MATH 5510 — Mathematical Models of Financial Derivatives

Topic 1 — Risk neutral pricing principles under discrete securities models

1.1 Law of one price and Arrow securities

1.2 No-arbitrage theory and risk neutral probability measure — Fundamental theorem of asset pricing

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1.1 Law of one price and Arrow securities

- The initial prices of $M$ risky securities, denoted by $S_1(0), \cdots, S_M(0)$, are positive scalars that are known at $t = 0$.

- Their values at $t = 1$ are random variables, which are defined with respect to a sample space $\Omega = \{\omega_1, \omega_2, \cdots, \omega_K\}$ of $K$ possible outcomes (or states of the world).

- At $t = 0$, the investors know the list of all possible outcomes, but which outcome does occur is revealed only at the end of the investment period $t = 1$.

- A probability measure $P$ satisfying $P(\omega) > 0$, for all $\omega \in \Omega$, is defined on $\Omega$.

- We use $S$ to denote the price process $\{S(t) : t = 0, 1\}$, where $S(t)$ is the row vector $S(t) = (S_1(t) \ S_2(t) \cdots S_M(t))$. 

Consider 3 risky assets with time-0 price vector

\[ S(0) = (S_1(0), S_2(0), S_3(0)) = (1, 2, 3). \]

At time 1, there are 2 possible states of the world:

\( \omega_1 = \) Hang Seng index is at or above 22,000
\( \omega_2 = \) Hang Seng index falls below 22,000.

If \( \omega_1 \) occurs, then

\[ S(1; \omega_1) = (1.2, 2.1, 3.4); \]

otherwise, \( \omega_2 \) occurs and

\[ S(1; \omega_2) = (0.8, 1.9, 2.9). \]
The possible values of the asset price process at $t = 1$ are listed in the following $K \times M$ matrix

$$
S(1; \Omega) = \begin{pmatrix}
S_1(1; \omega_1) & S_2(1; \omega_1) & \cdots & S_M(1; \omega_1) \\
S_1(1; \omega_2) & S_2(1; \omega_2) & \cdots & S_M(1; \omega_2) \\
\vdots & \vdots & \ddots & \vdots \\
S_1(1; \omega_K) & S_2(1; \omega_K) & \cdots & S_M(1; \omega_K)
\end{pmatrix}.
$$

Since the assets are limited liability securities, the entries in $S(1; \Omega)$ are non-negative scalars.

Existence of a strictly positive riskless security or bank account, whose value is denoted by $S_0$. Without loss of generality, we take $S_0(0) = 1$ and the value at time 1 to be $S_0(1) = 1 + r$, where $r \geq 0$ is the deterministic interest rate over one period.
• We define the discounted price process by

\[ S^*(t) = \frac{S(t)}{S_0(t)}, \quad t = 0, 1, \]

that is, we use the riskless security as the *numeraire* or *accounting unit*.

• The payoff matrix of the discounted price processes of the \( M \) risky assets and the riskless security can be expressed in the form

\[
\hat{S}^*(1; \Omega) = \begin{pmatrix}
1 & S_1^*(1; \omega_1) & \cdots & S_M^*(1; \omega_1) \\
1 & S_1^*(1; \omega_2) & \cdots & S_M^*(1; \omega_2) \\
\vdots & \vdots & \ddots & \vdots \\
1 & S_1^*(1; \omega_K) & \cdots & S_M^*(1; \omega_K)
\end{pmatrix}.
\]
Trading strategies

- An investor adopts a trading strategy by selecting a portfolio of the $M$ assets at time 0. A trading strategy is characterized by asset holding in the portfolio.

- The number of units of asset $m$ held in the portfolio from $t = 0$ to $t = 1$ is denoted by $h_m, m = 0, 1, \ldots, M$.

- The scalars $h_m$ can be positive (long holding), negative (short selling) or zero (no holding).

- An investor is endowed with an initial endowment $V_0$ at time 0 to set up the trading portfolio. How do we choose the portfolio holding of the assets such that the expected portfolio value at time 1 is maximized?
**Portfolio value process**

- Let $V = \{V_t : t = 0, 1\}$ denote the value process that represents the total value of the portfolio over time. It is seen that

$$V_t = h_0 S_0(t) + \sum_{m=1}^{M} h_m S_m(t), \quad t = 0, 1.$$ 

- Let $G$ be the random variable that denotes the total gain generated by investing in the portfolio. We then have

$$G = h_0 r + \sum_{m=1}^{M} h_m \Delta S_m, \quad \Delta S_m = S_m(1) - S_m(0).$$
Account balancing

• If there is no withdrawal or addition of funds within the investment horizon, then

\[ V_1 = V_0 + G. \]

• Suppose we use the bank account as the numeraire, and define the discounted value process by \( V_t^* = V_t/S_0(t) \) and discounted gain by \( G^* = V_1^* - V_0^* \), we then have

\[
\begin{align*}
V_t^* &= h_0 + \sum_{m=1}^{M} h_m S_m^*(t), \quad t = 0, 1; \\
G^* &= V_1^* - V_0^* = \sum_{m=1}^{M} h_m \Delta S_m^*.
\end{align*}
\]
Dominant trading strategies

A trading strategy $\mathcal{H}$ is said to be dominant if there exists another trading strategy $\tilde{\mathcal{H}}$ such that

$$V_0 = \tilde{V}_0 \quad \text{and} \quad V_1(\omega) > \tilde{V}_1(\omega) \quad \text{for all } \omega \in \Omega.$$

- Suppose $\mathcal{H}$ dominates $\tilde{\mathcal{H}}$, we define a new trading strategy $\mathcal{H} = \mathcal{H} - \tilde{\mathcal{H}}$. Let $\tilde{V}_0$ and $\tilde{V}_1$ denote the portfolio value of $\tilde{\mathcal{H}}$ at $t = 0$ and $t = 1$, respectively. We then have $\tilde{V}_0 = 0$ and $\tilde{V}_1(\omega) > 0$ for all $\omega \in \Omega$.

- This trading strategy is dominant since it dominates the strategy which starts with zero value and does no investment at all.

- Equivalent definition: A dominant trading strategy exists if and only if there exists a trading strategy satisfying $V_0 < 0$ and $V_1(\omega) \geq 0$ for all $\omega \in \Omega$. 
Asset span

- Consider two risky securities whose discounted payoff vectors are

\[ S^*_1(1) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \text{and} \quad S^*_2(1) = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}. \]

- The payoff vectors are used to form the discounted terminal payoff matrix

\[ S^*(1) = \begin{pmatrix} 1 & 3 \\ 2 & 1 \\ 3 & 2 \end{pmatrix}. \]

- Let the current prices be represented by the row vector \( S^*(0) = (1 \quad 2). \)
• We write $h$ as the column vector whose entries are the portfolio holding of the securities in the portfolio. The trading strategy is characterized by specifying $h$. The current portfolio value and the discounted portfolio payoff are given by $S^*(0)h$ and $S^*(1)h$, respectively.

• The set of all portfolio payoffs via different holding of securities is called the asset span $S$. The asset span is seen to be the column space of the payoff matrix $S^*(1)$, which is a subspace in $\mathbb{R}^K$ spanned by the columns of $S^*(1)$. 
asset span = column space of \( S^*(1) \)
\[ = \text{span}(S_1^*(1) \cdots S_M^*(1)) \]

Recall that

\[
\text{column rank} = \text{dimension of column space} = \text{number of independent columns.}
\]

It is well known that number of independent columns = number of independent rows, so column rank = row rank = rank \( \leq \min(K, M) \).

- In the above numerical example, the asset span consists of all vectors of the form
  \[
  h_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + h_2 \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix},
  \]
  where \( h_1 \) and \( h_2 \) are scalars.
Redundant security and complete model

- If the discounted terminal payoff vector of an added security lies inside $S$, then its payoff can be expressed as a linear combination of $S^*_1(1)$ and $S^*_2(1)$. In this case, it is said to be a redundant security. The added security is said to be replicable by some combination of existing securities.

- A securities model is said to be complete if every payoff vector lies inside the asset span. That is, all new securities can be replicated by existing securities. This occurs if and only if the dimension of the asset span equals the number of possible states, that is, the asset span becomes the whole $\mathbb{R}^K$. 
Given the securities model with 4 risky securities and 3 possible states of world:

\[ S^*(1; \Omega) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 5 & 7 \\ 3 & 5 & 8 & 11 \end{pmatrix}, \quad S^*(0) = (1 \quad 2 \quad 4 \quad 7). \]

asset span = span\((S^*_1(1), S^*_2(1))\), which has dimension = 2 < 3 = number of possible states. Hence, the securities model is not complete! For example, the following security

\[ S^*_\beta(1; \Omega) = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} \]

does not lie in the asset span of the securities model. There is no solution to

\[
\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 5 & 7 \\ 3 & 5 & 8 & 11 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}.
\]
Pricing problem

Given a new security that is replicable by existing securities, its price with reference to a given securities model is given by the cost of setting up the replicating portfolio.

Consider a new security with discounted payoff at $t = 1$ as given by

$$S^*_\alpha(1; \Omega) = \begin{pmatrix} 5 \\ 8 \\ 13 \end{pmatrix},$$

which is seen to be

$$S^*_\alpha(1; \Omega) = S^*_2(1; \Omega) + S^*_3(1; \Omega) = S^*_1(1; \Omega) + 2S^*_2(1; \Omega).$$

This new security is redundant. Unfortunately, the price of this security can be either

$$S^*_2(0) + S^*_3(0) = 6 \quad \text{or} \quad S^*_1(0) + 2S^*_2(0) = 5.$$

There are two possible prices, corresponding to two different choices of replicating portfolios.
**Question**

How to modify $S^*(0)$ so as to avoid the above ambiguity that portfolios with the same terminal payoff have different initial prices (failure of law of one price).

Note that $S_3^*(1; \Omega) = S_1^*(1; \Omega) + S_2^*(1; \Omega)$ and $S^*_4(1; \Omega) = S_1^*(1; \Omega) + S_3^*(1; \Omega)$, both the third and fourth security are redundant securities. To achieve the law of one price, we modify $S_3^*(0)$ and $S_4^*(0)$ such that

$$S_3^*(0) = S_1^*(0) + S_2^*(0) = 3 \quad \text{and} \quad S_4^*(0) = 2S_1^*(0) + S_2^*(0) = 4.$$  

**Conjecture**

If there are no redundant securities, then the law of one price holds. Mathematically, non-existence of redundant securities means $S^*(1; \Omega)$ has full column rank. That is, column rank = number of columns. This gives a sufficient condition for “law of one price”.  


Law of one price (pricing of securities that lie in the asset span)

1. The law of one price states that all portfolios with the same terminal payoff have the same initial price.

2. Consider two portfolios with different portfolio weights $h$ and $h'$. Suppose these two portfolios have the same discounted payoff, that is, $S^*(1)h = S^*(1)h'$, then the law of one price infers that $S^*(0)h = S^*(0)h'$.

3. The trading strategy $h$ is obtained by solving

$$S^*(1)h = S^*_\alpha(1).$$

Solution exists if $S^*_\alpha(1)$ lies in the asset span. Uniqueness of solution is equivalent to null space of $S^*(1)$ having zero dimension. There is only one trading strategy that replicates the security with discounted terminal payoff $S^*_\alpha(1)$. In this case, the law of one price always holds.
Law of one price and dominant trading strategy

If the law of one price fails, then it is possible to have two trading strategies $h$ and $h'$ such that $S^*(1)h = S^*(1)h'$ but $S^*(0)h > S^*(0)h'$. Let $G^*(\omega)$ and $G^{*'}(\omega)$ denote the respective discounted gain corresponding to the trading strategies $h$ and $h'$. We then have $G^{*'}(\omega) > G^*(\omega)$ for all $\omega \in \Omega$, so there exists a dominant trading strategy. The corresponding dominant trading strategy is $h' - h$ so that $V_0 < 0$ but $V_1^*(\omega) = 0$ for all $\omega \in \Omega$.

Hence, the non-existence of dominant trading strategy implies the law of one price. However, the converse statement does not hold.

[See later numerical example.]
Pricing functional

- Given a discounted portfolio payoff \( x \) that lies inside the asset span, the payoff can be generated by some linear combination of the securities in the securities model. We have \( x = S^*(1)h \) for some \( h \in \mathbb{R}^M \). Existence of the solution \( h \) is guaranteed since \( x \) lies in the asset span, or equivalently, \( x \) lies in the column space of \( S^*(1) \).

- The current value of the portfolio is \( S^*(0)h \), where \( S^*(0) \) is the initial price vector.

- We may consider \( S^*(0)h \) as a pricing functional \( F(x) \) on the payoff \( x \). If the law of one price holds, then the pricing functional is single-valued. Furthermore, it is a linear functional, that is,

\[
F(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 F(x_1) + \alpha_2 F(x_2)
\]

for any scalars \( \alpha_1 \) and \( \alpha_2 \) and payoffs \( x_1 \) and \( x_2 \).
Arrow security and state price

• Let $e_k$ denote the $k^{th}$ coordinate vector in the vector space $\mathbb{R}^K$, where $e_k$ assumes the value 1 in the $k^{th}$ entry and zero in all other entries. The vector $e_k$ can be considered as the discounted payoff vector of a security, and it is called the Arrow security of state $k$. This Arrow security has unit payoff when state $k$ occurs and zero payoff otherwise.

• Suppose the securities model is complete (all Arrow securities lie in the asset span) and the law of one price holds, then the pricing functional $F$ assigns unique value to each Arrow security. We write $s_k = F(e_k)$, which is called the state price of state $k$. Note that state price must be non-negative. Take

$$S^*_\alpha(1) = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_K \end{pmatrix} = \sum_{k=1}^{K} \alpha_k e_k,$$

then

$$S^*_\alpha(0) = F(S^*_\alpha(1)) = F\left( \sum_{k=1}^{K} \alpha_k e_k \right) = \sum_{k=1}^{K} \alpha_k F(e_k) = \sum_{k=1}^{K} \alpha_k s_k.$$
Summary

Given a securities model endowed with $S^*(1; \Omega)$ and $S^*(0)$, can we find a trading strategy to form a portfolio that replicates a new security $S^*_\alpha(1; \Omega)$ (also called a contingent claim) that is outside the universe of the $M$ available risky securities in the securities model?

Replication means the terminal payoff of the replicating portfolio matches with that of the contingent claim under all scenarios of occurrence of the state of the world at $t = 1$.

1. Formation of the replicating portfolio is possible if we have existence of solution $h$ to the following system

$$S^*(1; \Omega)h = S^*_\alpha(1; \Omega).$$
This is equivalent to the fact that \( S^\alpha_\Omega(1; \Omega) \) lies in the asset span (column space) of \( S^*(1; \Omega) \). The solution \( h \) is the corresponding trading strategy. Note that \( h \) may not be unique.

**Completeness of securities model**

If all contingent claims are replicable, then the securities model is said to be complete. This is equivalent to

\[
\text{dim(asset span)} = K = \text{number of possible states},
\]

that is, asset span = \( \mathbb{R}^K \). In this case, solution \( h \) always exists.
2. Uniqueness of trading strategy

If $h$ is unique, then there is only one trading strategy that generates the replicating portfolio. This occurs when the columns of $S^*(1; \Omega)$ are independent. Equivalently, column rank $= M$ and all securities are non-redundant. Mathematically, this is equivalent to observe that the homogeneous system

$$S^*(1; \Omega)h = 0$$

admits only the trivial zero solution. In other words, the dimension of the null space of $S^*(1; \Omega)$ is zero.

When we have unique solution $h$, the initial cost of setting up the replicating portfolio (price at time 0) as given by $S^*(0)h$ is unique. In this case, law of one price holds.
Matrix properties of $S^*(1)$ that are related to financial economics concepts

The securities model is endowed with

(i) discounted terminal payoff matrix $= \begin{pmatrix} S_1^*(1) & \cdots & S_M^*(1) \end{pmatrix}$, and
(ii) initial price vector; $S^*(0) = (S_1^*(0) \cdots S_M^*(0))$.

Recall that
column rank $\leq \min(K, M)$
where $K =$ number of possible states, $M =$ number of risky securities.

Given a risky security with the discounted terminal payoff $S_{\alpha}^*(1)$, we are interested to explore the existence and uniqueness of solution to

$S^*(1)h = S_{\alpha}^*(1)$.

Here, $h$ is the asset holding of the portfolio that replicates $S_{\alpha}^*(1)$. 
(i) column rank = $K$

asset span = $\mathbb{R}^K$, so the securities model is complete. Any risky securities is replicable. In this case, solution $h$ always exists.

(ii) column rank = $M$ (all columns of $S^*(1)$ are independent)

All securities are non-redundant. In this case, $h$ may or may not exist. However, if $h$ exists, then it must be unique. The price of any replicable security is unique.

(iii) column rank < $K$

Solution $h$ exists if and only if $S^*_\alpha(1)$ lies in the asset span. However, there is no guarantee for the uniqueness of solution.

(iv) column rank < $M$

Existence of redundant securities, so the law of one price may fail.
Law of one price revisited

Law of one price holds if and only if solution to

$$\pi S^*(1) = S(0) \quad (A)$$

exists.

1. Suppose solution to (A) exists, let \( h \) and \( h' \) be two trading strategies such that their respective discounted terminal payoff \( V \) and \( V' \) are the same. That is,

$$S^*(1)h = V = V' = S^*(1)h'.$$

Since \( \pi \) exists, we then have

$$\pi S^*(1)(h - h') = 0.$$

Noting that \( \pi S^*(1) = S(0) \), we obtain

$$S(0)(h - h') = 0 \quad \text{so that } V_0 = V'_0.$$
2. Suppose solution to (A) does not exist for the given $S(0)$, this implies that $S(0)$ that does not lie in the row space of $S^*(1)$. The row space of $S^*(1)$ does not span the whole $\mathbb{R}^M$. Therefore, $\dim(\text{row space of } S^*(1)) < M$, where $M$ is the number of securities = number of columns in $S^*(1)$.

Recall that

$$\dim(\text{null space of } S^*(1)) + \text{rank}(S^*(1)) = M$$

so that $\dim(\text{null space of } S^*(1)) > 0$.

Hence, there exists non-zero solution $h$ to

$$S^*(1)h = 0.$$

Note that $h$ is orthogonal to all rows of $S^*(1)$. This is consistent with the property that

row space = orthogonal complement of null space.
We claim that one can always find non-zero solution $h$ that is not orthogonal to $S(0)$. If otherwise, $S(0)$ lies in the orthogonal complement of the null space (that is, row space). This leads to a contradiction.

Consider the above choice of non-zero $h$, where $S^*(1)h = 0$. We split $h = h_1 - h_2$, where $h_1 \neq h_2$. Then there exist two distinct trading strategies such that

$$S^*(1)h_1 = S^*(1)h_2.$$ 

The two strategies have the same discounted terminal payoff under all states of the world. However, their initial prices are unequal since

$$S(0)h_1 \neq S(0)h_2,$$

by virtue of the property: $S(0)h \neq 0$. Hence, the law of one price does not hold.
Linear pricing measure

We consider securities models with the inclusion of the riskfree security. A non-negative row vector \( q = (q(\omega_1) \cdots q(\omega_K)) \) is said to be a linear pricing measure if for every trading strategy the portfolio values at \( t = 0 \) and \( t = 1 \) are related by

\[
V_0^* = \sum_{k=1}^{K} q(\omega_k) V_1^*(\omega_k).
\]

**Remark**
Here, the same initial price \( V_0^* \) is always resulted as there is no dependence of \( V_0^* \) on the asset holding of the portfolio. Two portfolios with the same terminal payoff for all states of the world would have the same price. Implicitly, this implies that the law of one price holds. The rigorous justification of the above statement will be presented later. Note that \( q \) is not required to be unique.
1. Suppose we take the holding amount of every risky security to be zero, thereby \( h_1 = h_2 = \cdots = h_M = 0 \), then

\[
V_0^* = h_0 = \sum_{k=1}^{K} q(\omega_k) h_0
\]

so that

\[
\sum_{k=1}^{K} q(\omega_k) = 1.
\]

2. Suppose that the securities model is complete. By taking the portfolio to have the same terminal payoff as that of the \( k^{th} \) Arrow security, we obtain

\[
s_k = q(\omega_k), \quad k = 1, 2, \cdots, K.
\]

That is, the state price of the \( k^{th} \) state is simply \( q(\omega_k) \). This is not surprising when we compare

\[
V_0^* = \sum_{k=1}^{K} q(\omega_k)V_1^*(\omega_k) \quad \text{and} \quad S_\alpha^*(0) = \sum_{k=1}^{K} \alpha_k s_k.
\]
• Since we have taken \( q(\omega_k) \geq 0, k = 1, \cdots, K \), and their sum is one, we may interpret \( q(\omega_k) \) as a probability measure on the sample space \( \Omega \).

