u}{\sigma \sqrt{u}} \right) - e^{\frac{2\mu m}{\sigma^2}} N \left( \frac{m + \mu u}{\sigma \sqrt{u}} \right) \right] \, du \, 1_{\{m<0\}}
= \frac{-m}{\sqrt{2\pi\sigma^2 u^3}} \exp \left( -\frac{(m - \mu u)^2}{2\sigma^2 u} \right) \, du \, 1_{\{m<0\}}.
$$

Let $Q(u; M)$ denote the first passage time density associated with the upstream barrier $M$. In a similar manner, we obtain

$$
Q(u; M) = -\frac{\partial}{\partial u} \left[ N \left( \frac{M - \mu u}{\sigma \sqrt{u}} \right) - e^{\frac{2\mu M}{\sigma^2}} N \left( -\frac{M + \mu u}{\sigma \sqrt{u}} \right) \right] 1_{\{M>0\}}
= \frac{M}{\sqrt{2\pi\sigma^2 u^3}} \exp \left( -\frac{(M - \mu u)^2}{2\sigma^2 u} \right) 1_{\{M>0\}}.
$$
Now, we consider $\ln \frac{S_t}{S}$ to be a Brownian motion with drift $r - \frac{\sigma^2}{2}$. We write $B$ as the option barrier, either upstream or downstream. The normalized barrier under the Brownian motion is $\ln \frac{B}{S}$. When the barrier is downstream (upstream), we have $\ln \frac{B}{S} < 0 \left( \ln \frac{B}{S} > 0 \right)$. The combined first passage time density function is given by

$$Q(u; B) = \frac{|\ln \frac{B}{S}|}{\sqrt{2\pi \sigma^2 u^3}} \exp \left( - \frac{\left[ \ln \frac{B}{S} - \left( r - \frac{\sigma^2}{2} \right) u \right]^2}{2\sigma^2 u} \right).$$

Suppose a rebate $R(t)$ is paid to the option holder upon breaching the barrier at level $B$ by the asset price path at time $t$. Since the expected rebate payment over the time interval $[u, u + du]$ is given by $R(u)Q(u; B) \, du$, so the expected present value of the rebate is given by

$$\text{rebate value} = \int_0^T e^{-ru} R(u)Q(u; B) \, du.$$
When \( R(t) = R_0 \), a constant value, direct integration of the above integral gives

\[
\text{rebate value} = R_0 \left[ \left( \frac{B}{S} \right)^{\alpha^+} N \left( \delta \frac{\ln \frac{B}{S} + \beta T}{\sigma \sqrt{T}} \right) + \left( \frac{B}{S} \right)^{\alpha^-} N \left( \delta \frac{\ln \frac{B}{S} - \beta T}{\sigma \sqrt{T}} \right) \right],
\]

where

\[
\beta = \sqrt{\left( r - \frac{\sigma^2}{2} \right)^2 + 2r\sigma^2}, \quad \alpha_\pm = \frac{r - \frac{\sigma^2}{2}}{\sigma^2} \pm \beta,
\]

\[
\delta = \text{sign} \left( \ln \frac{S}{B} \right).
\]

Here, \( \delta \) is a binary variable indicating whether the barrier is downstream (\( \delta = 1 \)) or upstream (\( \delta = -1 \)).
Two-sided barriers

We take the initial position $X_0 = 0$. Let $g(x, t; \ell, u)$ denote the density function of the restricted Brownian motion $X_t$ with two-sided absorbing barriers at $x = \ell$ and $x = u$, where the barriers are positioned such that $\ell < 0 < u$.

Recall that $X_t = \ln \frac{S_t}{S}$, and if $L$ and $U$ are the absorbing barriers of the asset price process $S_t$, respectively, then $\ell = \ln \frac{L}{S}$ and $u = \ln \frac{U}{S}$.

