MATH 5510 — Mathematical Models of Financial Derivatives

Topic 5 - Lookback style derivatives

5.1 Product nature of lookback options
5.2 Pricing formulas of European lookback options
   - Floating strike lookback options
   - Fixed strike lookback options
5.3 Rollover strategy and strike bonus premium
5.4 Partial differential equation formulation
5.5 Multistate lookback options
5.6 Dynamic fund protection
   - Finite number of resets
   - Automatic reset
5.1 Product nature of lookback options

The payoff of a lookback option depends on the minimum or maximum price of the underlying asset attained during certain period of the life of the option.

Let $T$ denote the time of expiration of the option and $[T_0, T]$ be the lookback period. We denote the minimum value and maximum value of the asset price realized from $T_0$ to the current time $t$ ($T_0 \leq t \leq T$) by

$$m_{T_0}^t = \min_{T_0 \leq \xi \leq t} S_{\xi}$$

and

$$M_{T_0}^t = \max_{T_0 \leq \xi \leq t} S_{\xi}$$
• A floating strike lookback call gives the holder the right to buy at the lowest realized price while a floating strike lookback put allows the holder to sell at the highest realized price over the lookback period.

• Since \( S_T \geq m_{T_0}^T \) and \( M_{T_0}^T \geq S_T \) so that the holder of a floating strike lookback option always exercise the option.

• Hence, the respective terminal payoff of the lookback call and put are given by \( S_T - m_{T_0}^T \) and \( M_{T_0}^T - S_T \).

• A fixed strike lookback call (put) is a call (put) option on the maximum (minimum) realized price. The respective terminal payoff of the fixed strike lookback call and put are \( \max(M_{T_0}^T - X, 0) \) and \( \max(X - m_{T_0}^T, 0) \), where \( X \) is the strike price.
• An interesting example is the Russian option, which is in fact a perpetual American lookback option. The owner of a Russian option on a stock receives the historical maximum value of the asset price when the option is exercised and the option has no pre-set expiration date.

Under the risk neutral measure, the stochastic price process of the underlying asset is governed by

$$\frac{dS_t}{S_t} = r \ dt + \sigma \ dZ_t \quad \text{or} \quad d\left( \ln \frac{S_t}{S_0} \right) = dU_t = \left( r - \frac{\sigma^2}{2} \right) \ dt + \sigma \ dZ_t,$$

where $U_t = \ln \frac{S_t}{S_0}$ and $\mu = r - \frac{\sigma^2}{2}$. 
5.2 Pricing formulas of European lookback options

We define the following stochastic variables

\[ y_T = \ln \frac{m_T^t}{S} = \min\{U_\xi, \xi \in [t, T]\} \]
\[ Y_T = \ln \frac{M_T^t}{S} = \max\{U_\xi, \xi \in [t, T]\}, \]

and write \( \tau = T - t \). Here, \( S \) is the asset price at the current time \( t \) (dropping the subscript \( t \) for brevity).

**Downstream barrier**

For \( y \leq 0 \) and \( y \leq u \), we can deduce the following joint distribution function of \( U_T \) and \( y_T \) from the transition density function of the Brownian motion with the presence of a downstream barrier

\[
P[U_T \geq u, y_T \geq y] = N \left( \frac{-u + \mu \tau}{\sigma \sqrt{\tau}} \right) - e^{\frac{2u y}{\sigma^2}} N \left( \frac{-u + 2y + \mu \tau}{\sigma \sqrt{\tau}} \right).
\]
**Illustration of** $[U_T \geq u, Y_T \geq y]$

$U_\xi = \ln \frac{S_\xi}{S_0}$ is visualized as the restricted Brownian motion with constant drift rate $\mu$ and downstream absorbing barrier $y$. 

![Diagram](image-url)
**Upstream barrier**

For \( y \geq 0 \) and \( y \geq u \), the corresponding joint distribution function of \( U_T \) and \( Y_T \) is given by

\[
P[U_T \leq u, Y_T \leq y] = N\left(\frac{u - \mu \tau}{\sigma \sqrt{\tau}}\right) - e^{\frac{2\mu y}{\sigma^2}} N\left(\frac{u - 2y - \mu \tau}{\sigma \sqrt{\tau}}\right).
\]