• Note that \( q(\omega_k) \) is not related to the actual probability of occurrence of the state \( k \), though the current security price is given by the discounted expectation of the security payoff one period later under the linear pricing measure.

• By taking the portfolio weights to be zero except for the \( m^{th} \) security, we have

\[
S_m^*(0) = \sum_{k=1}^{K} q(\omega_k) S_m^*(1; \omega_k), \quad m = 0, 1, \cdots, M.
\]

In matrix form, we have

\[
\hat{S}^*(0) = q \hat{S}^*(1; \Omega), \quad q \geq 0.
\]
Numerical example

Take $S^*(1) = \begin{pmatrix} 1 & 2 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $S^*(0) = (1 \ 1 \frac{1}{3})$, then

$$q = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

is a linear pricing measure. The linear pricing measure is not unique! Actually, we have $q(\omega_1) = \frac{1}{3}$ and $q_2(\omega_2) + q(\omega_3) = \frac{2}{3}$.

- The securities model is not complete. Though $e_1$ is replicable and its initial price is $\frac{1}{3}$, but $e_2$ and $e_3$ are not replicable so the state price of $\omega_2$ and $\omega_3$ do not exist.
Suppose we add the new risky security with discounted terminal payoff
\[
\begin{pmatrix}
1 \\
2 \\
1
\end{pmatrix}
\] and initial price \(\frac{2}{3}\) into the securities model, then the securities model becomes complete. We have the following state prices

\[
s_1 = \frac{1}{3}, \quad s_2 = -\frac{1}{3}, \quad s_3 = 1.
\]

In this case, law of one price holds but dominant trading strategy exists. For example, we may take

\[
V_1^*(\omega) = \begin{pmatrix}
3 \\
6 \\
1
\end{pmatrix} > 0, \quad V_0^* = 3s_1 + 6s_2 + s_3 = 0.
\]

Remark To explore “law of one price”, one has to consider the existence of solution to the linear system of equations: \(S^*(0) = \pi S^*(1)\).
Example – Law of one price

Take \( \hat{S}^*(1; \Omega) = \begin{pmatrix} 1 & 2 & 6 & 9 \\ 1 & 3 & 3 & 7 \\ 1 & 6 & 12 & 19 \end{pmatrix} \), the sum of the first 3 columns gives the fourth column. The first column corresponds to the discounted terminal payoff of the riskfree security under the 3 possible states of the world. The third risky security is a redundant security.

Let \( \hat{S}^*(0) = (1 \ 2 \ 3 \ k) \). We observe that solution to

\[
(1 \ 2 \ 3 \ k) = (\pi_1 \ \pi_2 \ \pi_3) \begin{pmatrix} 1 & 2 & 6 & 9 \\ 1 & 3 & 3 & 7 \\ 1 & 6 & 12 & 19 \end{pmatrix}
\]

exists if and only if \( k = 6 \). That is, \( S^*_3(0) = S^*_0(0) + S^*_1(0) + S^*_2(0) \).

When \( k \neq 6 \), the law of one price does not hold. The last equation: \( 9\pi_1 + 7\pi_2 + 19\pi_3 = k \neq 6 \) is inconsistent with the first 3 equations. One may check that \((1 \ 2 \ 3 \ 6)\) can be expressed as a linear combination of the rows of \( \hat{S}^*(1; \Omega) \).
We consider the linear system

\[ \hat{S}^*(0) = \pi \hat{S}^*(1; \Omega), \]

solution exists if and only if \( \hat{S}^*(0) \) lies in the row space of \( \hat{S}^*(1; \Omega) \). Uniqueness follows if the rows of \( \hat{S}^*(1; \Omega) \) are independent.

Since

\[ S^*_3(1; \Omega) = S^*_0(1; \Omega) + S^*_1(1; \Omega) + S^*_2(1; \Omega), \]

the third risky security is replicable by holding one unit of each of the riskfree security and the first two risky securities. The initial price must observe the same relation in order that the law of one price holds.

Here, we have redundant securities. Actually, one may show that the law of one price holds if and only if we have existence of solution to the linear system. In this example, when \( k = 6 \), we obtain

\[ \pi = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 3 & -6 \end{pmatrix}. \]

This is not a linear pricing measure.
Example – Law of one price holds while dominant trading strategies exist

Consider a securities model with 2 risky securities and the riskfree security, and there are 3 possible states. The current discounted price vector \( \hat{S}^*(0) \) is \( (1 \ 4 \ 2) \) and the discounted payoff matrix at \( t = 1 \) is \( \hat{S}^*(1) = \begin{pmatrix} 1 & 4 & 3 \\ 1 & 3 & 2 \\ 1 & 2 & 4 \end{pmatrix} \). Here, the law of one price holds since the only solution to \( \hat{S}^*(1)h = 0 \) is \( h = 0 \). This is because the columns of \( \hat{S}^*(1) \) are independent so that the dimension of the nullspace of \( \hat{S}^*(1) \) is zero.
The linear pricing probabilities $q(\omega_1), q(\omega_2)$ and $q(\omega_3)$, if exist, should satisfy the following equations:

\[
\begin{align*}
1 & = q(\omega_1) + q(\omega_2) + q(\omega_3) \\
4 & = 4q(\omega_1) + 3q(\omega_2) + 2q(\omega_3) \\
2 & = 3q(\omega_1) + 2q(\omega_2) + 4q(\omega_3).
\end{align*}
\]

Solving the above equations, we obtain $q(\omega_1) = q(\omega_2) = 2/3$ and $q(\omega_3) = -1/3$.

- Since not all the pricing probabilities are non-negative, the linear pricing measure does not exist for this securities model.
Existence of dominant trading strategies

• For convenience of graphical interpretation, we consider trading strategies that take $h_0 = 0$. Can we find a trading strategy $(h_1, h_2)$ such that $V_0^* = 4h_1 + 2h_2 = 0$ but $V_1^*(\omega_k) > 0, k = 1, 2, 3$? This is equivalent to ask whether there exist $h_1$ and $h_2$ such that $4h_1 + 2h_2 = 0$ and

\[
4h_1 + 3h_2 > 0 \\
3h_1 + 2h_2 > 0 \\
2h_1 + 4h_2 > 0.
\] (A)

• The region is found to be lying on the top right sides above the two bold lines: (i) $3h_1 + 2h_2 = 0, h_1 < 0$ and (ii) $2h_1 + 4h_2 = 0, h_1 > 0$. It is seen that all the points on the dotted half line: $4h_1 + 2h_2 = 0, h_1 < 0$ represent dominant trading strategies that start with zero wealth but end with positive wealth with certainty.
The region above the two bold lines represents trading strategies that satisfy inequalities (A). The trading strategies that lie on the dotted line: $4h_1 + 2h_2 = 0, h_1 < 0$ are dominant trading strategies.
Suppose the initial discounted price vector is changed from $(4 \ 2)$ to $(3 \ 3)$, the new set of linear pricing probabilities will be determined by

\[
\begin{align*}
1 & = q(\omega_1) + q(\omega_2) + q(\omega_3) \\
3 & = 4q(\omega_1) + 3q(\omega_2) + 2q(\omega_3) \\
3 & = 3q(\omega_1) + 2q(\omega_2) + 4q(\omega_3),
\end{align*}
\]

which is seen to have the solution: $q(\omega_1) = q(\omega_2) = q(\omega_3) = 1/3$. Now, all the pricing probabilities have non-negative values, the row vector $q = (1/3 \ 1/3 \ 1/3)$ represents a linear pricing measure.

- The line $3h_1 + 3h_2 = 0$ always lies outside the region above the two bold lines.

- We cannot find $(h_1 \ h_2)$ such that $3h_1 + 3h_2 = 0$ together with $h_1$ and $h_2$ satisfying all these inequalities.


**Theorem**

There exists a linear pricing measure if and only if there are no dominant trading strategies.

The above linear pricing measure theorem can be seen to be a direct consequence of the Farkas Lemma.

**Farkas Lemma**

There does not exist $h \in \mathbb{R}^M$ such that

$$\tilde{S}^*(1; \Omega) h > 0 \quad \text{and} \quad \tilde{S}^*(0) h = 0$$

if and only if there exists $q \in \mathbb{R}^K$ such that

$$\tilde{S}^*(0) = q \tilde{S}^*(1; \Omega) \quad \text{and} \quad q \geq 0.$$
Given that the security lies in the asset span, we can deduce that law of one price holds by observing either

(i) null space of $S^*(1)$ has zero dimension, or

(ii) existence of a linear pricing measure.

Both (i) and (ii) represent the various forms of sufficient condition for the law of one price.

Remarks

1. Condition (i) is equivalent to non-existence of redundant securities.

2. Condition (i) and condition (ii) are not equivalent.
Various forms of sufficient condition for the law of one price
1.2 No-arbitrage theory and risk neutral probability measure
— Fundamental theorem of asset pricing

• An arbitrage opportunity is some trading strategy that has the following properties: (i) $V_0^* = 0$, (ii) $V_1^*(\omega) \geq 0$ with strict inequality at least for one state.

• The existence of a dominant strategy requires a portfolio with initial zero wealth to end up with a strictly positive wealth in all states.

• The existence of a dominant trading strategy implies the existence of an arbitrage opportunity, but the converse is not necessarily true.
Risk neutral probability measure

A probability measure $Q$ on $\Omega$ is a risk neutral probability measure if it satisfies

(i) $Q(\omega) > 0$ for all $\omega \in \Omega$, and

(ii) $E_Q[\Delta S^*_m] = 0$, $m = 0, 1, \ldots, M$, where $E_Q$ denotes the expectation under $Q$. The expectation of the discounted gain of any security in the securities model under $Q$ is zero.

Note that $E_Q[\Delta S^*_m] = 0$ is equivalent to $S^*_m(0) = \sum_{k=1}^{K} Q(\omega_k) S^*_m(1; \omega_k)$.

- In financial markets with no arbitrage opportunities, every investor should use such risk neutral probability measure (though not necessarily unique) to find the fair value of a contingent claim, independent of the subjective assessment of the probabilities of occurrence of different states.
Fundamental theorem of asset pricing

No arbitrage opportunities exist if and only if there exists a risk neutral probability measure $Q$.

- The proof of the Theorem requires the Separating Hyperplane Theorem.
- The Separating Hyperplane Theorem states that if $A$ and $B$ are two non-empty disjoint convex sets in a vector space $V$, then they can be separated by a hyperplane.
The hyperplane (represented by a line in $\mathbb{R}^2$) separates the two convex sets $A$ and $B$ in $\mathbb{R}^2$. A set $C$ is convex if any convex combination $\lambda x + (1 - \lambda)y$, $0 \leq \lambda \leq 1$, of a pair of vectors $x$ and $y$ in $C$ also lies in $C$. 
The hyperplane \([ f, \alpha ]\) separates the sets \(A\) and \(B\) in \(\mathbb{R}^n\) if there exists \(\alpha\) such that \(f \cdot x \geq \alpha\) for all \(x \in A\) and \(f \cdot y < \alpha\) for all \(y \in B\). In \(\mathbb{R}^2\) and \(\mathbb{R}^3\), the vector \(f\) has the geometric interpretation that it is the normal vector to the hyperplane.

For example, the hyperplane \(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\) separates the two disjoint convex sets \(A = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : x_1 \geq 0, x_2 \geq 0, x_3 \geq 0 \right\}\)

and \(B = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : x_1 < 0, x_2 < 0, x_3 < 0 \right\}\) in \(\mathbb{R}^3\).

Note that the hyperplane is not necessarily unique. In the above example, \(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, 0\) is another possible choice of the separating hyperplane.
Proof

“⇐ part”.

Assume that a risk neutral probability measure $Q$ exists, that is, $\hat{S}^*(0) = \pi \hat{S}^*(1; \Omega)$, where $\pi = (Q(\omega_1) \cdots Q(\omega_K))$ and $\pi > 0$. Under such assumption, we would like to show that it is never possible to construct a trading strategy that represents an arbitrage opportunity.

Consider a trading strategy $h = (h_0 \ h_1 \ \cdots \ h_M)^T \in \mathbb{R}^{M+1}$ such that $\hat{S}^*(1; \Omega)h \geq 0$ in all $\omega \in \Omega$ and with strict inequality in at least one state. Now consider $\hat{S}^*(0)h = \pi \hat{S}^*(1; \Omega)h$, it is seen that $\hat{S}^*(0)h > 0$ since all entries in $\pi$ are strictly positive and entries in $\hat{S}^*(1; \Omega)h$ are either zero or strictly positive. It is then impossible to have $\hat{S}(0)h = 0$ and $S^*(1; \Omega)h \geq 0$ in all $\omega \in \Omega$, with strict inequality in at least one state. Hence, no arbitrage opportunities exist.
First, we define the subset $U$ in $\mathbb{R}^{K+1}$ which consists of vectors of the form
\[
\begin{pmatrix}
-\hat{S}^*(0) & h \\
\hat{S}^*(1; \omega_1) & h \\
\vdots \\
\hat{S}^*(1; \omega_K) & h
\end{pmatrix}
\]
where $\hat{S}^*(1; \omega_k)$ is the $k$th row in $\hat{S}^*(1; \Omega)$ and $h \in \mathbb{R}^{M+1}$ represents a trading strategy. This subset is seen to be a subspace since $U$ contains the zero vector and $\alpha_1 h_1 + \alpha_2 h_2$ remains to be a trading strategy for any scalar multiples $\alpha_1$ and $\alpha_2$. The convexity property of $U$ is obvious.

Consider another subset $\mathbb{R}_+^{K+1}$ defined by
\[
\mathbb{R}_+^{K+1} = \{ \mathbf{x} = (x_0 \  x_1 \ \cdots \ x_K)^T \in \mathbb{R}^{K+1} : x_i \geq 0 \quad \text{for all} \quad 0 \leq i \leq K \},
\]
which is a convex set in $\mathbb{R}^{K+1}$.

We claim that the non-existence of arbitrage opportunities implies that $U$ and $\mathbb{R}_+^{K+1}$ can only have the zero vector in common.
Assume the contrary, suppose there exists a non-zero vector \( x \in U \cap \mathbb{R}^{K+1}_+ \). Since there is a trading strategy vector \( h \) associated with every vector in \( U \), it suffices to show that the trading strategy \( h \) associated with \( x \) always represents an arbitrage opportunity.

We consider the following two cases: \( -\tilde{S}^*(0)h = 0 \) or \( -\tilde{S}^*(0)h > 0 \).

(i) When \( \tilde{S}^*(0)h = 0 \), since \( x \neq 0 \) and \( x \in \mathbb{R}^{K+1}_+ \), then the entries \( \tilde{S}(1; \omega_k)h, k = 1, 2, \ldots K \), must be all greater than or equal to zero, with at least one strict inequality. In this case, \( h \) is seen to represent an arbitrage opportunity.

(ii) When \( \tilde{S}^*(0)h < 0 \), all the entries \( \tilde{S}(1; \omega_k)h, k = 1, 2, \ldots K \) must be all non-negative. Correspondingly, \( h \) represents a dominant trading strategy and in turns \( h \) is an arbitrage opportunity.
Since $U \cap R^{K+1}_+ = \{0\}$, by the Separating Hyperplane Theorem, there exists a hyperplane that separates the pair of disjoint convex sets: $R^{K+1}_+ \backslash \{0\}$ and $U$. One can show easily that this hyperplane must go through the origin, so its equation is of the form $[f, 0]$. Let $f \in R^{K+1}_+$ be the normal to this hyperplane, then we have $f \cdot x > f \cdot y$, for all $x \in R^{K+1}_+ \backslash \{0\}$ and $y \in U$.

[Remark: We may have $f \cdot x < f \cdot y$, depending on the orientation of the normal vector $f$. However, the final conclusion remains unchanged.]
Two-dimensional case

(i) \( f \cdot y = 0 \) for all \( y \in U \);

(ii) \( f \cdot x > 0 \) for all \( x \in \mathbb{R}^2_+ \setminus \{0\} \).
Since $U$ is a linear subspace so that a negative multiple of $y \in U$ also belongs to $U$. Note that $f \cdot x > f \cdot y$ and $f \cdot x > f \cdot (-y)$ both holds only if $f \cdot y = 0$ for all $y \in U$.

**Remark**

Interestingly, all vectors in $U$ lie in the hyperplane $f \cdot y = 0$ through the origin. This hyperplane separates $U$ (hyperplane itself) and $R^{K+1}_+ \{0\}$.

We have $f \cdot x > 0$ for all $x$ in $R^{K+1}_+ \{0\}$. This requires all entries in $f$ to be strictly positive. Note that if at least one of the components (say, the $i^{th}$ component) of $f$ is zero or negative, then we choose $x$ to be the $i^{th}$ coordinate vector. This gives $f \cdot x \leq 0$, a violation of $f \cdot x > 0$. 
From $f \cdot y = 0$, we have

$$-f_0 \tilde{S}^*(0) h + \sum_{k=1}^{K} f_k \tilde{S}^*(1; \omega_k) h = 0$$

for all $h \in \mathbb{R}^{M+1}$, where $f_j, j = 0, 1, \cdots, K$ are the entries of $f$. We then deduce that

$$\tilde{S}^*(0) = \sum_{k=1}^{K} Q(\omega_k) \tilde{S}^*(1; \omega_k), \text{ where } Q(\omega_k) = f_k / f_0.$$

Consider the first component in the vectors on both sides of the above equation. They both correspond to the current price and discounted payoff of the riskless security, and all are equal to one. We then obtain

$$1 = \sum_{k=1}^{K} Q(\omega_k).$$
We obtain the risk neutral probabilities $Q(\omega_k), k = 1, \cdots, K,$ whose sum is equal to one and they are all strictly positive since $f_j > 0, j = 0, 1, \cdots, K.$

Corresponding to each risky asset, we have

$$S_m^*(0) = \sum_{k=1}^{K} Q(\omega_k) S_m^*(1; \omega_k), \quad m = 1, 2, \cdots, M.$$ 

Hence, the current price of any one of risky securities in the securities model is given by the expectation of the discounted payoff under the risk neutral measure $Q$.

**Remark**

The existence of the separating hyperplane leads to the existence of $Q(\omega_k), k = 1, \cdots, K,$ as determined by the ratio of some appropriate entries in the normal vector $f$ to the hyperplane. The non-uniqueness of the separating hyperplane leads to non-uniqueness of the risk neutral measure.
Remark The securities model contains the riskfree asset when we consider the linear pricing measure $q$ and martingale pricing measure $Q$. 

Financial Economics

Linear Algebra: $\pi S^*(1) = S(0)$
$q \hat{S}^*(1) = \hat{S}(0)$
$Q \hat{S}^*(1) = \hat{S}(0)$

Remark The securities model contains the riskfree asset when we consider the linear pricing measure $q$ and martingale pricing measure $Q$. 