The partial differential equation formulation for $g(x, t; \ell, u)$ is given by (see Problem 3.8 in Kwok’s text)

$$
\frac{\partial g}{\partial t} = -\mu \frac{\partial g}{\partial x} + \frac{\sigma^2}{2} \frac{\partial^2 g}{\partial x^2}, \quad \ell < x < u, \quad t > 0,
$$

with the homogeneous boundary conditions:

$$
g(\ell, t) = g(u, t) = 0 \quad \text{and} \quad g(x, 0^+) = \delta(x).
$$
Both $x = \ell$ and $x = u$ are the absorbing barriers (equivalent to say “particles are removed from the system once these barriers are hit”), so the probability of staying at each of these barriers is zero.

Defining the transformation

$$g(x, t) = e^{\mu x \sigma^2 - \frac{\mu^2 t}{2 \sigma^2}} \hat{g}(x, t),$$

we observe that $\hat{g}(x, t)$ satisfies the forward Fokker-Planck equation with zero drift:

$$\frac{\partial \hat{g}}{\partial t}(x, t) = \frac{\sigma^2}{2} \frac{\partial^2 \hat{g}}{\partial x^2}(x, t).$$

Note that the factor $e^{\mu x \sigma^2 - \frac{\mu^2 t}{2 \sigma^2}}$ resembles the Radon-Nikodym derivative: $\exp \left( \frac{\mu W_t^\mu}{\sigma^2} - \frac{\mu^2 t}{2 \sigma^2} \right)$. 
The auxiliary conditions for $\tilde{g}(x,t)$ are seen to remain the same as those for $g(x,t)$. Without the barriers, the infinite-domain fundamental solution to the above equation is known to be

$$\phi(x,t) = \frac{1}{\sqrt{2\pi\sigma^2 t}} \exp\left(-\frac{x^2}{2\sigma^2 t}\right).$$

Like the one-sided barrier case, we try to add extra terms to the above solution such that the homogeneous boundary conditions at $x=\ell$ and $x=u$ are satisfied.
Method of images revisited

We attempt to add the pair of negative terms $-\phi(x - 2\ell, t)$ and $-\phi(x - 2u, t)$ whereby

$$\left[\phi(x, t) - \phi(x - 2\ell, t)\right]_{x=\ell} = 0 \quad \text{and} \quad \left[\phi(x, t) - \phi(x - 2u, t)\right]_{x=u} = 0.$$ 

Note that $\phi(x - 2\ell, t)$ and $\phi(x - 2u, t)$ correspond to the fundamental soluton with initial condition: $\delta(x - 2\ell)$ and $\delta(x - 2u)$, respectively. Writing the above partial sum with three terms as

$$\hat{g}_3(x, t) = \phi(x, t) - \phi(x - 2\ell, t) - \phi(x - 2u, t),$$

we observe that the homogeneous boundary conditions are not yet satisfied since

$$\hat{g}_3(\ell, t) = -\phi(x - 2u, t) \bigg|_{x=\ell} \neq 0$$

$$\hat{g}_3(u, t) = -\phi(2 - 2\ell, t) \bigg|_{x=u} \neq 0.$$
To nullify the non-zero value of \(-\phi(x-2u, t)\bigg|_{x=\ell}\) and \(-\phi(x-2\ell, t)\bigg|_{x=u}\), we add a new pair of positive terms \(\phi(x - 2(u - \ell), t)\) and \(\phi(x + 2(u - \ell), t)\). Similarly, we write the partial sum with five terms as

\[
\hat{g}_5(x, t) = \hat{g}_3(x, t) + \phi(x - 2(u - \ell), t) + \phi(x + 2(u - \ell), t),
\]

and observe that

\[
\hat{g}_5(\ell, t) = \phi(x - 2(u - \ell), t)\bigg|_{x=\ell} \neq 0
\]

\[
\hat{g}_5(u, t) = \phi(x + 2(u - \ell), t)\bigg|_{x=u} \neq 0.
\]

Whenever a new pair of positive terms or negative terms are added, the value of the partial sum at \(x = \ell\) and \(x = u\) becomes closer to zero. In a recursive manner, we add successive pairs of positive and negative terms so as to come closer to the satisfaction of the homogeneous boundary conditions at \(x = \ell\) and \(x = u\).
The two absorbing barriers may be visualized as a pair of mirrors with the object placed at the origin (see Figure on the next page).