By taking \( y = u \) in the above two joint distribution functions, we obtain the respective distribution function for \( y_T \) and \( Y_T \)

\[
P(y_T \geq y) = N\left(\frac{-y + \mu \tau}{\sigma \sqrt{\tau}}\right) - e^{\frac{2\mu y}{\sigma^2}} N\left(\frac{y + \mu \tau}{\sigma \sqrt{\tau}}\right), \quad y \leq 0,
\]

\[
P(Y_T \leq y) = N\left(\frac{y - \mu \tau}{\sigma \sqrt{\tau}}\right) - e^{\frac{2\mu y}{\sigma^2}} N\left(\frac{-y - \mu \tau}{\sigma \sqrt{\tau}}\right), \quad y \geq 0.
\]

The density function of \( y_T \) and \( Y_T \) can be obtained by differentiating the above distribution functions.
Illustration of \([U_T \leq u, Y_T \leq y]\)

For convenience, we write \(S = S_t, M = M_{T_0}^t\) and \(m = m_{T_0}^t\).
European fixed strike lookback options

Consider a European fixed strike lookback call option whose terminal payoff is $\max(M^{T}_{T_0} - X, 0)$. The value of this lookback call option at the current time $t$ is given by

$$c_{fix}(S, M, t) = e^{-r\tau}E\left[\max(\max(M, M^T_t) - X, 0)\right],$$

where $S_t = S, M^T_{T_0} = M$ and $\tau = T - t$, and the expectation is taken under the risk neutral measure. The payoff function can be simplified into the following forms, depending on $M \leq X$ or $M > X$:

(i) $M \leq X$

$$\max(\max(M, M^T_t) - X, 0) = \max(M^T_t - X, 0)$$

(ii) $M > X$

$$\max(\max(M, M^T_t) - X, 0) = (M - X) + \max(M^T_t - M, 0).$$
Define the function $H$ by

$$H(S, \tau; K) = e^{-r\tau} E[\max(M^T_t - K, 0)],$$

where $K$ is a positive constant. Once $H(S, \tau; K)$ is determined, then

$$c_{fix}(S, M, \tau) = \begin{cases} 
H(S, \tau; X) & \text{if } M \leq X \\
 e^{-r\tau}(M - X) + H(S, \tau; M) & \text{if } M > X 
\end{cases}$$

$$= e^{-r\tau} \max(M - X, 0) + H(S, \tau; \max(M, X)).$$

- $c_{fix}(S, M, \tau)$ is independent of $M$ when $M \leq X$ because the terminal payoff is independent of $M$ when $M \leq X$.

- When $M > X$, the terminal payoff is guaranteed to have the floor value $M - X$. If we subtract the present value of this guaranteed floor value, then the remaining value of the fixed strike call option is equal to a new fixed strike call but with the strike being increased from $X$ to $M$. 
Recall that when $X$ is a non-negative random variable, we have

$$E[X] = \int_0^\infty [1 - F_X(t)] \, dt,$$  
if $X$ is continuous.

Since $\max(M_t^T - K, 0)$ is a non-negative random variable, its expected value is given by the integral of the tail probabilities where

$$H(S, \tau; K) = e^{-r\tau} E[\max(M_t^T - K, 0)] = e^{-r\tau} \int_0^\infty P[M_t^T - K \geq x] \, dx = e^{-r\tau} \int_K^\infty P\left[\ln\frac{M_t^T}{S} \geq \ln\frac{z}{S}\right] \, dz, \quad z = x + K$$

$$= e^{-r\tau} \int_{\ln\frac{K}{S}}^\infty S e^y P[Y_T \geq y] \, dy, \quad y = \ln\frac{z}{S} \left[dy = \frac{1}{z} \, dz, z = Se^y\right]$$

$$= e^{-r\tau} \int_{\ln\frac{K}{S}}^\infty S e^y \left[ N\left(\frac{-y + \mu \tau}{\sigma \sqrt{\tau}}\right) + e^{\frac{2\mu y}{\sigma^2}} N\left(\frac{-y - \mu \tau}{\sigma \sqrt{\tau}}\right) \right] \, dy$$
\[
SN(d) - e^{-r\tau} KN(d - \sigma \sqrt{\tau}) \\
+ e^{-r\tau} \frac{\sigma^2}{2r} S \left[ e^{r\tau} N(d) - \left( \frac{S}{K} \right)^{-\frac{2r}{\sigma^2}} N \left( d - \frac{2r}{\sigma} \sqrt{\tau} \right) \right],
\]

where

\[
d = \frac{\ln \frac{S}{K} + \left( r + \frac{\sigma^2}{2} \right) \tau}{\sigma \sqrt{\tau}}.
\]