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(Existence of arbitrage opportunities but non-existence of dominant trading strategies)

Consider the securities model

\[
(1 \ 2 \ 3 \ 6) = (\pi_1 \ \pi_2 \ \pi_3) \begin{pmatrix} 1 & 2 & 3 & 6 \\ 1 & 3 & 4 & 8 \\ 1 & 6 & 7 & 14 \end{pmatrix},
\]

where the number of non-redundant securities is only 2. Note that

\[
S^*_2(1; \Omega) = S^*_0(1; \Omega) + S^*_1(1; \Omega) \quad \text{and}
\]

\[
S^*_3(1; \Omega) = S^*_0(1; \Omega) + S^*_1(1; \Omega) + S^*_2(1; \Omega),
\]

and the initial prices have been set such that

\[
S^*_2(0) = S^*_0(0) + S^*_1(0) \quad \text{and} \quad S^*_3(0) = S^*_0(0) + S^*_1(0) + S^*_2(0),
\]

so we expect to have the existence of solution. However, since \(2 = \text{number of non-redundant securities} < \text{number of states} = 3\), we do not have uniqueness of solution. Indeed, we obtain

\[
(\pi_1 \ \pi_2 \ \pi_3) = (1 \ 0 \ 0) + t(3 \ -4 \ 1), \quad t \text{ any value.}
\]
For example, when we take $t = 1$, then

$$(\pi_1 \quad \pi_2 \quad \pi_3) = (4 \quad -4 \quad 1).$$

In terms of linear algebra, we have existence of solution if the equations are consistent. Consider the present example, we have

$$\begin{align*}
\pi_1 + \pi_2 + \pi_3 &= 1 \\
2\pi_1 + 3\pi_2 + 6\pi_3 &= 2 \\
3\pi_1 + 4\pi_2 + 7\pi_3 &= 3 \\
6\pi_1 + 8\pi_2 + 14\pi_3 &= 6
\end{align*}$$

Note that the last two redundant equations are consistent. Alternatively, we can interpret that the row vector $S^*(0) = (1 \quad 2 \quad 3 \quad 6)$ lies in the row space of $\hat{S}^*(1; \Omega)$, which is spanned by $\{(1 \quad 2 \quad 3 \quad 6), \quad (0 \quad 1 \quad 1 \quad 2)\}$. 
In this securities model, we cannot find a risk neutral measure where \((Q_1 \ Q_2 \ Q_3) > 0\). This is easily seen since \(\pi_2 = -4t\) and \(\pi_3 = t\), and they always have opposite sign. However, a linear pricing measure exists. One such example is \((q_1 \ q_2 \ q_3) = (1 \ 0 \ 0) > 0\).

Since \(Q\) does not exist, the securities model admits arbitrage opportunities. One such example is \(h = (-11 \ 1 \ 1 \ 1)^T\), where

\[
S^*(0)h = 0 \quad \text{and} \quad S^*(1; \Omega)h = \begin{pmatrix}
1 & 2 & 3 & 6 \\
1 & 3 & 4 & 8 \\
1 & 6 & 7 & 14
\end{pmatrix}\begin{pmatrix}
-11 \\
1 \\
1
\end{pmatrix} = \begin{pmatrix}
0 \\
14 \\
5
\end{pmatrix}.
\]

The discounted portfolio value at \(t = 1\) is guaranteed to be non-negative, with strict positivity for at least one state.
However, the securities model does not admit dominant trading strategies since a linear pricing measure exists. This is evidenced by showing that one cannot find a trading strategy \( \mathbf{h} = (h_0 \ h_1 \ h_2 \ h_3)^T \) such that

\[
    h_0 + 2h_1 + 3h_2 + 6h_3 = 0
\]

while

\[
    h_0 + 2h_1 + 3h_2 + 6h_3 > 0, \quad h_0 + 3h_1 + 4h_2 + 8h_3 > 0, \\
    h_0 + 6h_1 + 6h_2 + 14h_3 > 0.
\]

The first inequality can never be satisfied when we impose \( h_0 + 2h_1 + 3h_2 + 6h_3 = 0 \). Indeed, when \( S^*(0) = S^*(1; \omega_k) \) for some \( \omega_k \), then a linear pricing measure exists where \( \mathbf{q} = e_k^T \).
• Martingale property is defined for adapted stochastic processes*. In the context of one-period model, given the information on the initial prices and terminal payoff values of the security prices at \( t = 0 \),

\[
S_m^*(0) = E_Q[S_m^*(1; \Omega)] = \sum_{k=1}^{K} S_m^*(1; \omega_k)Q(\omega_k), \quad m = 1, 2, \cdots, M.
\]

(1)

The discounted security price process \( S_m^*(t) \) is said to be a martingale† under \( Q \).

Martingale is associated with the wealth process of a gambler in a fair game. In a fair game, the expected value of the gambler’s wealth after any number of plays is always equal to her initial wealth.

*A stochastic process is adapted to a filtration with respect to a measure. Suppose \( S_m^* \) is adapted to \( \mathcal{F} = \{ \mathcal{F}_t; t = 0, 1, \cdots, T \} \), we say \( S_m^*(t) \) is \( \mathcal{F}_t \)-measurable.

†Martingale property with respect to \( Q \) and \( \mathcal{F} \):

\[
S_m^*(t) = E_Q[S_m^*(s + t)|\mathcal{F}_t] \text{ for all } t \geq 0, s \geq 0.
\]
Equivalent martingale measure

• The risk neutral probability measure $Q$ is commonly called the equivalent martingale measure. “Equivalent” refers to the equivalence between the physical measure $P$ and martingale measure $Q$ [observing $P(\omega) > 0 \Leftrightarrow Q(\omega) > 0$ for all $\omega \in \Omega$]. The linear pricing measure falls short of this equivalence property since $q(\omega)$ can be zero.

*$P$ and $Q$ may not agree on the assignment of probability values to individual events, but they always agree as to which events are possible or impossible.
Martingale property of discounted portfolio value (assuming the existence of \( Q \) or equivalently, the absence of arbitrage in the securities model)

- Let \( V_1^*(\Omega) \) denote the discounted payoff of a portfolio. Since \( V_1^*(\Omega) = \tilde{S}^*(1; \Omega)h \) for some trading strategy \( h = (h_0 \cdots h_M)^T \), by Eq. (1),

\[
V_0^* = (S_0^*(0) \cdots S_M^*(0))h \\
= (E_Q[S_0^*(1; \Omega)] \cdots E_Q[S_M^*(1; \Omega)])h \\
= \sum_{m=0}^{M} \left[ \sum_{k=1}^{K} S_m^*(1; \omega_k)Q(\omega_k) \right] h_m \\
= \sum_{k=1}^{K} Q(\omega_k) \left[ \sum_{m=0}^{M} S_m^*(1; \omega_k)h_m \right] = E_Q[V_1^*(\Omega)].
\]

- The equivalent martingale measure \( Q \) is not necessarily unique. Since “absence of arbitrage opportunities” implies “law of one price”, the expectation value \( E_Q[V_1^*(\Omega)] \) is single-valued under all equivalent martingale measures.
Finding the set of risk neutral measures

Consider the earlier securities model with the riskfree security and only one risky security, where \( \hat{S}(1; \Omega) = \begin{pmatrix} 1 & 4 \\ 1 & 3 \\ 1 & 2 \end{pmatrix} \) and \( \hat{S}(0) = (1 \ 3) \). The risk neutral probability measure

\[
Q = (Q(\omega_1) \ Q(\omega_2) \ Q(\omega_3)),
\]

if exists, will be determined by the following system of equations

\[
\begin{pmatrix} Q(\omega_1) & Q(\omega_2) & Q(\omega_3) \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 1 & 3 \\ 1 & 2 \end{pmatrix} = (1 \ 3).
\]

Since there are more unknowns than the number of equations, the solution is not unique. The solution is found to be \( Q = (\lambda \ 1 - 2\lambda \ \lambda) \), where \( \lambda \) is a free parameter. Since all risk neutral probabilities are all strictly positive, we must have \( 0 < \lambda < 1/2 \).
Under market completeness, if the set of risk neutral measures is non-empty, then it must be a singleton.

Under market completeness, column rank of $\hat{S}^*(1; \Omega)$ equals the number of states. Since column rank = row rank, then all rows of $\hat{S}^*(1; \Omega)$ are independent. If solution $Q$ exists for

$$Q\hat{S}^*(1; \Omega) = \hat{S}^*(0),$$

then it must be unique. Note that $Q > 0$.

Conversely, suppose the set of risk neutral measures is a singleton, one can show that the securities model is complete (see later discussion).
Numerical example

Suppose we add the second risky security with discounted payoff

\[ S_2^*(1) = \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix} \]

and current discounted value \( S_2^*(0) = 3 \). With this new addition, the securities model becomes complete.

With the new equation \( 3Q(\omega_1) + 2Q(\omega_2) + 4Q(\omega_3) = 3 \) added to the system, this new securities model is seen to have the unique risk neutral measure \((1/3 \quad 1/3 \quad 1/3)\).

Indeed, when the securities model is complete, all Arrow securities are replicable. Their prices (called state prices) are simply equal to the risk neutral measures. In this example, we have

\[ s_1 = Q(\omega_1) = \frac{1}{3}, \quad s_2 = Q(\omega_2) = \frac{1}{3}, \quad s_3 = Q(\omega_3) = \frac{1}{3}. \]
Subspace of discounted gains

Let $W$ be a subspace in $\mathbb{R}^K$ which consists of discounted gains corresponding to some trading strategy $h$. Note that $W$ is spanned by the set of vectors representing discounted gains of the risky securities.

In the above securities model, the discounted gains of the first and second risky securities are

$$
\begin{pmatrix}
4 \\
3 \\
2
\end{pmatrix} - \begin{pmatrix}
3 \\
3 \\
3
\end{pmatrix} = \begin{pmatrix}
1 \\
0 \\
-1
\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}
3 \\
2 \\
4
\end{pmatrix} - \begin{pmatrix}
3 \\
3 \\
3
\end{pmatrix} = \begin{pmatrix}
0 \\
-1 \\
1
\end{pmatrix}, \text{respectively.}
$$

The discounted gain subspace is given by

$$
W = \left\{ h_1 \begin{pmatrix}
1 \\
0 \\
-1
\end{pmatrix} + h_2 \begin{pmatrix}
0 \\
-1 \\
1
\end{pmatrix}, \text{where } h_1 \text{ and } h_2 \text{ are scalars} \right\}.
$$
Orthogonality of the discounted gain vector and $Q$

Let $G^*$ denote the discounted gain of a portfolio. For any risk neutral probability measure $Q$, we have

$$E_QG^* = \sum_{k=1}^{K} Q(\omega_k) \left[ \sum_{m=1}^{M} h_m \Delta S^*_m(\omega_k) \right]$$

$$= \sum_{m=1}^{M} h_m E_Q[\Delta S^*_m] = 0.$$

Under the absence of arbitrage opportunities, the expected discounted gain from any risky portfolio is simply zero. Apparently, there is no risk premium derived from the risky investment. Therefore, the financial economics term “risk neutrality” is adopted under this framework of asset pricing.

For any $G^* = (G(\omega_1) \cdots G(\omega_K))^T \in W$, we have

$$QG^* = 0,$$

where $Q = (Q(\omega_1) \cdots Q(\omega_K)).$
Characterization of the set of neutral measures

Since the sum of risk neutral probabilities must be one and all probability values must be positive, the risk neutral probability vector $Q$ must lie in the following subset

$$P^+ = \{ y \in \mathbb{R}^K : y_1 + y_2 + \cdots + y_K = 1 \quad \text{and} \quad y_k > 0, k = 1, \ldots, K \}.$$

Also, the risk neutral probability vector $Q$ must lie in the orthogonal complement $W^\perp$. Let $R$ denote the set of all risk neutral measures, then $R = P^+ \cap W^\perp$.

In the above numerical example, $W^\perp$ is the line through the origin in $\mathbb{R}^3$ which is perpendicular to $(1 \ 0 \ -1)^T$ and $(0 \ -1 \ 1)^T$. The line should assume the form $\lambda(1 \ 1 \ 1)$ for some scalar $\lambda$. We obtain the risk neutral probability vector $Q = (1/3 \ 1/3 \ 1/3)$. 
1.3 Valuation of contingent claims: complete and incomplete markets

- A contingent claim can be considered as a random variable $Y$ that represents a terminal payoff whose value depends on the occurrence of a particular state $\omega_k$, where $\omega_k \in \Omega$.

- Suppose the holder of the contingent claim is promised to receive the preset contingent payoff, how much should the writer of such contingent claim charge at $t = 0$ so that the price is fair to both parties.

- Consider the securities model with the riskfree security whose values at $t = 0$ and $t = 1$ are $S_0(0) = 1$ and $S_0(1) = 1.1$, respectively, and a risky security with $S_1(0) = 3$ and $S_1(1) = \begin{pmatrix} 4.4 \\ 3.3 \\ 2.2 \end{pmatrix}$. 
The set of $t = 1$ payoffs that can be generated by certain trading strategy is given by
$h_0 \begin{pmatrix} 1.1 \\ 1.1 \\ 1.1 \end{pmatrix} + h_1 \begin{pmatrix} 4.4 \\ 3.3 \\ 2.2 \end{pmatrix}$ for some scalars $h_0$ and $h_1$.

For example, the contingent claim $\begin{pmatrix} 5.5 \\ 4.4 \\ 3.3 \end{pmatrix}$ can be generated by the trading strategy: $h_0 = 1$ and $h_1 = 1$, while the other contingent claim $\begin{pmatrix} 5.5 \\ 4.0 \\ 3.3 \end{pmatrix}$ cannot be generated by any trading strategy associated with the given securities model.
A contingent claim $Y$ is said to be *attainable* if there exists some trading strategy $h$, called the *replicating portfolio*, such that $V_1 = Y$ for all possible states occurring at $t = 1$.

The price at $t = 0$ of the replicating portfolio is given by

\[ V_0 = h_0 S_0(0) + h_1 S_1(0) = 1 \times 1 + 1 \times 3 = 4. \]

Suppose there are no arbitrage opportunities (equivalent to the existence of a risk neutral probability measure), then the law of one price holds and so $V_0$ is unique.
Pricing of attainable contingent claims

Let $V_1^*(1; \Omega)$ denote the value of the replicating portfolio that matches with the payoff of the attainable contingent claim at every state of the world. Suppose the associated trading strategy to generate the replicating portfolio is $h$, then

$$V_1^* = \hat{S}^*(1; \Omega)h.$$ 

The initial cost of setting up the replicating portfolio is

$$V_0^* = \hat{S}^*(0)h.$$ 

Assuming $\pi$ exists, where $\hat{S}^*(0) = \pi\hat{S}^*(1; \Omega)$ so that

$$V_0^* = \pi\hat{S}^*(1; \Omega)h = \pi V_1^*(1; \Omega)$$

$$= \sum_{k=1}^{K} \pi_k V_1^*(1; \omega_k), \text{ independent of } h.$$ 

Even when $\pi$ is not a risk neutral measure or linear pricing measure, the above pricing relation remains valid. Though $\pi$ may not be unique, by virtue of law of one price, we have the same value for $V_0^*$. 

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Consider a given attainable contingent claim $Y$ which is generated by certain trading strategy. The associated discounted gain $G^*$ of the trading strategy is given by $G^* = \sum_{m=1}^{M} h_m \Delta S_m^*$. Now, suppose a risk neutral probability measure $Q$ associated with the securities model exists, we have

$$V_0 = E_Q V_0^* = E_Q[V_1^* - G^*].$$

Since $E_Q[G^*] = 0$ and $V_1^* = Y/S_0(1)$, we obtain

$$V_0 = E_Q[Y/S_0(1)].$$

**Risk neutral valuation principle**

The price at $t = 0$ of an attainable claim $Y$ is given by the expectation under any risk neutral measure $Q$ of the discounted value of the contingent claim.
Attainability of a contingent claim and uniqueness of $E_Q[Y^*]$

- Recall that the existence of the risk neutral probability measure implies the law of one price. Does $E_Q[Y/S_0(1)]$ assume the same value for every risk neutral probability measure $Q$?

Provided that $Y$ is attainable, this must be true by virtue of the law of one price since we cannot have two different values for $V_0$ corresponding to the same attainable contingent claim $Y$.

**Theorem**

Suppose the securities model admits no arbitrage opportunities. The contingent claim $Y$ is attainable if and only if $E_Q[Y^*]$ takes the same value for every $Q \in M$, where $M$ is the set of risk neutral measures.
Proof

\[ \implies \text{part} \]

existence of \( Q \iff \text{absence of arbitrage} \implies \text{law of one price}. \] For an attainable \( Y \), \( E_Q[Y^*] \) is constant with respect to all \( Q \in M \), otherwise this leads to violation of the law of one price.

\[ \iff \text{part} \]

It suffices to show that if the contingent claim \( Y \) is not attainable then \( E_Q[Y^*] \) does not take the same value for all \( Q \in M \).
Let $y^* \in \mathbb{R}^K$ be the discounted payoff vector corresponding to $Y^*$. Since $Y$ is not attainable, then there is no solution to

$$\hat{S}^*(1)h = y^*$$

(non-existence of trading strategy $h$). It then follows that there must exist a non-zero row vector $\pi \in \mathbb{R}^K$ such that

$$\pi\hat{S}^*(1) = 0 \quad \text{and} \quad \pi y^* \neq 0.$$

**Remark**

Recall that the orthogonal complement of the column space is the left null space. The dimension of the left null space equals $K$–column rank, and it is non-zero when the column space does not span the whole $\mathbb{R}^K$. The above result indicates that when $y^*$ is not in the column space of $S^*(1)$, then there exists a non-zero vector $\pi$ in the left null space of $S^*(1)$ such that $y^*$ and $\pi$ are not orthogonal.
Write $\pi = (\pi_1 \cdots \pi_K)$. Let $\hat{Q} \in M$ be arbitrary, and let $\lambda > 0$ be small enough such that

$$Q(\omega_k) = \hat{Q}(\omega_k) + \lambda \pi_k > 0, \quad k = 1, 2, \ldots, K.$$ 

We would like to show that $Q(\omega_k)$ is also a risk neutral measure by virtue of the relation: $\pi \hat{S}^*(1) = 0$.

1. Note that $\pi \mathbf{1} = \sum_{k=1}^{K} \pi_k = 0$, so $\sum_{k=1}^{K} Q(\omega_k) = 1$.

2. For the discounted price process $S^*_n$ of the $n^{\text{th}}$ risky securities in the securities model, we have

$$E_Q[S^*_n(1)] = \sum_{k=1}^{K} Q(\omega_k) S^*_n(1; \omega_k)$$

$$= \sum_{k=1}^{K} \hat{Q}(\omega_k) S^*_n(1; \omega_k) + \lambda \sum_{k=1}^{K} \pi_k S^*_n(1; \omega_k)$$

$$= \sum_{k=1}^{K} \hat{Q}(\omega_k) S^*_n(1; \omega_k) = S_n(0).$$
$Q$ satisfies the martingale property, together with $Q(\omega_k) > 0$ and
\[ \sum_{k=1}^{K} Q(\omega_k) = 1 \] so it is also a risk neutral measure.

Lastly, we consider
\[
E_Q[Y^*] = \sum_{k=1}^{K} Q(\omega_k)Y^*(\omega_k)
\]
\[= \sum_{k=1}^{K} \hat{Q}(\omega_k)Y^*(\omega_k) + \lambda \sum_{k=1}^{K} \pi_k Y^*(\omega_k).\]

The last term is non-zero since $\pi y^* \neq 0$ and $\lambda > 0$. Therefore, we have
\[ E_Q[Y^*] \neq E_{\hat{Q}}[Y^*]. \]

Thus, when $Y$ is not attainable, $E_Q[Y^*]$ does not take the same value for all risk neutral measures.
**Corollary** Given that the set of risk neutral measures $R$ is non-empty. The securities model is complete if and only if $R$ consists of exactly one risk neutral measure.

An earlier proof of “$\implies$ part” has been shown on p.69. Alternatively, we may prove by contradiction: non-uniqueness of $Q \implies$ non-completeness.

Suppose there exist two distinct $Q$ and $\hat{Q}$, that is, $Q(\omega_k) \neq \hat{Q}(\omega_k)$ for some state $\omega_k$. Let $Y^* = \begin{cases} 1 & \text{if } \omega = \omega_k \\ 0 & \text{otherwise} \end{cases}$, which is the $k^{th}$ Arrow security. Obviously,

$$E_Q[Y^*] = Q(\omega_k) \neq \hat{Q}(\omega_k) = E_{\hat{Q}}[Y^*],$$

so $E_Q[Y^*]$ is not unique. By the theorem, $Y^*$ is not attainable so the securities model is not complete.