The source at the origin generates a sink at \( x = 2\ell \) due to the mirror at \( x = \ell \) and another sink at \( x = 2u \) due to the mirror at \( x = u \).

To continue, the sink at \( x = 2\ell \) (\( x = 2u \)) generates a source at \( x = 2(u - \ell) \) [\( x = 2(\ell - u) \)] due to the mirror at \( x = u \) (\( x = \ell \)).

As the procedure continues, this leads to the sum of an infinite number of positive and negative terms.
Infinite number of images

The double-mirror analogy provides the intuitive argument showing why $g(x, t)$ involves an infinite number of terms.

A graphical representation of an infinite number of sources and sinks due to a pair of absorbing barriers (mirrors) with the object placed at the origin. The positions of the sources and sinks are $\alpha_j = 2(u - \ell)j$ and $\beta_j = 2\ell + 2(u - \ell)(j - 1)$, respectively, $j = 0, \pm 1, \pm 2, \ldots$. 
The solution to \( g(x, t) \) is deduced to be

\[
g(x, t) = e^{\frac{\mu x - \mu^2 t}{2\sigma^2}} \hat{g}(x, t)
\]

\[
= e^{\frac{\mu x - \mu^2 t}{2\sigma^2}} \sum_{n=-\infty}^{\infty} \left[ \phi(x - 2n(u - \ell), t) - \phi(x - 2\ell - 2n(u - \ell), t) \right]
\]

\[
= \frac{\mu x - \mu^2 t}{e \sigma^2 - \mu^2 2\sigma^2} \sum_{n=-\infty}^{\infty} \left[ \exp \left( -\frac{[x - 2n(u - \ell)]^2}{2\sigma^2 t} \right) \right.
\]

\[
- \exp \left( -\frac{[(x - 2\ell) - 2n(u - \ell)]^2}{2\sigma^2 t} \right) \right]
\]

\[
= \sum_{n=-\infty}^{\infty} \left[ \frac{2\mu}{e \sigma^2 n(u - \ell)} \right]
\]

\[
\exp \left( -\frac{[x - \mu t - 2n(u - \ell)]^2}{2\sigma^2 t} \right)
\]

\[
- \frac{2\mu}{e \sigma^2 [\ell + n(u - \ell)]} \exp \left( -\frac{[(x - \mu t - 2\ell) - 2n(u - \ell)]^2}{2\sigma^2 t} \right) \right].
\]
Alternative representation: eigenfunction expansion

Let $P(x, t; x_0, t_0)$ denote the transition density function of the restricted Brownian process $W^\mu_t = \mu t + \sigma Z_t$ with two absorbing barriers at $x = 0$ and $x = \ell$, where $\ell > 0$. We take the convenience of setting one of the absorbing barriers to be $x = 0$. Using the method of separation of variables, the solution to $P(x, t; x_0, t_0)$ admits the following eigenfunction expansion

$$P(x, t; x_0, t_0) = e^{\frac{\mu}{\sigma^2}(x-x_0)-\frac{\mu^2}{2\sigma^2}(t-t_0)} \sum_{k=1}^{\infty} e^{-\lambda_k(t-t_0)} \sin \frac{k\pi x}{\ell} \sin \frac{k\pi x_0}{\ell}$$

where the eigenvalues are given by

$$\lambda_k = \frac{k^2\pi^2\sigma^2}{2\ell^2}.$$

$P(x, t; x_0, t_0)$ satisfies the forward Fokker-Planck equation with auxiliary conditions: $P(0, t) = P(\ell, t) = 0$ and $P(x, t^+_0; x_0, t_0) = \delta(x-x_0)$, $0 < x_0 < \ell$. 