$\iff$ part: If the risk neutral measure is unique, then for any contingent claim $Y$, $E_Q[Y^*]$ takes the same value for any $Q$ (actually single $Q$). Hence, any contingent claim is attainable so the market is complete.
Remarks

- When the securities model is complete and admits no arbitrage opportunities, all Arrow securities lie in the asset span and risk neutral measures exist. The state price of state $\omega_k$ exists for any state and it is equal to the unique risk neutral probability $Q(\omega_k)$. This represents the best scenario of applying the risk neutral valuation procedure for pricing any contingent claim (which is always attainable due to completeness).

- On the other hand, suppose there are two risk neutral probability values for the same state $\omega_k$, the state price of that state cannot be defined in proper sense without contradicting the law of one price. Actually, by the theorem, the Arrow security of that state would not be attainable, so the state price of that state is not defined. Furthermore, we deduce that the securities model cannot be complete.
Example

Suppose

\[ Y^* = \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix} \quad \text{and} \quad \hat{S}^*(1; \Omega) = \begin{pmatrix} 1 & 4 \\ 1 & 3 \\ 1 & 2 \end{pmatrix}, \]

\( Y^* \) is seen to be attainable. We have seen that the risk neutral probability is given by

\[ Q = (\lambda \quad 1 - 2\lambda \quad \lambda), \] where \( 0 < \lambda < 1/2. \)

The price at \( t = 0 \) of the contingent claim is given by

\[ V_0 = 5\lambda + 4(1 - 2\lambda) + 3\lambda = 4, \]

which is independent of \( \lambda \). This verifies the earlier claim that \( E_Q[Y/S_0(1)] \) assumes the same value for any risk neutral measure \( Q \).

Suppose \( Y^* \) is changed to \( (5 \quad 4 \quad 4)^T \), then \( V_0 = E_Q[Y^*] = 4 + \lambda \), which is not unique. This is expected since the new \( Y^* \) is non-attainable.
Complete markets - summary of results

Recall that a securities model is complete if every contingent claim $Y$ lies in the asset span, that is, $Y$ can be generated by some trading strategy.

Consider the augmented terminal payoff matrix

$$\widehat{S}(1; \Omega) = \begin{pmatrix} S_0(1; \omega_1) & S_1(1; \omega_1) & \cdots & S_M(1; \omega_1) \\ \vdots & \vdots & \ddots & \vdots \\ S_0(1; \omega_K) & S_1(1; \omega_K) & \cdots & S_M(1; \omega_K) \end{pmatrix},$$

$Y$ always lies in the asset span if and only if the column space of $\widehat{S}(1; \Omega)$ is equal to $\mathbb{R}^K$.

- Since the dimension of the column space of $\widehat{S}(1; \Omega)$ cannot be greater than $M + 1$, a necessary condition for market completeness is that $M + 1 \geq K$. 
• When $\tilde{S}(1; \Omega)$ has independent columns and the asset span is the whole $\mathbb{R}^K$, then $M + 1 = K$. Now, the trading strategy that generates $Y$ must be unique since there are no redundant securities. In this case, any contingent claim is replicable and its price is unique. Though law of one price holds, there is no guarantee that arbitrage opportunities do not exist.

• When the asset span is the whole $\mathbb{R}^K$ but some securities are redundant, the trading strategy that generates $Y$ would not be unique. Suppose absence of arbitrage is observed, the price at $t = 0$ of the contingent claim is unique under risk neutral pricing, independent of the chosen trading strategy. This is a consequence of the law of one price, which holds since a risk neutral measure exists.

• Non-existence of redundant securities is a sufficient but not necessary condition for law of one price.
Non-attainable contingent claim

Suppose a risk neutral measure $Q$ exists, risk neutral valuation fails when we price a non-attainable contingent claim. However, we may specify an interval $(V_-(Y), V_+(Y))$ where a reasonable price at $t = 0$ of the contingent claim should lie. The lower and upper bounds are given by

$$\begin{align*}
V_+(Y) &= \inf\{E_Q[\tilde{Y}/S_0(1)]: \tilde{Y} \geq Y \text{ and } \tilde{Y} \text{ is attainable}\} \\
V_-(Y) &= \sup\{E_Q[\tilde{Y}/S_0(1)]: \tilde{Y} \leq Y \text{ and } \tilde{Y} \text{ is attainable}\}.
\end{align*}$$

Here, $V_+(Y)$ is the minimum value among all prices of attainable contingent claims that dominate the non-attainable claim $Y$, while $V_-(Y)$ is the maximum value among all prices of attainable contingent claims that are dominated by $Y$.

Note that there exists a sufficiently large scalar $\lambda$ such that $\lambda S_0(1) > Y$, so $V_+(Y)$ is finite and well defined. Since $E_Q[\tilde{Y}/S_0(1)]$ is constant with respect to all $Q \in R$ and $\tilde{Y} \geq Y$, so $V_+(Y)$ is bounded below by $\sup\{E_Q[Y^*]: Q \in R\}$. 
Proof of the upper bound

Suppose $V(Y) > V_+(Y)$, then an arbitrageur can lock in riskless profit by selling the contingent claim to receive $V(Y)$ and use $V_+(Y)$ to construct the replicating portfolio that generates the attainable $\tilde{Y}$. The upfront positive gain is $V(Y) - V_+(Y)$ and the terminal gain is $\tilde{Y} - Y$.

Alternatively, based on the linear programming duality theory, we have the following results:

If $R \neq \phi$, then for any contingent claim $Y$, we have

$$V_+(Y) = \sup \{E_Q[Y^*] : Q \in R\},$$

$$V_-(Y) = \inf \{E_Q[Y^*] : Q \in R\}.$$

If $Y$ is attainable, then $V_+(Y) = V_-(Y)$. 
Example

Consider the securities model: \( \tilde{S}(0) = (1 \ 3) \) and \( \tilde{S}^*(1; \Omega) = \begin{pmatrix} 1 & 4 \\ 1 & 3 \\ 1 & 2 \end{pmatrix} \), and the non-attainable contingent claim \( Y^* = \begin{pmatrix} 5 \\ 4 \\ 4 \end{pmatrix} \). The risk neutral measure is

\[
Q = (\lambda \ 1 - 2\lambda \ \lambda), \quad \text{where } 0 < \lambda < 1/2.
\]

Note that \( E_Q[Y^*] = 4 + \lambda \) so that

\[
V_+ = \sup\{E_Q[Y^*] : Q \in R\} = 9/2 \quad \text{and} \quad V_- = \inf\{E_Q[Y^*] : Q \in R\} = 4.
\]
The attainable contingent claim corresponding to \( V_+ \) is

\[
\tilde{Y}^* = \begin{pmatrix} 5 \\ 4.5 \\ 4 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 0.5 \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}, \quad \text{where } E_Q[\tilde{Y}^*_+] = 4.5.
\]

On the other hand, the attainable contingent claim corresponding to \( V_- \) is

\[
\tilde{Y}^* = \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}, \quad \text{where } E_Q[\tilde{Y}^*_-] = 4.
\]

Any reasonable initial price of the non-attainable contingent claim \( Y^* = (5 \ 4 \ 4)^T \) should lie between the interval (4, 4.5).
Linear programming formulation

Recall that the set of all risk neutral measures $R$ is given by

$$R = W^\perp \cap P^+;$$

and $W = \{x \in \mathbb{R}^K : x = G^* \text{ for some trading strategy } h\}$, where

$$\text{discount gain } = G^* = \sum_{m=1}^{M} h_m \Delta S_m^*.$$ 

$$W^\perp = \{y \in \mathbb{R}^K : x^T y = 0 \text{ for all } x \in W\}$$

$$P^+ = \{x \in \mathbb{R}^K : x_1 + \cdots + x_K = 1, x_1 > 0, \cdots, x_K > 0\}.$$ 

Let $J$ be the dimension of $W^\perp$, $Q_j \in R = W^\perp \cap P^+, j = 1, \cdots J$; and they are chosen to be independent vectors, thus forming a basis of $W^\perp$. Then

$$W = \{x \in \mathbb{R}^K : x^T Q_j = 0, \quad j = 1, 2, \cdots, J\}.$$ 

For an attainable contingent claim $X$, whose terminal payoff vector is $x$, how to find the upper bound $V_+(X)$?

Solve the following linear program

$$\begin{align*}
\text{minimize} & \quad \lambda \\
\text{subject to} & \quad y \geq x \\
& \quad y^* = y/S_0(1) \\
& \quad \lambda = y^T Q_1 \\
& \quad \vdots \\
& \quad \lambda = y^T Q_J \\
\end{align*}$$

$\lambda \in \mathbb{R}, y \in \mathbb{R}^K$.

We enforce the condition that $E_Q[Y/S_0(1)]$ takes the same value for every risk neutral measure $Q$. 
Justification of the linear programming formulation

Let \( e \) denote the vector whose components are all one.

Suppose \( Y \) is an attainable contingent claim with initial price \( \lambda \). Since \( V_1^* = V_0 + G^* \), this is equivalent to say

\[
y^* - \lambda e \in W.
\]

Since \( e^T Q_j = 1 \), so we have \( y^{*T} Q_j = \lambda \) for \( j = 1, 2, \ldots, J \).

The feasible region is the set of all attainable contingent claims \( Y \) with \( y \geq x \).

If \( \lambda \) and \( Y \) are part of an optimal solution of the linear programming problem, then \( V_+(X) = \lambda \) and \( Y \) is an attainable contingent claim with \( y \geq x \) and initial price is \( V_+(X) \).

An optimal solution always exists since the feasible region is nonempty and the objective function is bounded below.
Summary  

An arbitrage strategy is requiring no initial investment, having no chance of occurrence of negative value at expiration, and yet having some possibility of a positive terminal portfolio value.

- It is commonly assumed that there are no arbitrage opportunities in well functioning and competitive financial markets.

1. absence of arbitrage opportunities
   ⇒ absence of dominant trading strategies
   ⇒ law of one price
2. absence of arbitrage opportunities ⇔ existence of risk neutral measure
   absence of dominant trading strategies ⇔ existence of linear pricing measure.

3. Under market completeness, the state prices are non-negative when a linear pricing measure exists and they become strictly positive when a risk neutral measure exists.

4. Under the absence of arbitrage opportunities, the risk neutral valuation principle can be applied to find the fair price of an attainable contingent claim.
1.4 Binomial option pricing model

By buying the asset and borrowing cash (in the form of riskless money market account) in appropriate proportions, one can replicate the position of a call.

Under the binomial random walk model, the asset prices after one period $\Delta t$ will be either $uS$ or $dS$ with probability $q$ and $1 - q$, respectively.

We assume $u > 1 > d$ so that $uS$ and $dS$ represent the up-move and down-move of the asset price, respectively. The jump parameters $u$ and $d$ will be related to the asset price dynamics.
Let $R$ denote the growth factor of riskless investment over one period so that $1$ invested in a riskless money market account will grow to $R$ after one period. In order to avoid riskless arbitrage opportunities, we must have $u > R > d$.

For example, suppose $u > d > R$, then we borrow as much as possible for the riskfree asset and use the loan to buy the risky asset. Even the downward move of the risky asset generates a return better than the riskfree rate. This represents an arbitrage.

Suppose we form a portfolio which consists of $\alpha$ units of asset and cash amount $M$ in the form of riskless money market account. After one period $\Delta t$, the value of the portfolio becomes

$$\begin{cases} 
\alpha u S + RM & \text{with probability } q \\
\alpha d S + RM & \text{with probability } 1 - q.
\end{cases}$$
Valuation of a call option using the approach of replication

The portfolio is used to replicate the long position of a call option on a non-dividend paying asset.

As there are two possible states of the world: asset price goes up or down, the call is thus a contingent claim.

Suppose the current time is only one period $\Delta t$ prior to expiration. Let $c$ denote the current call price, and $c_u$ and $c_d$ denote the call price after one period (which is the expiration time in the present context) corresponding to the up-move and down-move of the asset price, respectively.
Let $X$ denote the strike price of the call. The payoff of the call at expiry is given by

\[
\begin{aligned}
    c_u &= \max(uS - X, 0) \quad \text{with probability } q \\
    c_d &= \max(dS - X, 0) \quad \text{with probability } 1 - q.
\end{aligned}
\]

One can establish easily that $uc_d - dc_u \leq 0$.

\[
\begin{aligned}
    \alpha S + M \\
    \alpha uS + RM \quad \text{with probability } q \\
    \alpha dS + RM \quad \text{with probability } 1 - q
\end{aligned}
\]

*Evolution of the asset price $S$ and money market account $M$ after one time period under the binomial model.*
**Concept of replication revisited**

The above portfolio containing the risky asset and money market account is said to replicate the long position of the call if and only if the values of the portfolio and the call option match for each possible outcome, that is,

\[
\alpha u S + RM = c_u \quad \text{and} \quad \alpha d S + RM = c_d.
\]

Solving the equations, we obtain

\[
\alpha = \frac{c_u - c_d}{(u - d)S} > 0, \quad M = \frac{uc_d - dc_u}{(u - d)R} < 0.
\]

- Apparently, we are fortunate to have two instruments in the replicating portfolio and two states of the world so that the number of equations equals the number of unknowns. The securities model is complete.
1. The parameters $\alpha$ and $M$ are seen to have opposite sign since cash is paid to acquire stock when the call is exercised.

2. $u/d < c_u/c_d$ due to the leverage effect inherited in the call option. That is, when a given upside growth/downside drop is experienced in the stock, the corresponding ratio is higher in the call.

- The number of units of asset held is seen to be the ratio of the difference of call values $c_u - c_d$ to the difference of asset values $uS - dS$.

- The call option can be replicated by a portfolio of the two basic securities: risky asset and riskfree money market account.
Binomial option pricing formula

By no-arbitrage argument, the current value of the call is given by the current value of the portfolio, that is,

\[
c = \alpha S + M = \frac{R-d}{u-d}cu + \frac{u-R}{u-d}cd = \frac{pcu + (1-p)cd}{R}
\]

where \( p = \frac{R-d}{u-d} \).

- The probability \( q \), which is the subjective probability about upward or downward movement of the asset price, does not appear in the call value. The parameter \( p \) can be shown to be \( 0 < p < 1 \) since \( u > R > d \) and so \( p \) can be interpreted as a probability.
Query  Why not perform the simple discounted expectation procedure using the subjective probabilities $q$ and $1 - q$, where

$$c = \frac{qc_u + (1 - q)c_d}{R}?$$

Answer  This price depends on the subjective probabilities taken by individual investors and cannot enforce the price. The replication procedure enforces the price.

The relation

$$puS + (1 - p)dS = \frac{R - d}{u - d} uS + \frac{u - R}{u - d} dS = RS$$

shows that the expected rate of returns on the asset with $p$ as the probability of upside move is just equal to the riskless interest rate:

$$S = \frac{1}{R} E^*[S^{\Delta t}|S],$$

where $E^*$ is expectation under this probability measure. We may view $p$ as the risk neutral probability.
Treating the binomial model as a one-period securities model

The securities model consists of the riskfree asset and one risky asset with initial price vector: \( S^*(0) = (1 \ S) \) and discounted terminal payoff matrix: \( S^*(1) = \begin{pmatrix} 1 & uS \\ dS & R \end{pmatrix} \).

The risk neutral probability measure \( Q(\omega) = (Q(\omega_u) \ Q(\omega_d)) \) is obtained by solving

\[
(Q(\omega_u) \ Q(\omega_d)) \begin{pmatrix} 1 & uS \\ 1 & dS \end{pmatrix} = (1 \ S).
\]

We obtain

\[
Q(\omega_u) = 1 - Q(\omega_d) = \frac{R - d}{u - d}.
\]

The securities model is complete since there are two states and two securities. Provided that the securities model admits no arbitrage opportunities, we have uniqueness of the risk neutral measure and all contingent claims are attainable.
**Condition on u, d and R for absence of arbitrage**

The set of risk neutral measures is given by $= P^+ \cap W^\perp$, where $W$ is the subspace of discounted gains. In the binomial world, $W$ is spanned by the single vector $\left(\frac{u}{R} - 1\right)\left(\frac{d}{R} - 1\right)S$ since there is only one risky asset. Given that $u > d$, we require

$$\frac{u}{R} - 1 > 0 \quad \text{and} \quad \frac{d}{R} - 1 < 0 \quad \Leftrightarrow \quad u > R > d$$

in order that the unique risk neutral measure exists (equivalent to absence of arbitrage). To derive the above “no-arbitrage” condition using geometrical intuition, a vector normal to $\left(\frac{u}{R} - 1\right)\left(\frac{d}{R} - 1\right)S$ lies in the first quadrant of $Q(\omega_u)$-$Q(\omega_d)$ plane if and only if $u > R > d$.

By the risk neutral valuation formula, we have

$$c = \frac{Q(\omega_u)c_u + Q(\omega_d)c_d}{R} = \frac{1}{R}E^*[c^{\Delta t}|S].$$
Two equations for the determination of $Q(\omega_u)$ and $Q(\omega_d)$

\[
Q(\omega_u) \left( \frac{u}{R} - 1 \right) S + Q(\omega_d) \left( \frac{d}{R} - 1 \right) S = 0
\]
\[
Q(\omega_u) + Q(\omega_d) = 1.
\]
Extension to the trinomial model with 3 states of the world

When we extend the two-jump assumption to the three-jump model:

\[
\begin{align*}
\text{S} & \quad \text{with} \quad u > m > d \\
\text{uS} & \\
\text{mS} & \\
\text{dS} &
\end{align*}
\]

We lose market completeness if we only have the money market account and the underlying risky asset in the securities model. We expect non-uniqueness of risk neutral measures, if they do exist. The system of equations for the determination of the set of risk neutral measures is given by

\[
\begin{pmatrix}
1 & uS/R \\
1 & mS/R \\
1 & dS/R
\end{pmatrix}
\begin{pmatrix}
Q(\omega_u) \\
Q(\omega_m) \\
Q(\omega_d)
\end{pmatrix}
= (1 \quad S).
\]
Summary

• The binomial call value formula can be expressed by the following risk neutral valuation formulation:

\[ c = \frac{1}{R} E^*[c^{\Delta t}|S], \]

where \( c \) denotes the call value at the current time, and \( c^{\Delta t} \) denotes the random variable representing the call value one period later. The call price can be interpreted as the expectation of the payoff of the call option at expiry under the risk neutral probability measure \( E^* \) discounted at the riskless interest rate.

• Since there are 3 states of the world in a trinomial model, the application of the principle of replication of claims fails to derive the trinomial option pricing formula. Alternatively, one may use the risk neutral valuation approach for the direct determination of the risk neutral measures.
Determination of the jump parameters

• For the continuous asset price dynamics of Geometric Brownian motion under the risk neutral measure, we have $d \ln S_t = \left( r - \frac{\sigma^2}{2} \right) dt + \sigma dZ_t$ so that $\ln \frac{S_{t+\Delta t}}{S_t}$ becomes normally distributed with mean $\left( r - \frac{\sigma^2}{2} \right) \Delta t$ and variance $\sigma^2 \Delta t$, where $r$ is the riskless interest rate and $\sigma^2$ is the variance rate.

• The mean and variance of $\frac{S_{t+\Delta t}}{S_t}$ are $R$ and $R^2(e^{\sigma^2 \Delta t} - 1)$, respectively, where $R = e^{r \Delta t}$.

• For the one-period binomial option model under the risk neutral measure, the mean and variance of the asset price ratio $\frac{S_{t+\Delta t}}{S_t}$ are

$$pu + (1 - p)d \quad \text{and} \quad pu^2 + (1 - p)d^2 - [pu + (1 - p)d]^2,$$

respectively.
• By equating the mean and variance of the asset price ratio in both the continuous and discrete models, we obtain

\[ pu + (1 - p)d = R \]
\[ pu^2 + (1 - p)d^2 - R^2 = R^2(e^{\sigma^2 \Delta t} - 1). \]

The first equation leads to \( p = \frac{R - d}{u - d} \), the usual risk neutral probability.

• A convenient choice of the third condition is the tree-symmetry condition

\[ u = \frac{1}{d}, \]

so that the lattice nodes associated with the binomial tree are symmetrical. Writing \( \tilde{\sigma}^2 = R^2 e^{\sigma^2 \Delta t} \), the solution is found to be

\[ u = \frac{1}{d} = \frac{\tilde{\sigma}^2 + 1 + \sqrt{\left(\tilde{\sigma}^2 + 1\right)^2 - 4R^2}}{2R}, \quad p = \frac{R - d}{u - d}. \]
How to obtain a nice approximation to the above daunting expression?

- By expanding $u$ in Taylor series in powers of $\sqrt{\Delta t}$, we obtain
  \[ u = 1 + \sigma \sqrt{\Delta t} + \frac{\sigma^2}{2} \Delta t + \frac{4r^2 + 4\sigma^2 r + 3\sigma^4}{8\sigma} \Delta t^2 + O(\Delta t^2). \]

- Observe that the first three terms in the above Taylor series agree with those of $e^{\sigma \sqrt{\Delta t}}$ up to $O(\Delta t)$ term.

- This suggests the judicious choice of the following set of parameter values
  \[ u = e^{\sigma \sqrt{\Delta t}}, \quad d = e^{-\sigma \sqrt{\Delta t}}, \quad p = \frac{R - d}{u - d}. \]

- With this new set of parameters, the variance of the price ratio $\frac{S_{t+\Delta t}}{S_t}$ in the continuous and discrete models agree up to $O(\Delta t)$. 

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Continuous limit of the binomial model

We consider the asymptotic limit $\Delta t \to 0$ of the binomial formula

$$c = [pc_u^{\Delta t} + (1 - p)c_d^{\Delta t}] e^{-r\Delta t}.$$

In the continuous analog, the binomial formula can be written as

$$c(S, t - \Delta t) = [pc(uS, t) + (1 - p)c(dS, t)] e^{-r\Delta t}.$$

Assuming sufficient continuity of $c(S, t)$, we perform the Taylor expansion of the binomial scheme at $(S, t)$ as follows:
\[-c(S, t - \triangle t) + [pc(uS, t) + (1 - p)c(dS, t)]e^{-r\triangle t}
\]
\[= \frac{\partial c}{\partial t}(S, t)\triangle t - \frac{1}{2} \frac{\partial^2 c}{\partial t^2}(S, t)\triangle t^2 + \cdots - (1 - e^{-r\triangle t})c(S, t)
\]
\[+ e^{-r\triangle t} \left\{ [p(u - 1) + (1 - p)(d - 1)]S\frac{\partial c}{\partial S}(S, t)
\right.
\[+ \frac{1}{2} [p(u - 1)^2 + (1 - p)(d - 1)^2]S^2 \frac{\partial^2 c}{\partial S^2}(S, t)
\]
\[+ \frac{1}{6} [p(u - 1)^3 + (1 - p)(d - 1)^3]S^3 \frac{\partial^3 c}{\partial S^3}(S, t) + \cdots \right\}.
\]

First, we observe that

\[1 - e^{-r\triangle t} = r\triangle t + O(\triangle t^2),\]

and it can be shown that

\[e^{-r\triangle t} [p(u - 1) + (1 - p)(d - 1)] = r\triangle t + O(\triangle t^2),\]
\[e^{-r\triangle t} [p(u - 1)^2 + (1 - p)(d - 1)^2] = \sigma^2 \triangle t + O(\triangle t^2),\]
\[e^{-r\triangle t} [p(u - 1)^3 + (1 - p)(d - 1)^3] = O(\triangle t^2).\]
Combining the results, we obtain

\[-c(S, t - \Delta t) + [pc(uS, t) + (1 - p)c(dS, t)] e^{-r\Delta t} = \left[ \frac{\partial c}{\partial t}(S, t) + rS \frac{\partial c}{\partial S}(S, t) + \frac{\sigma^2}{2} S^2 \frac{\partial^2 c}{\partial S^2}(S, t) - rc(S, t) \right] \Delta t + O(\Delta t^2).\]

Since \(c(S, t)\) satisfies the binomial formula, so we obtain

\[0 = \frac{\partial c}{\partial t}(S, t) + rS \frac{\partial c}{\partial S}(S, t) + \frac{\sigma^2}{2} S^2 \frac{\partial^2 c}{\partial S^2}(S, t) - rc(S, t) + O(\Delta t).\]

In the limit \(\Delta t \to 0\), the binomial call value \(c(S, t)\) satisfies the Black-Scholes equation.
1.5 Consumption-investment models: viability and risk neutral valuation

**Key question** How to choose the best strategy for transforming wealth invested at time $t = 0$ into wealth at $t = 1$, with possible endowment and a portion of wealth being consumed at time $t = 0$?

**Measure of performance** – expected utility criterion

Let $u(W, \omega)$ represent the utility of amount $W$, $P$ be a probability measure on $\Omega$, with $P(\omega) > 0$ for all $\omega \in \Omega$.

$u : \mathbb{R} \times \Omega \rightarrow \mathbb{R}$ is a function such that $W \rightarrow u(W, \omega)$ is differentiable, concave and strictly increasing for each $\omega \in \Omega$, $\omega$ is the state and

$$E[u(V_1)] = \sum_{\omega \in \Omega} P(\omega)u(V_1(\omega), \omega).$$

In most applications, it suffices for $u$ to depend on $W$ only.
Assumptions

• Uncertainty – listing of all basic events or states that could occur and their probabilities; sample space is $\Omega = \{\omega_1, \ldots, \omega_K\}$; each state $\omega \in \Omega$ occurs with a positive (physical) probability $P(\omega)$.

• Securities – contracts for a future delivery of cash, contingent on the prevailing state.

• Endowments – cash that the traders receive from sources other than trading.

• There is a finite number $N$ of endogenous securities (all securities are created by traders, no entities outside the model).
Payoff matrix of the securities

\[
D = \begin{pmatrix}
  d_1(\omega_1) & \cdots & d_N(\omega_1) \\
  \vdots & \ddots & \vdots \\
  d_1(\omega_K) & \cdots & d_N(\omega_K)
\end{pmatrix}
\]

\(d_n\) is a random variable defined on the sample space \(\Omega\), \(1 \leq n \leq N\).

- There are a finite number \(I\) of traders. At time 0, traders know only the set of possible states \(\Omega\), and at time \(T\) they know the prevailing state \(\omega \in \Omega\).

- All traders are price takers. They determine their demands and supplies of securities without paying attention to the impact that their actions on the ultimate market prices of securities.
• Endowment process

At time 0, trader $i$ receives an endowment $e_i^i(0)$. At time $T$, he receives the endowment $e_i^i(T, \omega)$ contingent on the prevailing state $\omega$.

$$e_i^i = \{e_i^i(0), e_i^i(T)\}$$ is the endowment process of trader $i$.

• Consumption process

The uncertain terminal endowments and payouts of securities introduce uncertainty into the consumption at time $T$.

At time 0, given security prices $P_1, \cdots, P_N$, each trader $i$ faces constraints on consumption imposed by her endowment process $e_i^i = \{e_i^i(0), e_i^i(T)\}$. 
**Budget set**

For an endowment process $e^i$ and security prices

$$P = (P_1, \cdots, P_N)$$

the budget set $B(e^i, P)$ of trader $i$ is the subset of the consumption set $X$ such that $c \in B(e^i, P)$ if and only if there are scalars $\theta_1, \cdots, \theta_N$ such that

$$c(0) = e^i(0) - \sum_{n=1}^{N} \theta_n P_n$$

$$c(T) = e^i(T) + \sum_{n=1}^{N} \theta_n d_n.$$ 

$\theta = (\theta_1 \cdots \theta_N)$ is called the trading strategy.

The consumption process $\{c(0), c(T)\}$ is said to be generated by the endowment process $e^i$ and the trading strategy $\theta$. 

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Example \( K = 2, N = 4 \)

\[
D = \begin{pmatrix}
100 & 40 & 60 & 120 \\
100 & 0 & 40 & 80
\end{pmatrix}
\]

\[
P = (50 \ 4 \ 22 \ 44).
\]

A trader has the endowment process

\[
e(0) = 9, \quad e(T, \omega_1) = 10, \quad e(T, \omega_2) = 20.
\]

The consumption set \( X = \mathbb{R}^3_+ \). A consumption process \( \{c(0), c(T, \omega_1), c(T, \omega_2)\} \) belongs to the trader’s budget set if and only if the following system of equations

\[
\begin{align*}
-50\theta_1 - 4\theta_2 - 22\theta_3 - 44\theta_4 &= c(0) - 9 \\
100\theta_1 + 40\theta_2 + 60\theta_3 + 120\theta_4 &= c(T, \omega_1) - 10 \\
100\theta_1 + 40\theta_3 + 80\theta_4 &= c(T, \omega_2) - 20
\end{align*}
\]

does not have a solution \( \theta_1, \theta_2, \theta_3 \) and \( \theta_4 \). The solvability condition may dictate certain condition on the consumption process.
Portfolio optimization

Assume that there are $N$ risky assets and single riskfree asset.

Let $\mathbb{H}$ denote the set of all trading strategies, $\mathbb{H} = \mathbb{R}^{N+1}$. Let $v \in \mathbb{R}$ be a specified scalar representing the initial wealth at $t = 0$.

Optimal portfolio problem:

\[
\begin{align*}
\text{maximize} & \quad E[u(V_{1})] \\
\text{subject to} & \quad V_{0} = v \\
\end{align*}
\]

Since $V_{1} = B_{1}V_{1}^*$ and $V_{1}^* = V_{0}^* + G^*$, the above is the same as

\[
\text{maximize } E[u(B_{1}[v + H_{1}\Delta S_{1}^* + \cdots + H_{N}\Delta S_{N}^*])].
\] (1)

Here, $B_{1}$ is the money market account at $t = 1$, with $B_{0} = 1, V_{1}^*$ is the discount value process.
Theorem  If there exists an optimal solution of the portfolio problem (2), then there are no arbitrage opportunities.

Proof  We prove by contradiction. Suppose $\widehat{H}$ is an optimal solution and $H$ is an arbitrage opportunity. Write $\widehat{H} = \widehat{H} + H$, then

$$v + \sum_{n=1}^{N} \widehat{H}_n \Delta S_n^* = v + \sum_{n=1}^{N} \widehat{H}_n \Delta S_n^* + \sum_{n=1}^{N} H_n \Delta S_n^* \geq v + \sum_{n=1}^{N} \widehat{H}_n \Delta S_n^*. \uparrow$$

$H$ is an arbitrage opportunity

The inequality is strict for at least one $\omega \in \Omega$.

Since $u$ is strictly increasing in wealth and $P(\omega) > 0$ for all $\omega \in \Omega$, the objective value in (1) is strictly greater under $\widehat{H}$ than under $\widehat{H}$. Hence, $\widehat{H}$ cannot be an optimal solution.
**Remark**

If there exists an optimal solution to (1), then there exists a risk neutral probability measure.

**Theorem** If \((H, v)\) is a solution of the optimal portfolio problem, then a risk neutral probability measure exists, which is related to the optimal solution \(V_1(\omega)\) as follows:

\[
Q(\omega) = \frac{P(\omega)B_1(\omega)u'(V_1(\omega), \omega)}{E[B_1u'(V_1)]}, \quad \omega \in \Omega,
\]

**Remark**

Eq. (2) gives the relation between \(Q(\omega)\) and the optimal solution \(V_1(\omega)\) for any utility function \(u\). Recall that \(Q(\omega)\) depends only on \(S^*(0)\) and \(S^*(1; \omega)\) but not \(u\). In case when \(Q\) is not unique, this would imply multiple optimal solutions for any given \(u\).
Proof    Rewrite the objective function \( E[u(V_1)] \) as
\[
E[u(V_1)] = \sum_{\omega \in \Omega} P(\omega)u(B_1(\omega)[v + H_1 \Delta S^*_1(\omega) + \cdots + H_N \Delta S^*_N(\omega)]), \omega).
\]

The choice variables are \( H_1, H_2, \ldots, H_N \). The first order necessary condition is
\[
0 = \frac{\partial}{\partial H_n} E[u(V_1)]
= \sum_{\omega \in \Omega} P(\omega)u'(B_1(\omega)[v + H_1 \Delta S^*_1(\omega) + \cdots + H_N \Delta S^*_N(\omega)], \omega)B_1(\omega) \Delta S^*_n(\omega)
= E[B_1 u'(V_1) \Delta S^*_n], \quad n = 1, \ldots, N.
\]

On the other hand, a risk neutral probability measure must satisfy
\[
0 = E_Q[\Delta S^*_n] = \sum_{\omega \in \Omega} Q(\omega) \Delta S^*_n(\omega), \quad n = 1, \ldots, N.
\]
From the first order condition, we may write

\[(P(\omega_1)B_1(\omega_1)u'(V_1(\omega_1)) \cdots P(\omega_K)B_1(\omega_K)u'(V_1(\omega_K)))\]

\[= \ E[B_1u'(V_1)](S^*_1(0) \cdots S^*_N(0)),\]

and the risk neutral probability values satisfy

\[(Q_1(\omega_1) \cdots Q(\omega_K))S^* = (S^*_1(0) \cdots S^*_N(0))\].

Assuming that a right inverse of \(S\) exists (not necessarily unique), then we can deduce the following relation between \(Q(\omega)\) and \(u'(V_1(\omega))\) as follow

\[Q(\omega) = \frac{P(\omega)B_1(\omega)u'(V_1(\omega))}{E[B_1u'(V_1)]}, \quad \omega \in \Omega.\]

Note that \(P(\omega)B_1(\omega)u'(V_1(\omega))/E[Bu'(V_1)] > 0\) for all \(\omega \in \Omega\) since \(u\) is strictly increasing and

\[\sum_{k=1}^{K} \frac{P(\omega)B_1(\omega)u'(V_1(\omega))}{E[B_1u'(V_1)]} = 1.\]
1. Converse of the above theorem: if there exists a risk neutral measure $Q$, then does the optimal portfolio problem have a solution?

2. The direct solution of the non-linear system

$$E[B_1 u'(V_1) \Delta S^*_n] = 0, \quad n = 1, 2, \cdots, N$$

for $H$ is complicated. How to find some convenient mean to get around it?

**Definition**

A securities market is said to be **viable** if there exists a function $u : \mathbb{R} \times \Omega \rightarrow \mathbb{R}$ and an initial wealth $v$ such that $W \rightarrow u(W, \omega)$ is concave and strictly increasing for each $\omega \in \Omega$ and the corresponding optimal portfolio problem has an optimal solution $H$.

**Theorem**  

The securities market model is **viable** if and only if there exists a risk neutral probability measure $Q$.  


Proof  “⇒” part is shown by eq. (2). We only need to consider “⇐” part. It suffices to show by assuming the existence of a risk neutral probability measure, cleverly select $u$ and $v$, then demonstrate the existence of the optimal solution to the portfolio problem.

Now, we choose $u(W, \omega) = W \frac{Q(\omega)}{P(\omega)B_1(\omega)}$ while $v$ will be arbitrary.

For an arbitrary $(H_1, \cdots, H_N)$, we have

$$E[u(B_1\{v + H_1\Delta S_1^* + \cdots + H_N\Delta S_N^*\}, \omega)] = \sum P(\omega)B_1(\omega)\{v + H_1\Delta S_1^* + \cdots + H_N\Delta S_N^*\}Q(\omega)/[P(\omega)B_1(\omega)]$$

$$= \sum Q(\omega)\{v + H_1\Delta S_1^* + \cdots + H_N\Delta S_N^*\}$$

$$= v + H_1E_Q[\Delta S_1^*] + \cdots + H_NE_Q[\Delta S_N^*] = v.$$  

Hence, every vector $(H_1, \cdots, H_N)$ with the same initial wealth $v$ gives rise to the same objective function. That is, all such trading strategies are optimal. Hence, the theorem is true by this clever choice of utility function.
**Example**  Consider the following discounted price process

<table>
<thead>
<tr>
<th>n</th>
<th>$S_n^*(0)$</th>
<th>$S_n^*(1)$</th>
<th>$\omega_1$</th>
<th>$\omega_2$</th>
<th>$\omega_3$</th>
</tr>
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<tbody>
<tr>
<td>1</td>
<td>6</td>
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<td>2</td>
<td>10</td>
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<td>9</td>
<td>8</td>
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</tr>
</tbody>
</table>

Solve for the risk neutral probability measure

\[
\begin{align*}
6 &= 6Q(\omega_1) + 8Q(\omega_2) + 4Q(\omega_3) \\
10 &= 13Q(\omega_1) + 9Q(\omega_2) + 8Q(\omega_3) \\
1 &= Q(\omega_1) + Q(\omega_2) + Q(\omega_3)
\end{align*}
\]

we obtain the unique solution:

\[(Q(\omega_1) \ Q(\omega_2) \ Q(\omega_3)) = (1/3 \ 1/3 \ 1/3).\]
Suppose we choose the utility function: \( u(W) = -\exp(-W) \) so that \( u'(W) = \exp(-W) \). The necessary conditions:

\[
E[B_1u'(V_1)\Delta S^*_n] = 0, \quad n = 1, 2, \ldots, N,
\]

become the following system of non-linear algebraic equations

\[
0 = P(\omega_1) \exp \left\{ -\frac{10}{9}(v + 0H_1 + 3H_2) \right\} \frac{10}{9} \cdot 0
+ P(\omega_2) \exp \left\{ -\frac{10}{9}(v + 2H_1 - H_2) \right\} \frac{10}{9} \cdot 2
+ P(\omega_3) \exp \left\{ -\frac{10}{9}(v - 2H_1 - 2H_2) \right\} \frac{10}{9}(-2)
\]

\[
0 = P(\omega_1) \exp \left\{ -\frac{10}{9}(v + 0H_1 + 3H_2) \right\} \frac{10}{9}(3)
+ P(\omega_2) \exp \left\{ -\frac{10}{9}(v + 2H_1 - H_2) \right\} \frac{10}{9}(-1)
+ P(\omega_3) \exp \left\{ -\frac{10}{9}(v - 2H_1 - 2H_2) \right\} \frac{10}{9}(-2).
\]

It is quite complicated to solve for \( H_1 \) and \( H_2 \).
More efficient computational technique

The objective function $H \rightarrow E[u(V_1)]$ can be viewed as the composition of two functions:

$$H \rightarrow V_1 \quad \text{(maps trading strategies into random variables)}$$
$$V_1 \rightarrow E[u(V_1)] \quad \text{(maps random variables into real numbers)}$$

Two-step process

1. Identify the optimal random variable $V_1$, the value of $V_1$ that maximizes $E[u(V_1)]$ over the subset of feasible random variables.

2. Compute the trading strategy $H$ that generates this $V_1$. This is related to the solution of a linear system of equations.
\[ Eu(B_1(v + \Sigma H_n \Delta S_n^*)) \]

\[ B_1(v + \Sigma H_n \Delta S_n^*) \quad Eu(V_1) \]

set of trading strategies

set of attainable wealths

real line
For step one, we start with the specification of the subset of feasible random variables correctly and conveniently.

If the securities model is complete (every contingent claim lies in the asset span), the subset is simply

$$
W_v = \{W \in \mathbb{R}^K : E_Q[W/B_1] = v \}.
$$

$W_v$ is called the set of attainable wealths.

(i) Under any trading strategy $H$ with $V_0 = v$, one has $E_Q[V_1/B_1] = v$ by the risk neutral valuation principle.

(ii) For any contingent claim $W \in W_v$, there exists a trading strategy $H$ such that $V_0 = v$ and $V_1 = W$. 
Solution of the first subproblem

maximize $E[u(W)]$ subject to $W \in W_v$

When the model is complete, the Lagrangian formulation is

maximize $E[u(W)] - \lambda\{E_Q[W/B_1] - v\}$.