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Proof (Separation of variables):

The eigenfunctions \( \sin \frac{k\pi x}{\ell} \), \( k = 1, 2, \ldots \), are seen to satisfy the homogeneous boundary conditions at \( x = 0 \) and \( x = \ell \). The solution in the form of eigenfunction expansion assumes an infinite series of the form

\[
P(x, t; x_0, t_0) = \sum_{k=1}^{\infty} A_k e^{-\lambda_k(t-t_0)} \sin \frac{k\pi x}{\ell},
\]

where the eigenvalues \( \lambda_k \), \( k = 1, 2, \ldots \), are determined so that each term \( e^{-\lambda_k(t-t_0)} \sin \frac{k\pi x}{\ell} \) satisfies the governing differential equation:

\[
\frac{\partial P}{\partial t} = \sigma^2 \frac{\partial^2 P}{\partial x^2}.
\]

This requires that the eigenvalues should be given by

\[
-\lambda_k = -\frac{\sigma^2 k^2 \pi^2}{2 \ell^2} \quad \text{or} \quad \lambda_k = \frac{k^2 \pi^2 \sigma^2}{2 \ell^2}, \quad k = 1, 2, \ldots.
\]
Lastly, we determine the constants $A_k$, $k = 1, 2, \ldots$, using the initial condition:

$$\delta(x - x_0) = \sum_{k=1}^{\infty} A_k \sin \frac{k\pi x}{\ell}.$$  

By virtue of the orthogonality of the eigenfunctions, we have

$$\int_0^\ell A_k \sin^2 \frac{k\pi x}{\ell} \, dx = \int_0^\ell \delta(x - x_0) \sin \frac{k\pi x}{\ell} \, dx, \quad 0 < x_0 < \ell.$$  

Lastly, we obtain $A_k = \frac{2}{\ell} \sin \frac{k\pi x_0}{\ell}, \ k = 1, 2, \ldots$.

The solution of the density function can be expressed either as an infinite series of Gaussian kernel functions using the method of images or the eigenfunction expansion approach. These two solutions are equivalent by virtue of the Poisson summation formula. It has been shown that the Gaussian kernel series has a faster rate of convergence to the exact value with respect to the number of terms $n$ used.
The density function of the first passage time to either barrier is defined by

\[ q(t; \ell, u) \, dt = P(\min(\tau_\ell, \tau_u) \in dt), \]

where \( \tau_\ell = \inf\{t | X_t = \ell\} \) and \( \tau_u = \inf\{t | X_t = u\} \). We consider the corresponding distribution function

\[ P(\min(\tau_\ell, \tau_u) \leq t) = 1 - P(\min(\tau_\ell, \tau_u) > t) = 1 - \int_\ell^u g(x, t) \, dx \]

where \( \int_\ell^u g(x, t) \, dx \) is the total probability that \( W_{t}^{\mu} \) stays within \((\ell, u)\). The density function of the first passage time is given by

\[
q(t; \ell, u) = -\frac{\partial}{\partial t} \int_\ell^u g(x, t) \, dx = \frac{1}{\sqrt{2\pi\sigma^2 t^3}} \left\{ \sum_{n=-\infty}^{\infty} \left[ 2n(u - \ell) - \ell \right] \exp \left( \frac{\mu \ell}{\sigma^2} - \frac{\mu^2 t}{2\sigma^2} \right) \exp \left( -\frac{(2n(u - \ell) - \ell)^2}{2\sigma^2 t} \right) + \left[ 2n(u - \ell) + u \right] \exp \left( \frac{\mu u}{\sigma^2} - \frac{\mu^2 t}{2\sigma^2} \right) \exp \left( -\frac{[2n(u - \ell) + u]^2}{2\sigma^2 t} \right) \right\}. \]
Exit time to a barrier