Introducing $L = Q/P$ (state price density)

$$
E[u(W)] - \lambda\{E_Q[W/B_1] - v\} = E[u(W) - \lambda\{LW/B_1 - v\}] = \sum_{\omega} P(\omega)[u(W(\omega)) - \lambda\{L(\omega)W(\omega)/B_1(\omega) - v\}].
$$

We may visualize $W(\omega), \omega \in \Omega$, as the choice variables. If $W$ maximizes the above quantity, then the necessary conditions are (one equation for each $\omega \in \Omega$)

$$
u'(W(\omega)) = \lambda L(\omega)/B_1(\omega), \quad \omega \in \Omega.$$

Recall the earlier relation between $Q$ and $u'(V_1)$, where

$$Q(\omega) = \frac{P(\omega)B_1(\omega)u'(V_1(\omega), \omega)}{E[B_1u'(V_1)]}, \quad \omega \in \Omega,$$

so that $\lambda$ is identified as $E[B_1u'(\widehat{W})]$, where $\widehat{W}$ is the optimal solution.

Let $I$ denote the inverse function corresponding to $u'$, we then have

$$W(\omega) = I(\lambda L(\omega)/B_1).$$

How to compute $\lambda$? From the constraint condition: $E_Q[W/B_1] = v$, we obtain

$$E_Q[I(\lambda L/B_1)/B_1] = v.$$
State prices

Assuming no discount effect, let

\[ Q(\omega_k) = \text{state price of state } \omega_k \]
\[ = \text{price of the Arrow-Debreu security } s_k \]
\[ \text{which pays off }$1 \text{ if } \omega_k \text{ occurs} \]

\[ Q(\omega_k) = e_{\omega}^T \mathbf{Q} = E_Q[s_k] = E[LS_k] \]

where \( L(\omega) = Q(\omega)/P(\omega) \) is called the state price density (also called pricing kernel).
**Example**  Take \( u(W) = -\exp(-W) \) so that \( u'(W) = \exp(-W) \). We have \( u'(W) = i \) if and only if \( W = -\ln i \) so that \( I(i) = -\ln i \). Note that

\[
W = -\ln(\lambda L/B_1) = -\ln \lambda - \ln L/B_1.
\]

To solve for \( \lambda \), we use

\[
v = -E_Q[B_1^{-1}\ln(\lambda L/B_1)] = -(\ln \lambda)E_QB_1^{-1} - E_Q[\ln(L/B_1)/B_1].
\]

Hence, the correct value of \( \lambda \) is

\[
\lambda = \exp\left(\frac{-v - E_Q[B_1^{-1}\ln(L/B_1)]}{E_QB_1^{-1}}\right)
\]

so that

\[
W = \frac{v + E_Q[B_1^{-1}\ln(L/B_1)]}{E_QB_1^{-1}} - \ln(L/B_1).
\]

It is seen that \( E_Q[W/B_1] = v \) is satisfied.
Putting back into $u(W) = -\exp(-W)$, and observing $u' = -u$ we have

$$u(W) = -\exp \left\{ \frac{-\nu + \ln(L/B_1)E_QB_1^{-1} - E_Q[B_1^{-1}\ln(L/B_1)]}{E_QB_1^{-1}} \right\} = -\frac{\lambda L}{B_1}$$

so that the optimal value of the objective function is

$$E[u(W)] = -\lambda E[L/B_1] = -\lambda E_QB_1^{-1}.$$ 

The optimal wealth $W$ is obtained and it depends on the underlying securities market model only via the probability measures $P$ and $Q$. That is, the true measure and risk neutral measure comprise what can be thought of as a sufficient statistic for the optimal portfolio subproblem.
Continued with the numerical example, recall $Q(\omega_1) = Q(\omega_2) = Q(\omega_3) = \frac{1}{3}$.

Let $P(\omega_1) = 1/2, P(\omega_2) = P(\omega_3) = 1/4$ so that $L(\omega_1) = 2/3, L(\omega_2) = L(\omega_3) = 4/3$. With $r = 1/9$ so that $B_1 = 10/9$. Furthermore, we compute

$$E_Q[\ln(L/B_1)] = \frac{1}{3} \left[ \ln \left( \frac{2}{3} \cdot \frac{9}{10} \right) + 2 \ln \left( \frac{4}{3} \cdot \frac{9}{10} \right) \right] = -0.04873$$

so that the optimal attainable wealth is

$$W = v(1 + r) + E_Q[\ln(L/B_1)] - \ln(L/B_1)$$

$$= \begin{cases} v(10/9) + 0.46209 & \omega = \omega_1 \\ v(10/9) - 0.23105 & \omega = \omega_2 \text{ or } \omega = \omega_3 \end{cases}.$$
Now
\[\lambda = \exp\left(-\frac{10}{9}v + 0.04873\right)\]
so the optimal value of the objective function is
\[E[u(W)] = -\lambda E_Q B_1^{-1} = -\frac{9}{10}\lambda.\]

Note that this is consistent to \(\lambda = E[B_1 u'(V_1)].\) Once the optimal attainable wealth \(W\) is computed, we solve for the optimal trading strategy \(H\) by solving \(W/B_1 = v + G^*.\) Here, we have
\[G^* = \begin{pmatrix} 0.46209 & \frac{9}{10} \\ -0.23105 & \frac{9}{10} \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}.\]

Once \(h_1\) and \(h_2\) are obtained, we compute \(h_0\) using
\[h_0 S_0(0) + h_1 S_1(0) + h_2 S_2(0) = v.\]
Consumption-investment problem

- A consumption process $C = (C_0, C_1)$ consists of a non-negative scalar $C_0$ and a non-negative random variable $C_1$.

- A consumption-investment plan consists of a pair $(C, H)$ where $C$ is a consumption process and $H$ is a trading strategy.

1. $C_0 = \text{time zero consumption}$

   $V_0 = H_0 + \sum H_n S_n(0) = \text{amount invested at time zero. Amount of money available at time zero} = \nu \geq C_0 + V_0.$

2. $V_1 = H_0 B_1 + \sum H_n S_n(1) = \text{amount of money available at } t = 1$ so $C_1 \leq V_1.$

We assume that a sensible investor who can consume only at $t = 0$ and $t = 1$ would not leave money "lying in the drawer".
Throughout the subsequent analysis, we always assume the absence of arbitrage opportunities in the securities model so that a risk neutral probability measure exists.

A consumption-investment plan is said to be *admissible* if

(1) $C_0 + V_0 = \nu$ and (2) $C_1 = V_1$. We always assume $\nu \geq 0$.

If $(C, H)$ is admissible, then $C_1$ is an attainable contingent claim with

$$E_Q[C_1/B_1] = E_Q[V_1/B_1] = V_0$$

for every risk neutral measure $Q$. Adding $C_0$ to both sides, we obtain

$$E_Q[C_0 + C_1/B_1] = \nu.$$

This is one important constraint on the admissibility of the consumption-investment plan.
Example

<table>
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The securities model is complete with unique risk neutral measure $Q = (1/3, 1/3, 1/3)$. In order for $(C_0, C_1)$ to be a part of an admissible consumption-investment plan, we must have

$$\nu - C_0 = \frac{9}{10} E_Q[C_1] = \frac{3}{10} [C_1(\omega_1) + C_1(\omega_2) + C_1(\omega_3)].$$

Admissibility leads to this constraint involving $C_0$ and $C_1(\omega)$. 

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Maximization problem

An investor starts with initial wealth $\nu$ and wants to choose an admissible consumption-investment plan so as to maximize the expected value of the utility of consumption at both times zero and one.

$$u : \mathbb{R}_+ \rightarrow \mathbb{R}$$ is concave, differentiable and strictly increasing.

Maximize $u(C_0) + E[u(C_1)]$

subject to

$$C_0 + H_0 B_0 + \sum_{n=1}^{N} H_n S_n(0) = \nu$$

$$C_1 - H_0 B_1 - \sum_{n=1}^{N} H_n S_n(1) = 0 \quad \text{for all } \omega \in \Omega$$

$$C_0 \geq 0, \quad C_1 \geq 0, \quad H \in \mathbb{R}^{N+1}$$
Example Consider the above securities model:

Take \( u(C) = \ln C \). Since \( \lim_{C \to 0^+} \ln C \) is \( -\infty \), we may drop the explicit non-negativity constraint.

Given \( P(\omega_1) = 1/2, P(\omega_2) = P(\omega_3) = 1/4 \) and \( r = 1/9 \):

Maximize \( \ln C_0 + \frac{1}{2} \ln C_1(\omega_1) + \frac{1}{4} \ln C_1(\omega_2) + \frac{1}{4} \ln C_1(\omega_3) \)

subject to \( C_0 = \nu - H_0 - 6H_1 - 10H_2 \)

\[
\begin{align*}
C_1(\omega_1) &= \frac{10}{9} H_0 + \frac{60}{9} H_1 + \frac{130}{9} H_2 \\
C_1(\omega_2) &= \frac{10}{9} H_0 + \frac{80}{9} H_1 + \frac{90}{9} H_2 \\
C_1(\omega_3) &= \frac{10}{9} H_0 + \frac{40}{9} H_1 + \frac{80}{9} H_2.
\end{align*}
\]
Maximize

\[
\ln(\nu - H_0 - 6H_1 - 10H_2) + \frac{1}{2} \ln \left( \frac{10}{9} H_0 + \frac{60}{9} H_1 + \frac{130}{9} H_2 \right) \\
+ \frac{1}{4} \ln \left( \frac{10}{9} H_0 + \frac{80}{9} H_1 + \frac{90}{9} H_2 \right) + \frac{1}{4} \ln \left( \frac{10}{9} H_0 + \frac{40}{9} H_1 + \frac{80}{9} H_2 \right).
\]

The first order conditions give

\[
-\frac{1}{C_0} + \frac{110}{6} \frac{1}{C_1(\omega_1)} + \frac{110}{180} \frac{1}{C_1(\omega_2)} + \frac{110}{140} \frac{1}{C_1(\omega_3)} = 0
\]

\[
-\frac{1}{C_0} + \frac{29}{160} \frac{1}{C_1(\omega_1)} + \frac{49}{180} \frac{1}{C_1(\omega_2)} + \frac{49}{140} \frac{1}{C_1(\omega_3)} = 0
\]

\[
-\frac{1}{C_0} + \frac{1130}{190} \frac{1}{C_1(\omega_1)} + \frac{190}{180} \frac{1}{C_1(\omega_2)} + \frac{180}{49} \frac{1}{C_1(\omega_3)} = 0.
\]

Solve for \( H_0, H_1 \) and \( H_2 \), then obtain \( C_0, C_1(\omega_1), C_1(\omega_2) \) and \( C_1(\omega_3) \).
Solution of the maximization problem

Differentiate the objective function with respect to \( H_0, \ldots, H_N \) successively and substitute for \( C_0 \) and \( C_1 \). The following \( N + 1 \) first order necessary conditions are obtained:

\[
\begin{align*}
    u'(C_0) &= E[B_1 u'(C_1)] \\
    u'(C_0) S_n(0) &= E[u'(C_1) S_n(1)], \quad n = 1, \ldots, N.
\end{align*}
\]

Note that \( H_0 \) is also a choice variable since \( C_0 + V_0 = v \) in the consumption-investment model (unlike \( V_0 = v \) in the portfolio maximization model).

Recall that \( C_0 \) and \( C_1 \) must be both non-negative. If \( u \) is chosen such that \( u(C) \to -\infty \) as \( C \to 0^+ \), then these constraints will not be binding.
**Theorem**

If $C$ is a part of a solution to the optimal consumption-investment problem with $C_0 \geq 0$ and $C_1(\omega) \geq 0$ for all $\omega$, then

$$Q(\omega) = P(\omega)B_1(\omega)\frac{u'(C_1(\omega))}{u'(C_0)}.$$ 

The proof is similar to that of the theorem on portfolio maximization, except that $V_1$ is replaced by $C_1$ and $E[B_1u'(V_0)]$ is replaced by $u'(C_0)$.

The level of consumption at $t = 0$ and $t = 1$ are related by $u'(C_0) = E[B_1u'(C_1)]$, where the marginal consumption at $t = 0$ is equal to the discounted expected marginal consumption at $t = 1$. 
Risk neutral computational approach for the consumption-investment problem

Alternative formulation: maximize \( u(C_0) + E[u(C_1)] \)

subject to \( C_0 + E_Q[C_1/B_1] = \nu \)

\[ C_0 \geq 0 \text{ and } C_1 \geq 0. \]

First, we analyze the constrained problem:

Maximize \( u(C_0) + E[u(C_1)] - \lambda \{ C_0 + E[C_1L/B_1] - \nu \} \).

Assume that we choose an utility function such that the optimal solution features strictly positive consumption values.
Consider the satisfaction of the first order conditions:

\[ u'(C_0) = \lambda \quad \text{and} \quad u'(C_1(\omega)) = \lambda L/B_1, \]

we have

\[ C_0 = I(\lambda) \quad \text{and} \quad C_1(\omega) = I(\lambda L/B_1). \]

Solve for \( \lambda \) using the constraint condition:

\[ I(\lambda) + EQ[I(\lambda L/B_1)/B_1] = \nu. \]

Solution generally exists if \( I(\lambda) \) is monotonic.
Example

Suppose \( u(C) = \ln C \) so that \( u'(C) = 1/C \) and \( I(i) = 1/i \).

\[
C_0 = 1/\lambda \quad \text{and} \quad C_1(\omega) = \frac{1}{\lambda L/B_1}
\]

and

\[
\frac{1}{\lambda} + \frac{1}{\lambda} E_Q[L^{-1}] = \frac{1}{\lambda} + \frac{1}{\lambda} E[1] = \frac{2}{\lambda} = \nu,
\]

so \( \lambda = 2/\nu \) and \( C_0 = \nu/2, C_1(\omega) = \nu B_1(\omega) P(\omega)/[2Q(\omega)] \). Both \( C_0 \) and \( C_1 \) are non-negative if \( \nu \geq 0 \). The maximum value of the objective function is

\[
\ln \frac{\nu}{2} + \frac{1}{2} E \left[ \ln \frac{\nu B_1}{L} \right].
\]

With \( L(\omega_1) = 2/3, L(\omega_2) = L(\omega_3) = 4/3 \) and \( r = 1/9 \)

\[
C_1(\omega) = \nu \frac{5}{9} L^{-1} = \left\{
\begin{array}{ll}
\frac{5}{6}\nu & \text{if } \omega = \omega_1 \\
\frac{5}{12}\nu & \text{if } \omega = \omega_2 \text{ or } \omega = \omega_3
\end{array}
\right.
\]
Note that the first order conditions are satisfied. Lastly, we compute the optimal $H_1$ and $H_2$ using $\frac{C_1}{B_1} = V_0 + G^* = \frac{\nu}{2} + G^*$:

\[
\begin{align*}
\frac{3}{4} \nu &= \frac{1}{2} \nu + 0H_1 + 3H_2 \\
\frac{3}{8} \nu &= \frac{1}{2} \nu + 2H_1 - H_2 \\
\frac{3}{8} \nu &= \frac{1}{2} \nu - 2H_1 - 2H_2
\end{align*}
\]

There are 2 unknowns: $H_1, H_2$ but 3 equations. Solution exists provided that $C_1$ is attainable. Attainability is guaranteed if the market is complete. If otherwise, by enforcing $E_Q[C_1/B_1]$ to have the same value for all $Q$, $C_1$ is guaranteed to be attainable.

The solution is given by: $H_1 = -\nu/48, H_2 = \nu/12$.

From $\frac{\nu}{2} = H_0 + 6H_1 + 10H_2$, we obtain $H_0 = -\frac{5}{24} \nu$. 
Generalization

1. The objective function is given as $u(C_0) + \beta E[u(C_1)]$, where $0 < \beta \leq 1$; here $\beta$ is considered as the discount factor.

2. Allow the consumer to have endowment (income) $\tilde{E}$ at time $t = 1$, where $\tilde{E}$ is a specified random variable. The second constraint becomes

   $$ C_1 - H_0 B_1 - \sum_{n=1}^{N} H_n S_n(1) = \tilde{E}. $$

   The pair $(\nu, \tilde{E})$ is called the endowment process for the consumer.

The consumption-investment plan $(C, H)$ is admissible if and only if

$$ C_0 + E_Q[(C_1 - \tilde{E})/B_1] = \nu. $$
Optimal portfolios in incomplete markets

Under incomplete market, it is necessary to properly identify the set of attainable wealths.

A contingent claim (or wealth) $W$ is attainable if and only if $E_Q[W/B_1]$ takes the same value for every risk neutral probability measure $Q \in M$.

$$W_v = \{W \in \mathbb{R}^K : E_Q[W/B_1] = v, \quad Q \in M\},$$

where $W_v$ is the set of wealths that can be generated starting with initial capital $v$.

If there exists a finite number of independent vectors $Q(1), \cdots, Q(J)$ such that every element of $M$ can be expressed as a linear combination of these $J$ vectors. We have

$$E_Q[W/B_1] = v \text{ for all } Q \in M \text{ if and only if } E_{Q(j)}[W/B_1] = v, \quad j = 1, 2, \cdots, J.$$
To generate $J$ equations for the Lagrange multipliers, we substitute (2) into the constraint equation (1):

$$E[L_j I\left[\sum_{k=1}^{I} \lambda_k L_k(\omega)/B_1(\omega)\right]] = v, \quad j = 1, 2, \ldots J.$$ 

It can be shown that if the utility function is strictly concave, then the solution to the optimal portfolio problem (1) is unique.

**Alternative approach**

Introduce fictitious securities to the market in such a way so as to make the model complete, and then use constraints to prohibit any positions in these fictitious securities.
The optimal portfolio problem becomes

\[
\text{maximize } E u(W) \\
\text{subject to } E Q(j)[W/B_1] = v, j = 1, 2, \cdots, J.
\]

Define \( L_j = Q(j)/P \), and we introduce \( J \) Lagrange multipliers.

Maximize \( E[u(W)] - \sum_{j=1}^{J} \lambda_j \{ E[L_j W/B_1] - v \} \).

The first order conditions, one for each \( \omega \in \Omega \), are given by

\[
u'(W(\omega)) = \sum_{j=1}^{J} \lambda_j L_j(\omega)/B_1(\omega), \quad \omega \in \Omega,
\]

or

\[
W(\omega) = I \left[ \sum_{j=1}^{J} \lambda_j L_j(\omega)/B_1(\omega) \right], \quad \omega \in \Omega,
\]

where \( I \) is the inverse function of \( u' \).
Example

\( K = 3, N = 1, r = 1/9, \) so that \( B_1 = 10/9, \) and \( S_1(0) = 5. \) There is only one risky security.

\[
\begin{array}{cccc}
\omega & S_1(\omega) & S_1^*(\omega) & P(\omega) \\
\hline
\omega_1 & 20/3 & 6 & 1/3 \\
\omega_2 & 40/9 & 4 & 1/3 \\
\omega_3 & 30/9 & 3 & 1/3 \\
\end{array}
\]

\[
\left\{ \begin{aligned}
Q_1 - Q_2 - 2Q_3 &= 0 \\
Q_1 + Q_2 + Q_3 &= 1
\end{aligned} \right. 
\]

The model is incomplete with \( M \) consisting of all probability measures of the form

\[
Q = (\theta, 2 - 3\theta, -1 + 2\theta) \quad \text{where} \quad \frac{1}{2} < \theta < \frac{2}{3}.
\]

Note that \( \text{dim } W = 1, \text{dim } W^\perp = 2 \) so that the set of risk neutral measures is generated by one free parameter.
A contingent claim $X = (X_1 \ X_2 \ X_3)$ is attainable if and only if

$$X_1 - 3X_2 + 2X_3 = 0.$$ 

This is the plane spanned by $(1 \ 1 \ 1)^T$ and $(6 \ 4 \ 3)^T$.

Say, take $\theta = 1/2$ and $\theta = 1/3$, we obtain

$$Q(1) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \quad \text{and} \quad Q(2) = \begin{pmatrix} \frac{2}{3} & 0 & \frac{1}{3} \end{pmatrix}.$$ 

For computational convenience, we obtain $Q(1)$ and $Q(2)$ by taking the two endpoints in the range $\frac{1}{2} < \theta < \frac{2}{3}$. Though they are linear pricing measures, the attainability condition remains to be observed.

Now, we have

$$L_1 = \begin{pmatrix} \frac{3}{2} & \frac{3}{2} & 0 \end{pmatrix} \quad \text{and} \quad L_2 = (2 \ 0 \ 1).$$
Recall $W(\omega) = I \left[ \sum_{j=1}^{2} \lambda_j L_j(\omega)/B_1(\omega) \right]$.

Taking $u(W) = \ln W, u'(W) = 1/W$ and $I(i) = 1/i$.

The Lagrange multipliers are determined by applying

$$E_Q(J)[W/B_1] = v.$$ 

The equations for $\lambda_1$ and $\lambda_2$ are given by

$$L_1(\omega_1) \frac{1}{\lambda_1 \frac{L_1(\omega_1)}{B_1}} + L_1(\omega_2) \frac{1}{\lambda_1 \frac{L_1(\omega_2)}{B_1} + \lambda_2 \frac{L_2(\omega_2)}{B_1}} + L_1(\omega_3) \frac{1}{\lambda_1 \frac{L_1(\omega_3)}{B_1} + \lambda_2 \frac{L_2(\omega_3)}{B_1}} = v$$

$$L_2(\omega_1) \frac{1}{\lambda_1 \frac{L_1(\omega_1)}{B_1} + \lambda_2 \frac{L_2(\omega_1)}{B_1}} + L_2(\omega_2) \frac{1}{\lambda_1 \frac{L_1(\omega_2)}{B_1} + \lambda_2 \frac{L_2(\omega_2)}{B_1}} + L_2(\omega_3) \frac{1}{\lambda_1 \frac{L_1(\omega_3)}{B_1} + \lambda_2 \frac{L_2(\omega_3)}{B_1}} = v.$$ 

We solve for $\lambda_1$ and $\lambda_2$ and obtain

$$W(\omega) = \frac{1}{\lambda_1 L_1(\omega)/B_1 + \lambda_2 L_2(\omega)/B_1}.$$
One may check that \( W(\omega_1) - 3W(\omega_2) + 2W(\omega_3) = 0 \) is observed. Therefore, \( W(\omega) \) is attainable.

We find the optimal trading strategy by solving
\[
\begin{pmatrix} 1 & 6 \\ 1 & 4 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} H_0 \\ H_1 \end{pmatrix} = \begin{pmatrix} W(\omega_1) \\ W(\omega_2) \\ W(\omega_3) \end{pmatrix}.
\]

For numerical calculations, we have
\[
\frac{1}{3\lambda_1 + 4\lambda_2} + \frac{1}{3\lambda_1} = v
\]
\[
\frac{1}{9\lambda_1 + 12\lambda_2} + \frac{1}{3\lambda_2} = v.
\]

The unique (non-negative) solution is
\[
\lambda_1 = \frac{0.46482}{v} \quad \text{and} \quad \lambda_2 = \frac{0.53519}{v}.
\]
\[
W(\omega) = \frac{v}{0.46482 \left(\frac{9}{10}\right) L_1(\omega) + 0.53519 \left(\frac{9}{10}\right) L_2(\omega)} = \begin{cases} 
0.62860v, & \omega = \omega_1 \\
1.59360v, & \omega = \omega_2 \\
2.07611v, & \omega = \omega_3 
\end{cases}.
\]

Note that \(W(\omega)\) satisfies \(X_1 - 3X_2 + 2X_3 = 0\).

Solve \(H_1\) and \(H_0\) from
\[
\begin{cases}
H_0 + 6H_1 = \left(\frac{9}{10}\right)(0.6286)v \\
H_0 + 4H_1 = \left(\frac{9}{10}\right)(1.5936)v
\end{cases},
\]
this yields
\[
H_0 = 3.17124v \text{ and } H_1 = -0.43425v.
\]

The optimal objective value is \(E[\ln W] = 0.24409 + \ln v\).
Alternative approach

- Add one or more securities to the model such that it is made to be complete. The computation is easier since it is performed under a complete market.

- Solve the optimal portfolio problem with the constraint that no position can be taken in any of the added fictitious securities.

Be careful that when adding new fictitious securities, one has to maintain the absence of arbitrage opportunities in the securities model.
1.6 Extension to multiperiod model

- Securities models with \( T+1 \) trading dates: \( t = 0, 1, \cdots, T, T > 1 \).
- A finite sample space \( \Omega \) of \( K \) elements, \( \Omega = \{\omega_1, \omega_2, \cdots, \omega_K\} \), which represents all possible states of the world within the trading horizon.
- A probability measure \( P \) defined on the sample space with \( P(\omega) > 0 \) for all \( \omega \in \Omega \).
- \( M \) risky securities whose price processes are non-negative stochastic processes, as denoted by \( S_m = \{S_m(t); t = 0, 1, \cdots, T\}, m = 1, \cdots, M \).
- Riskfree security whose price process \( S_0(t) \) is deterministic, with \( S_0(t) \) strictly positive and possibly non-decreasing.
• We may visualize $S_0(t)$ as the money market account, and the quantity $r_t = \frac{S_0(t) - S_0(t - 1)}{S_0(t - 1)}, t = 1, \cdots, T$, is visualized as the interest rate over the time interval $(t - 1, t)$.

• Specify how the investors learn about the true state of the world on intermediate trading dates in a multiperiod model. This is done through the construction of some information structure that models how information is revealed to investors in terms of the partitions of the sample space $\Omega$.

• We form a partition $\mathcal{P}$ of $\Omega$, which is a collection of disjoint subsets (events) of $\Omega$. In the dice tossing experiment, $\mathcal{P}_{\text{coarse}} = \{\{1, 4\}, \{2, 3, 5, 6\}\}$ (concerned with either red face or dark face occurs) and $\mathcal{P}_{\text{fine}} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\}$ (precisely which face occurs).
Outline

• How can an information structure be described by a filtration (nested sequence of algebras)? Understand how the security price processes can be adapted to a given filtration $\{F_t\}_{t=0,1,\ldots,T}$ $[S_m(t)$ is measurable with respect to the algebra $F_t]$.

• We introduce martingales, which are defined with reference to conditional expectation. In the multiperiod setting, risk neutral probability measures are defined such that all discounted price processes of the risky securities are martingales under a risk neutral measure.

• Multiperiod version of the Fundamental Theorem of Asset Pricing.

• Multiperiod binomial models.
Information structures and filtrations

Consider the sample space $\Omega = \{\omega_1, \omega_2, \cdots, \omega_{10}\}$ with 10 elements. We can construct various partitions of the set $\Omega$.

A *partition* of $\Omega$ is a collection $\mathcal{P} = \{B_1, B_2, \cdots, B_n\}$ such that $B_j, j = 1, \cdots, n$, are subsets of $\Omega$, where $B_i \cap B_j = \emptyset, i \neq j$, and $\bigcup_{j=1}^{n} B_j = \Omega$.

Each of the sets $B_1, \cdots, B_n$ is called an *atom* of the partition. For example, we may form the partitions as

$$
\mathcal{P}_0 = \{\Omega\}
$$

$$
\mathcal{P}_1 = \{\{\omega_1, \omega_2, \omega_3, \omega_4\}, \{\omega_5, \omega_6, \omega_7, \omega_8, \omega_9, \omega_{10}\}\}
$$

$$
\mathcal{P}_2 = \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}, \{\omega_5, \omega_6\}, \{\omega_7, \omega_8, \omega_9\}, \{\omega_{10}\}\}
$$

$$
\mathcal{P}_3 = \{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_4\}, \{\omega_5\}, \{\omega_6\}, \{\omega_7\}, \{\omega_8\}, \{\omega_9\}, \{\omega_{10}\}\}.
$$
Information tree of a three-period securities model with 10 possible states. The partitions of $\Omega$ form a sequence of successively finer partitions. The information structure describes the arrival of information as time lapses.
Consider a three-period securities model that consists of a sequence of successively finer partitions: \( \{ \mathcal{P}_k : k = 0, 1, 2, 3 \} \). The pair \((\Omega, \mathcal{F})\) is called a \textit{filtered space}, which consists of a sample space \(\Omega\) and a sequence of partitions of \(\Omega\) (denoted by \(\mathcal{F}\)). The filtered space is used to model the unfolding of information through time.

At time \(t = 0\), the investors know only the set of all possible states of the world, so \(\mathcal{P}_0 = \{\Omega\}\).

At time \(t = 1\), the investors receive a bit more information: the actual state \(\omega\) is in either \(\{\omega_1, \omega_2, \omega_3, \omega_4\}\) or \(\{\omega_5, \omega_7, \omega_8, \omega_9, \omega_{10}\}\).
**Algebra**

Let $\Omega$ be a finite set and $\mathcal{F}$ be a collection of subsets of $\Omega$. The collection $\mathcal{F}$ is an *algebra* on $\Omega$ if

(i) $\Omega \in \mathcal{F}$

(ii) $B \in \mathcal{F} \Rightarrow B^c \in \mathcal{F}$

(iii) $B_1$ and $B_2 \in \mathcal{F} \Rightarrow B_1 \cup B_2 \in \mathcal{F}$. 
Filtration as a sequence of nested algebras

- Given an algebra $\mathcal{F}$ on $\Omega$, one can always find a unique collection of disjoint subsets $B_n$ such that each $B_n \in \mathcal{F}$ and the union of the subsets equals $\Omega$.

- The algebra $\mathcal{F}$ generated by a partition $\mathcal{P} = \{B_1, \cdots, B_n\}$ is a set of subsets of $\Omega$. Actually, when $\Omega$ is a finite sample space, there is a one-to-one correspondence between partitions of $\Omega$ and algebras on $\Omega$.

- The information structure defined by a sequence of partitions can be visualized as a sequence of algebras. We define a filtration $\mathcal{F} = \{\mathcal{F}_k; k = 0, 1, \cdots, T\}$ to be a nested sequence of algebras satisfying $\mathcal{F}_k \subseteq \mathcal{F}_{k+1}$.
Given the algebra $\mathcal{F} = \{\phi, \{\omega_1\}, \{\omega_2, \omega_3\}, \{\omega_4\}, \{\omega_1, \omega_2, \omega_3\}, \{\omega_2, \omega_3, \omega_4\}, \{\omega_1, \omega_4\}, \{\omega_1, \omega_2, \omega_3, \omega_4\}\}$, the corresponding partition $\mathcal{P}$ is found to be $\{\{\omega_1\}, \{\omega_2, \omega_3\}, \{\omega_4\}\}$.

The atoms of $\mathcal{P}$ are $B_1 = \{\omega_1\}, B_2 = \{\omega_2, \omega_3\}$ and $B_3 = \{\omega_4\}$. For a non-empty event which is an union of atoms in $\mathcal{P}$, its occurrence can be revealed through revelation of $\mathcal{P}$.

Take the event $A = \{\omega_1, \omega_2, \omega_3\}$ in the algebra $\mathcal{F}$, which is the union of $B_1$ and $B_2$. Given that $B_2 = \{\omega_2, \omega_3\}$ of $\mathcal{P}$ has occurred, we can decide whether $A$ or its complement $A^c$ has occurred. However, for another event $\tilde{A} = \{\omega_1, \omega_2\}$ which is NOT an union of atoms in $\mathcal{P}$, even though we know that $B_2$ has occurred, we cannot determine whether $\tilde{A}$ or $\tilde{A}^c$ has occurred. The event $\tilde{A}$ whose occurrence cannot be revealed through revelation of $\mathcal{P}$. 
**Probability measure and filtered probability space**

Consider a probability measure $P$ defined on an algebra $\mathcal{F}$. The probability measure $P$ is a function

$$P : \mathcal{F} \to [0, 1]$$

such that

1. $P(\Omega) = 1$.

2. If $B_1, B_2, \cdots$ are pairwise disjoint sets belonging to $\mathcal{F}$, then

$$P[B_1 \cup B_2 \cup \cdots] = P[B_1] + P[B_2] + \cdots.$$ 

Equipped with a probability measure, the elements of $\mathcal{F}$ are called measurable events. Given the sample space $\Omega$ and a probability measure $P$ defined on $\Omega$, together with the filtration $\mathbb{F}$ associated with $\mathcal{F}$, the triplet $(\Omega, \mathcal{F}, P)$ is called a *filtered probability space*. 
Measurability of random variables

- Consider an algebra $\mathcal{F}$ generated by a partition $\mathcal{P} = \{B_1, \ldots, B_n\}$, a random variable $X$ is said to be measurable with respect to $\mathcal{F}$ (denoted by $X \in \mathcal{F}$) if $X(\omega)$ is constant for all $\omega \in B_i, B_i$ is any atom in $\mathcal{P}$. For example, consider the algebra $\mathcal{F}_1$ generated by $\mathcal{P}_1 = \{\{\omega_1, \omega_2, \omega_3, \omega_4\}, \{\omega_5, \omega_6, \omega_7, \omega_8, \omega_9, \omega_{10}\}\}$, $X \in \mathcal{P}_1$ means $X(\omega_1) = X(\omega_2) = X(\omega_3) = X(\omega_4)$ and $X(\omega_5) = X(\omega_6) = \cdots = X(\omega_{10})$. If $X(\omega_1) = 3$ and $X(\omega_4) = 5$, then $X$ is not measurable with respect to $\mathcal{F}_1$.

- Consider an example where $\mathcal{P} = \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}, \{\omega_5\}\}$ and $X$ is measurable with respect to the algebra $\mathcal{F}$ generated by $\mathcal{P}$. Let $X(\omega_1) = X(\omega_2) = 3, X(\omega_3) = X(\omega_4) = 5$ and $X(\omega_5) = 7$. Suppose the random experiment associated with the random variable $X$ is performed, giving $X = 5$. This tells the information that the event $\{\omega_3, \omega_4\}$ has occurred.
The information of outcome from the random experiment is revealed through the random variable $X$. We may say that $\mathcal{F}$ is being generated by $X$.

A stochastic process $S_m = \{S_m(t); t = 0, 1, \cdots, T\}$ is said to be adapted to the filtration $\mathbb{F} = \{\mathcal{F}_t; t = 0, 1, \cdots, T\}$ if the random variables $S_m(t)$ is $\mathcal{F}_t$-measurable for each $t = 0, 1, \cdots, T$.

For the bank account process $S_0(t)$, the interest rate is normally known at the beginning of the period so that $S_0(t)$ is $\mathcal{F}_{t-1}$-measurable, $t = 1, \cdots, T$. In this case, we say that the process $S_0(t)$ is predictable.
Conditional expectation

- Consider the filtered probability space defined by the triplet $(\Omega, \mathcal{F}, P)$. Recall that a random variable is a mapping $\omega \to X(\omega)$ that assigns a real number $X(\omega)$ to each $\omega \in \Omega$.

- A random variable $X$ measurable in $\mathcal{F}$ can be decomposed into the form

$$X(\omega) = \sum_{j=1}^{n} a_j \mathbf{1}_{B_j}(\omega)$$

where $\{B_1, \cdots B_n\}$ is the finite partition of $\Omega$ that generates $\mathcal{F}$. The indicator of $B_j$ is defined by

$$\mathbf{1}_{B_j}(\omega) = \begin{cases} 1 & \text{if } \omega \in B_j \\ 0 & \text{if otherwise} \end{cases}.$$
• The expectation of $X$ with respect to the probability measure $P$ is defined as

$$E[X] = \sum_{j=1}^{n} a_j E[1_{B_j}(\omega)] = \sum_{j=1}^{n} a_j P[B_j],$$

where $P[B_j]$ is the probability that a state $\omega$ contained in $B_j$ occurs. The expectation $E[X]$ is a weighted average of values taken by $X$, weighted according to the various probabilities of occurrence of events. The set of events run through the whole sample space $\Omega$.

• The conditional expectation of $X$ given that event $B$ has occurred is defined to be

$$E[X|B] = \sum_{x} xP[X = x|B]$$

$$= \sum_{x} xP[X = x, B]/P[B]$$

$$= \frac{1}{P[B]} \sum_{\omega \in B} X(\omega)P[\omega].$$
Numerical example

Consider the sample space $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$. The probabilities of occurrence of the states are given by

$$P[\omega_1] = 0.2, P[\omega_2] = 0.3, P[\omega_3] = 0.35 \text{ and } P[\omega_4] = 0.15.$$ 

Consider the two-period price process $S$ whose values are given by

$$S(1; \omega_1) = 3, \quad S(1; \omega_2) = 3, \quad S(1; \omega_3) = 5, \quad S(1; \omega_4) = 5,$$
$$S(2; \omega_1) = 4, \quad S(2; \omega_2) = 2, \quad S(2; \omega_3) = 4, \quad S(2; \omega_4) = 6.$$
The asset price process of a two-period securities model. The filtration $\mathcal{F}$ is revealed through the asset price process that is adapted to $\mathcal{F}$. Here, the partitions are: $\mathcal{P}_2 = \{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_4\}\}, \mathcal{P}_1 = \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\} \text{ and } \mathcal{P}_0 = \{\Omega\}.$
The conditional expectations

\[ E[S(2)|S(1) = 3] \quad \text{and} \quad E[S(2)|S(1) = 5] \]

are calculated by

\[
E[S(2)|S(1) = 3] = \frac{S(2; \omega_1)P[\omega_1] + S(2; \omega_2)P[\omega_2]}{P[\omega_1] + P[\omega_2]} = \frac{(4 \times 0.2 + 2 \times 0.3)}{0.5} = 2.8;
\]

\[
E[S(2)|S(1) = 5] = \frac{S(2; \omega_3)P[\omega_3] + S(2; \omega_4)P[\omega_4]}{P[\omega_3] + P[\omega_4]} = \frac{(4 \times 0.35 + 6 \times 0.15)}{0.5} = 4.6.
\]

Note that “\(S(1) = 3\)” is equivalent to the occurrence of either \(\omega_1\) or \(\omega_2\).
$E[X|\mathcal{F}]$ as a random variable measurable on $\mathcal{F}$

We consider all conditional expectations of the form $E[X|B]$ where the atom $B$ runs through the partition associated with the algebra $\mathcal{F}$. We define the quantity $E[X|\mathcal{F}]$ by

$$E[X|\mathcal{F}] = \sum_{j=1}^{n} E[X|B_j] \mathbb{1}_{B_j}.$$

We see that $E[X|\mathcal{F}]$ is actually a random variable that is measurable with respect to the algebra $\mathcal{F}$. In the above numerical example, suppose we write $\mathcal{F}_1 = \{\phi, \{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}, \Omega\}$, and the atoms of the partition associated with $\mathcal{F}_1$ are $B_1 = \{\omega_1, \omega_2\}$ and $B_2 = \{\omega_3, \omega_4\}$. Since we have

$$E[S(2)|S(1) = 3] = 2.8 \quad \text{and} \quad E[S(2)|S(1) = 5] = 4.6,$$

so

$$E[S(2)|\mathcal{F}_1] = 2.8 \mathbb{1}_{B_1} + 4.6 \mathbb{1}_{B_2}.$$
A useful formula for conditional expectation

Suppose that the random variable $X$ is $\mathcal{F}$-measurable, we would like to show $E[XY|\mathcal{F}] = XE[Y|\mathcal{F}]$ for any random variable $Y$.