The density function of the exit time to the respective lower barrier and upper barrier are defined by

\[ q^-(t; \ell, u) \, dt = P(\tau_\ell \in dt, \tau_\ell < \tau_u) \]
\[ q^+(t; \ell, u) \, dt = P(\tau_u \in dt, \tau_u < \tau_\ell). \]

Since \( \{\tau_\ell \in dt, \tau_\ell < \tau_u\} \cup \{\tau_u \in dt, \tau_u < \tau_\ell\} = \{\min(\tau_\ell, \tau_u) \in dt\} \), we deduce that

\[ q(t; \ell, u) = q^-(t; \ell, u) + q^+(t; \ell, u). \]
A judicious decomposition of $q(t; \ell, u)$ into its two components would suggest

$$q^-(t; \ell, u) = \frac{1}{\sqrt{2\pi\sigma^2 t^3}} \sum_{n=-\infty}^{\infty} [2n(u - \ell) - \ell]$$
$$\exp\left(\frac{\mu\ell}{\sigma^2} - \frac{\mu^2 t}{2\sigma^2}\right) \exp\left(-\frac{[2n(u - \ell) - \ell]^2}{2\sigma^2 t}\right)$$

$$q^+(t; \ell, u) = \frac{1}{\sqrt{2\pi\sigma^2 t^3}} \sum_{n=-\infty}^{\infty} [2n(u - \ell) + u]$$
$$\exp\left(\frac{\mu u}{\sigma^2} - \frac{\mu^2 t}{2\sigma^2}\right) \exp\left(-\frac{[2n(u - \ell) + u]^2}{2\sigma^2 t}\right).$$
To show the claim, we define the probability flow by

\[ J(x, t) = \mu g(x, t) - \frac{\sigma^2}{2} \partial x g(x, t), \]

where the negative sign is chosen for the diffusion term since the probability flow is in the negative direction when \( \partial g / \partial x > 0 \) (diffusion tends to make probability concentration to spread evenly). Also, recall that

\[
q(t; \ell, u) = -\frac{\partial}{\partial t} \int_{\ell}^{u} g(x, t) \, dx = \int_{\ell}^{u} -\frac{\partial g}{\partial t} \, dx.
\]

Since \( g \) satisfies the forward Fokker-Planck equation, we have

\[
q(t; \ell, u) = \int_{\ell}^{u} \left( \mu \frac{\partial g}{\partial x} - \frac{\sigma^2}{2} \frac{\partial^2 g}{\partial x^2} \right) \, dx = J(u, t) - J(\ell, t).
\]

One may visualize the probability flow across \( x = \ell \) and \( x = u \) as

\[
-J(\ell, t) = P(\tau_\ell \in dt, \tau_\ell < \tau_u)
\]

\[
J(u, t) = P(\tau_u \in dt, \tau_u < \tau_\ell).
\]
Note that $J(\ell, t)$ is negative since the probability flow is outward from the interval $(\ell, u)$ through $x = \ell$ along the negative $x$-direction.

The exit time densities $q^-(t; \ell, u)$ and $q^+(t; \ell, u)$ are seen to satisfy

$$q^-(t; \ell, u) = -J(\ell, t) = - \left[ \mu g(x, t) - \frac{\sigma^2}{2} \frac{\partial g(x, t)}{\partial x} \right] \bigg|_{x=\ell}$$

$$q^+(t; \ell, u) = J(u, t) = \mu g(x, t) - \frac{\sigma^2}{2} \frac{\partial g(x, t)}{\partial x} \bigg|_{x=u}.$$

### Rebate payment

Suppose a rebate $R^-(t) [R^+(t)]$ is paid when the lower (upper) barrier is first breached during the life of the option, $0 < t < T$, the value of the rebate portion of the double-barrier option is then given by

$$\text{rebate value} = \int_0^T e^{-r\xi} \left[ R^-(\xi)q^-(\xi; \ell, u) + R^+(\xi)q^+(\xi; \ell, u) \right] d\xi.$$