Recall that $X = \sum_{B_j \in \mathcal{P}} a_j 1_{B_j}$, where $X(\omega) = a_j$ when $\omega \in B_j$ and $\mathcal{P}$ is the partition corresponding to the algebra $\mathcal{F}$. We obtain

$$E[XY|\mathcal{F}] = \sum_{B_j \in \mathcal{P}} E[XY|B_j] 1_{B_j} = \sum_{B_j \in \mathcal{P}} E[a_jY|B_j] 1_{B_j}$$

$$= \sum_{B_j \in \mathcal{P}} a_j E[Y|B_j] 1_{B_j} = XE[Y|\mathcal{F}].$$

Note that $X$ is known with regard to the information provided by $\mathcal{F}$. In the above two-period securities model, we obtain

$$E[S(1)S(2)|\mathcal{F}_1] = S(1)E[S(2)|\mathcal{F}_1] = \begin{cases} 3 \times 2.8 & \text{if } \omega_1 \text{ or } \omega_2 \text{ occurs} \\ 5 \times 4.6 & \text{if } \omega_3 \text{ or } \omega_4 \text{ occurs} \end{cases}.$$ 

Both $S(1)$ and $E[S(2)|\mathcal{F}_1]$ are $\mathcal{F}_1$-measurable random variables.
**Tower property of nested conditional expectations**

Since $E[X|\mathcal{F}]$ is a random variable, we may compute its expectation. Recall $E[X|\mathcal{F}] = \sum_{j=1}^{n} E[X|B_j]1_{B_j}$, and $E[1_{B_j}] = P[B_j]$, so

$$
E[E[X|\mathcal{F}]] = \sum_{j=1}^{n} E[X|B_j]P[B_j]
$$

$$
= \sum_{j=1}^{n} \left( \sum_{\omega \in B_j} \frac{X(\omega)P[\omega]}{P[B_j]} \right) P[B_j] = E[X].
$$

In general, if $\mathcal{F}_1 \subset \mathcal{F}_2$, then

$$
E[E[X|\mathcal{F}_2]|\mathcal{F}_1] = E[X|\mathcal{F}_1].
$$

If we condition first on the information up to $\mathcal{F}_2$ and later on the information $\mathcal{F}_1$ at an earlier time, then it is the same as conditioning originally on $\mathcal{F}_1$. This is called the *tower property* of nested conditional expectations.
Tower property of nested conditional expectations

Since $E[X|\mathcal{F}]$ is a random variable, we may compute its expectation. Recall $E[X|\mathcal{F}] = \sum_{j=1}^{n} E[X|B_j]1_{B_j}$, and $E[1_{B_j}] = P[B_j]$, so

$$E[E[X|\mathcal{F}]] = \sum_{j=1}^{n} E[X|B_j]P[B_j] = \sum_{j=1}^{n} \left( \sum_{\omega \in B_j} \frac{X[\omega]P[\omega]}{P[B_j]} \right) P[B_j] = E[X].$$

In general, if $\mathcal{F}_1 \subset \mathcal{F}_2$, then

$$E[E[X|\mathcal{F}_2]|\mathcal{F}_1] = E[X|\mathcal{F}_1].$$

If we condition first on the information up to $\mathcal{F}_2$ and later on the information $\mathcal{F}_1$ at an earlier time, then it is the same as conditioning originally on $\mathcal{F}_1$. This is called the tower property of nested conditional expectations.
Notion of martingales: Discounted gain process and self-financing strategy

Martingales are related to models of fair gambling. For example, let $X_n$ represent the amount of money a player possesses at stage $n$ of the game. The martingale property means that the expected amount of the player would have at stage $n + 1$ given that $X_n = \alpha_n$, is equal to $\alpha_n$, regardless of his past history of fortune.

Consider a filtered probability space with filtration $\mathbb{F} = \{\mathcal{F}_t; t = 0, 1, \cdots, T\}$. A stochastic process $S = \{S(t); t = 0, 1, \cdots, T\}$ adapted to $\mathbb{F}$ is said to be martingale if it observes

$$E[S(t + s)|\mathcal{F}_t] = S(t) \quad \text{for all } t \geq 0 \text{ and } s \geq 0.$$
Supermartingales and submartingales

We define an adapted stochastic process $S$ to be a supermartingale if

$$E[S(t + s) | \mathcal{F}_t] \leq S(t) \quad \text{for all } t \geq 0 \text{ and } s \geq 0;$$

and a submartingale if

$$E[S(t + s) | \mathcal{F}_t] \geq S(t) \quad \text{for all } t \geq 0 \text{ and } s \geq 0.$$

1. All martingales are supermartingales, but not vice versa. The same observation is applied to submartingales.

2. An adapted stochastic process $S$ is a submartingale if and only if $-S$ is a supermartingale; $S$ is a martingales if and only if it is both a supermartingale and a submartingale.
Example

Recall in the earlier two-period security model

\[
E[S(2)|\mathcal{F}_1] = \begin{cases} 
2.8 & \text{if } \omega_1 \text{ or } \omega_2 \text{ occurs} \\
4.6 & \text{if } \omega_3 \text{ or } \omega_4 \text{ occurs}
\end{cases}
\]

\[
\leq S(1) = \begin{cases} 
3 & \text{if } \omega_1 \text{ or } \omega_2 \text{ occurs} \\
5 & \text{if } \omega_3 \text{ or } \omega_4 \text{ occurs}
\end{cases}
\]

Also

\[
E[S(1)|\mathcal{F}_0] = 0.5 \times 3 + 0.5 \times 5 = 4 = S(0).
\]

Hence \( S(t) \) is a supermartingale.

- If the price process of a security is a supermartingale, after the arrival of new information, we expect a price decrease. Supermartingales are thus associated with “unfavorable” games, that is, games where wealth is expected to decrease.
Martingale transforms

Suppose $S$ is a martingale and $H$ is a predictable process with respect to the filtration $\mathbb{F} = \{\mathcal{F}_t; t = 0, 1, \cdots , T\}$, we define the process

$$G_t = \sum_{u=1}^{t} H_u \Delta S_u,$$

where $\Delta S_u = S_u - S_{u-1}$. If $S$ and $H$ represent the asset price process and trading strategy, respectively, then $G$ can be visualized as the gain process.

Note that trading strategy is a predictable process, that is, $H_t$ is $\mathcal{F}_{t-1}$-measurable. This is because the number of units held for each security is determined at the beginning of the trading period by taking into account all the information available up to that time.
We call $G$ to be the martingale transform of $S$ by $H$, as $G$ itself is also a martingale.

To show the claim, it suffices to show that $E[G_{t+s} | \mathcal{F}_t] = G_t, t \geq 0, s \geq 0$. We consider

$$E[G_{t+s} | \mathcal{F}_t] = E[G_{t+s} - G_t + G_t | \mathcal{F}_t]$$
$$= E[H_{t+1} \Delta S_{t+1} + \cdots + H_{t+s} \Delta S_{t+s} | \mathcal{F}_t] + E[G_t | \mathcal{F}_t]$$
$$= E[H_{t+1} \Delta S_{t+1} | \mathcal{F}_t] + \cdots + E[H_{t+s} \Delta S_{t+s} | \mathcal{F}_t] + G_t.$$ 

Consider the typical term $E[H_{t+u} \Delta S_{t+u} | \mathcal{F}_t]$, $u \geq 1$, we can express it as $E[E[H_{t+u} \Delta S_{t+u} | \mathcal{F}_{t+u-1}] | \mathcal{F}_t]$.

Further, since $H_{t+u}$ is $\mathcal{F}_{t+u-1}$-measurable and $S$ is a martingale, we have

$$E[H_{t+u} \Delta S_{t+u} | \mathcal{F}_{t+u-1}] = H_{t+u} E[\Delta S_{t+u} | \mathcal{F}_{t+u-1}] = 0.$$ 

Collecting all the calculations, we obtain the desired result.
Discounted gain process

- There is a sample space $\Omega = \{\omega_1, \omega_2, \ldots, \omega_K\}$ of $K$ possible states of the world.
- Let $S$ denote the asset price process $\{S(t); t = 0, 1, \ldots, n\}$, where $S(t)$ is the row vector $S(t) = (S_1(t) \ S_2(t) \cdots S_M(t))$ and whose components are security prices. Also, there is a bank account process $S_0(t)$, whose value is given by
  $$S_0(t) = (1 + r_1)(1 + r_2) \cdots (1 + r_t),$$
  where $r_u$ is the interest rate applied over one time period starting at time $u, u = 0, 1, \ldots, t - 1.$
A trading strategy is the rule taken by an investor that specifies the investor’s position in each security at each time and in each state of the world based on the available information as prescribed by a filtration. Hence, one can visualize a trading strategy as a predictable stochastic process.

We prescribe a trading strategy by a vector stochastic process

\[ H(t) = (h_1(t) \ h_2(t) \cdots h_M(t))^T, \quad t = 1, 2, \cdots, T \]  

(represented as a column vector), where \( h_m(t) \) is the number of units held in the portfolio for the \( m^{th} \) security over the time period from \((t-1)^+\) to \(t^-\). Once \( S_m(t) \) is revealed exactly at time \( t^+ \), \( h_m(t) \) is then adjusted to \( h_m(t+1) \) at \( t^+ \).

The value of the portfolio is a stochastic process given by

\[
V(t) = h_0(t)S_0(t) + \sum_{m=1}^{M} h_m(t)S_m(t), \quad t = 1, 2, \cdots, T,
\]

which gives the portfolio value at the moment right after the asset prices are observed but before changes in portfolio weights are made.
Note that \( h_m(t) \) is \( \mathcal{F}_{t-1} \)-measurable, \( m = 0, 1, \cdots, M \). We write \( \Delta S_m(t) = S_m(t) - S_m(t - 1) \) as the change in value of one unit of the \( m^{th} \) security between times \( t - 1 \) and \( t \). The cumulative gain associated with investing in the \( m^{th} \) security from time zero to time \( t \) is given by

\[
\sum_{u=1}^{t} h_m(u) \Delta S_m(u), \quad m = 0, 1, \cdots, M.
\]

We define the gain process \( G(t) \) to be the total cumulative gain in holding the portfolio consisting of the \( M \) risky securities and the bank account up to time \( t \). The gain process \( G(t) \) is found to be

\[
G(t) = \sum_{u=1}^{t} h_0(u) \Delta S_0(u) + \sum_{m=1}^{M} \sum_{u=1}^{t} h_m(u) \Delta S_m(u), \quad t = 1, 2, \cdots, T.
\]
If we define the discounted price process $S_m^*(t)$ by

$$S_m^*(t) = \frac{S_m(t)}{S_0(t)}, \quad t = 0, 1, \cdots, T, \quad m = 1, 2, \cdots, M,$$

and write $\Delta S_m^*(t) = S_m^*(t) - S_m^*(t-1)$, then the discounted portfolio value process $V^*(t)$ and discounted gain process $G^*(t)$ are given by

$$V^*(t) = h_0(t) + \sum_{m=1}^{M} h_m(t) S_m^*(t), \quad t = 1, 2, \cdots, T,$$

$$G^*(t) = \sum_{m=1}^{M} \sum_{u=1}^{t} h_m(u) \Delta S_m^*(u), \quad t = 1, 2, \cdots, T.$$

Once the asset prices, $S_m(t), m = 1, 2, \cdots, M$, are revealed to the investor at time $t$, he changes the trading strategy from $H(t)$ to $H(t+1)$ as a response to arrival of the new information at time $t$.

Given that $H(t)$ is predictable, $G^*(t)$ is a $Q$-martingale since each quantity $\sum_{u=1}^{t} h_m(u) \Delta S_m^*(u), m = 1, 2, \ldots, M$ is a $Q$-martingale.
Self-financing strategy

Recall that

\[ V(t) = h_0(t)S_0(t) + \sum_{m=1}^{M} h_m(t)S_m(t), \quad t = 1, 2, \ldots, T. \]

The purchase of additional units of one particular security is financed by the sales of other securities. In this case, the trading strategy is said to be self-financing. As a result, we have

\[ V(t) = h_0(t + 1)S_0(t) + \sum_{m=1}^{M} h_m(t + 1)S_m(t). \]

If there were no addition or withdrawal of funds at all trading times, then the cumulative change of portfolio value \( V(t) - V(0) \) should be equal to the gain \( G(t) \) associated with price changes of the securities on all trading dates (in agreement with financial intuition). We postulate that a trading strategy \( H \) is self-financing if and only if

\[ V(t) = V(0) + G(t). \]
No arbitrage principle and martingale pricing measure

A trading strategy $H$ represents an arbitrage opportunity if and only if the value process $V(t)$ associated with $H$ satisfy the following properties:

(i) $V(0) = 0$,
(ii) $V(T) \geq 0$ and $EV(T) > 0$, and
(iii) $H$ is self-financing.

The self-financing trading strategy $H$ is an arbitrage opportunity if and only if (i) $G^*(T) \geq 0$ and (ii) $EG^*(T) > 0$. Here, the expectation $E$ is taken with respect to the actual probability measure $P$, with $P(\omega) > 0$.

Similar to the result obtained for single period models, we expect that arbitrage opportunity does not exist if and only if there exists a risk neutral probability measure. In multiperiod models, risk neutral probabilities are found using the properties that discounted risky asset processes are martingales.
Martingale measure

The measure $Q$ is called a martingale measure (or called a risk neutral probability measure) if it has the following properties:

1. $Q(\omega) > 0$ for all $\omega \in \Omega$.

2. Every discounted price process $S^*_m$ in the securities model is a martingale under $Q$, $m = 1, 2, \cdots, M$, that is,

$$
E_Q[S^*_m(t + s)|\mathcal{F}_t] = S^*_m(t) \quad \text{for all} \ t \geq 0 \ \text{and} \ s \geq 0.
$$

Recall that the conditional expectation $E_Q[S^*_m(t + s)|\mathcal{F}_t]$ is a $\mathcal{F}_t$-measurable random variable, so does $S^*_m(t)$. We call the discounted price process $S^*_m(t)$ to be a $Q$-martingale.
**Numerical example**

We determine the martingale measure $Q$ associated with the earlier two-period securities model. Let $r \geq 0$ be the constant riskless interest rate over one period, and write $Q(\omega_j)$ as the martingale measure associated with the state $\omega_j, j = 1, 2, 3, 4$.

(i) $t = 0$ and $s = 1$

\[
4 = \frac{3}{1 + r} [Q(\omega_1) + Q(\omega_2)] + \frac{5}{1 + r} [Q(\omega_3) + Q(\omega_4)]
\]

(ii) $t = 0$ and $s = 2$

\[
4 = \frac{4}{(1 + r)^2} Q(\omega_1) + \frac{2}{(1 + r)^2} Q(\omega_2) + \frac{4}{(1 + r)^2} Q(\omega_3) + \frac{6}{(1 + r)^2} Q(\omega_4)
\]
(iii) $t = 1$ and $s = 1$

$$3 = \frac{4}{1 + r Q(\omega_1) + Q(\omega_2)} Q(\omega_1) + \frac{2}{1 + r Q(\omega_1) + Q(\omega_2)} Q(\omega_2)$$

$$5 = \frac{4}{1 + r Q(\omega_3) + Q(\omega_4)} Q(\omega_3) + \frac{6}{1 + r Q(\omega_3) + Q(\omega_4)} Q(\omega_4).$$

Recall that $\frac{Q(\omega_1)}{Q(\omega_1) + Q(\omega_2)}$ is the risk neutral conditional probability of going upstate to state $\{\omega_1\}$ at $t = 2$ when the state $\{\omega_1, \omega_2\}$ is reached at $t = 1$.

The calculation procedure can be simplified by observing that $Q(\omega_j)$ is given by the product of the conditional probabilities along the path from the node at $t = 0$ to the node $\omega_j$ at $t = 2$. 
We start with the probability $p$ associated with the upper branch $\{\omega_1, \omega_2\}$. The corresponding probability $p$ is given by

$$4 = \frac{3}{1 + r} p + \frac{5}{1 + r} (1 - p)$$

so that $p = \frac{1 - 4r}{2}$.

Similarly, the conditional probability $p'$ associated with the branch $\{\omega_1\}$ from the node $\{\omega_1, \omega_2\}$ is given by

$$3 = \frac{4}{1 + r} p' + \frac{2}{1 + r} (1 - p')$$

giving $p' = \frac{1 - 3r}{2}$. In a similar manner, the conditional probability $p''$ associated with $\{\omega_3\}$ from $\{\omega_3, \omega_4\}$ is found to be $\frac{1 - 5r}{2}$. 
Determine all the risk neutral conditional probabilities along the path from the node at $t = 0$ to the terminal node $(T, \omega)$. 

\[ S(0; \Omega) = 4 \]

\[ S(1; \omega_1, \omega_2) = 3 \]

\[ S(1; \omega_3, \omega_4) = 5 \]

\[ S(2; \omega_1) = 4 \]

\[ S(2; \omega_2) = 2 \]

\[ S(2; \omega_3) = 4 \]

\[ S(2; \omega_4) = 6 \]
Recall:  
\[ p = Q(\omega_1) + Q(\omega_2) \text{ and } p' = \frac{Q(\omega_1)}{Q(\omega_1) + Q(\omega_2)}. \]

The risk neutral probabilities are then found to be

\[
Q(\omega_1) = pp' = \frac{1-4r}{2} \frac{1-3r}{2},
\]

\[
Q(\omega_2) = p(1-p') = \frac{1-4r}{2} \frac{1+3r}{2},
\]

\[
Q(\omega_3) = (1-p)p'' = \frac{1+4r}{2} \frac{1-5r}{2},
\]

\[
Q(\omega_4) = (1-p)(1-p'') = \frac{1+4r}{2} \frac{1+5r}{2}.
\]

In order that the martingale probabilities remain positive, we have to impose the restriction:  \( r < 0.2 \).
Martingale property of portfolio value processes under self-financing trading strategies

Suppose $H$ is a self-financing trading strategy and $Q$ is a martingale measure with respect to a filtration $\mathcal{F}$, then the portfolio value process $V(t)$ is a $Q$-martingale. Recall the relation

$$V^*(t) = V^*(0) + G^*(t)$$

when $H$ is self-financing. Since $G^*(t)$ is a $Q$-martingale (sum of martingale transform of discounted risky asset price processes), so $V^*(t)$ itself is also a $Q$-martingale.
existence of $Q \Rightarrow$ non-existence of arbitrage opportunities

- Assume that $Q$ exists. Consider a self-financing trading strategy with $V^*(T) \geq 0$ and $E[V^*(T)] > 0$. Here, $E$ is the expectation under the actual probability measure $P$, with $P(\omega) > 0$. That is, $V^*(T)$ is strictly positive for some states of the world.

- As $Q(\omega) > 0$, given that $V^*(T) \geq 0$, we then have $E_Q[V^*(T)] > 0$. However, since $V^*(T)$ is a $Q$-martingale so that $V^*(0) = E_Q[V^*(T)]$, and by virtue of $E_Q[V^*(T)] > 0$, we always have $V^*(0) > 0$.

- It is then impossible to have $V^*(T) \geq 0$ and $E[V^*(T)] > 0$ while $V^*(0) = 0$. If otherwise, this would contradict $V^*(T) = V^*(0) = 0$. Hence, the self-financing strategy $H$ cannot be an arbitrage opportunity.
non-existence of arbitrage opportunities $\Rightarrow$ existence of $Q$

- If there are no arbitrage opportunities in the multiperiod model, then there will be no arbitrage opportunities in any underlying single period. If otherwise, an investor can seek an arbitrage opportunity in the multiperiod model by extracting arbitrage opportunity in this particular single period that admits arbitrage and doing nothing in other periods.

- Since each single period does not admit arbitrage opportunities, one can construct the one-period risk neutral conditional probabilities.

- The risk neutral probability measure $Q(\omega)$ is then obtained by multiplying all the risk neutral conditional probabilities along the path from the node at $t = 0$ to the terminal node $(T, \omega)$. 
Theorem

A multiperiod securities model is arbitrage free if and only if there exists a probability measure $Q$ such that the discounted asset price processes are $Q$-martingales.

Additional remarks

1. The martingale measure is unique if and only if the multiperiod securities model is complete. Here, completeness means all contingent claims ($\mathcal{F}_T$-measurable random variables) can be replicated by a self-financing trading strategy.

2. In an arbitrage free complete market, the arbitrage price of a contingent claim is then given by the discounted expected value under the martingale measure of the portfolio that replicates the claim.
Valuation of an attainable contingent claim

Let $Y$ denote an attainable contingent claim at maturity $T$ and $V(t)$ denote the arbitrage price of the contingent claim at time $t, t < T$. We then have

$$V(t) = \frac{S_0(t)}{S_0(T)} E_Q[Y|\mathcal{F}_t],$$

where $S_0(t)$ is the riskless asset and the ratio $S_0(t)/S_0(T)$ is the discount factor over the period from $t$ to $T$.

To show the claim, since $Y$ is attainable, so there exists a replicating portfolio whose time-$T$ value $V(T)$ equals $Y$. Under a self-financing strategy, the time-$t$ portfolio value $V(t)$ is dictated by the $Q$-martingale property where

$$\frac{V(t)}{S_0(t)} = E_Q \left[ \frac{V(T)}{S_0(T)} \right] = E_Q \left[ \frac{Y}{S_0(T)} \right].$$

The time-$t$ arbitrage price of the contingent claim is equal to the time-$t$ value of the replicating portfolio.