The European fixed strike lookback put option with terminal payoff \( \max(X - m^T_{T_0}, 0) \) can be priced in a similar manner. Write \( m = m^T_{T_0} \) and define the function

\[
h(S, \tau; K) = e^{-r\tau} E[\max(K - m^T_t, 0)].
\]
The value of this lookback put can be expressed as

\[ p_{fix}(S, m, \tau) = e^{-r\tau} \max(X - m, 0) + h(S, \tau; \min(m, X)), \]

where

\[
\begin{align*}
    h(S, \tau; K) &= e^{-r\tau} \int_{0}^{\infty} P[\max(K - m_t^T, 0) \geq x] \, dx \\
    &= e^{-r\tau} \int_{0}^{K} P[K - m_t^T \geq x] \, dx \quad 0 \leq \max(K - m_t^T, 0) \leq K \\
    &= e^{-r\tau} \int_{0}^{K} P[m_t^T \leq z] \, dz, \quad z = K - x \\
    &= e^{-r\tau} \int_{0}^{\ln \frac{K}{S}} Se^y P[y_T \leq y] \, dy, \quad y = \ln \frac{z}{S} \\
    &= e^{-r\tau} \int_{0}^{\ln \frac{K}{S}} Se^y \left[ N \left( \frac{y - \mu\tau}{\sigma\sqrt{\tau}} \right) + e^{\frac{2\mu y}{\sigma^2}} N \left( \frac{y + \mu\tau}{\sigma\sqrt{\tau}} \right) \right] \, dy \\
    &= e^{-r\tau} KN(-d + \sigma\sqrt{\tau}) - SN(-d) + e^{-r\tau} \frac{\sigma^2}{2r} S \\
    &\quad \left[ \left( \frac{S}{K} \right)^{-2r/\sigma^2} N \left( -d + \frac{2r}{\sigma\sqrt{\tau}} \right) - e^{r\tau} N(-d) \right].
\end{align*}
\]
European floating strike lookback options

By exploring the pricing relations between the fixed and floating lookback options, we can deduce the price functions of floating strike lookback options from those of fixed strike options. Consider a European floating strike lookback call option whose terminal payoff is $S_T - m_{T_0}$, the present value of this call option is given by

$$c_{f\ell}(S, m, \tau) = e^{-r\tau} E[S_T - \min(m, m^T_t)]$$
$$= e^{-r\tau} E[(S_T - m) + \max(m - m^T_t, 0)]$$
$$= S - me^{-r\tau} + h(S, \tau; m)$$

$$= SN\left(d_m\right) - e^{-r\tau} mN\left(d_m - \sigma\sqrt{\tau}\right) + e^{-r\tau} \frac{\sigma^2}{2r} S$$
$$\left[\left(\frac{S}{m}\right)^{-\frac{2r}{\sigma^2}} N\left(-d_m + \frac{2r}{\sigma} \sqrt{\tau}\right) - e^{r\tau} N(-d_m)\right],$$

where

$$d_m = \frac{\ln \frac{S}{m} + \left(r + \frac{\sigma^2}{2}\right) \tau}{\sigma \sqrt{\tau}}.$$
Consider a European floating strike lookback put option whose terminal payoff is \( M^T_{T_0} - S_T \), the present value of this put option is given by

\[
p_{f\ell}(S, M, \tau) = e^{-r\tau} E[\max(M, M^T_t) - S_T]
\]

\[
= e^{-r\tau} E[\max(M^T_t - M, 0) - (S_T - M)]
\]

\[
= H(S, \tau; M) - (S - Me^{-r\tau})
\]

\[
= e^{-r\tau} MN(-d_M + \sigma \sqrt{\tau}) - SN(-d_M) + e^{-r\tau} \frac{\sigma^2}{2r} S \left[ e^{r\tau} N(d_M) - \left( \frac{S}{M} \right)^{-\frac{2r}{\sigma^2}} N \left( d_M - \frac{2r}{\sigma} \sqrt{\tau} \right) \right],
\]

where

\[
d_M = \ln \frac{S}{M} + \left( r + \frac{\sigma^2}{2} \right) \tau
\]

Remark

Through \( H(S, \tau; M) \) and \( H(S, \tau; X) \), we can deduce the fixed-floating relation between lookback call and put; the form of which is dependent on either \( M \leq X \) or \( M > X \).
Boundary condition at $S = m$

Consider the scenario when $S = m$, that is, the current asset price happens to be at the minimum value realized so far. The probability that the current minimum value remains to be the realized minimum value at expiration is seen to be zero. In other words, the probability that $S_t$ touches the minimum value $m$ once (one-touch) and remains above $m$ at all subsequent times is zero.

Recall the distribution formula for $m_t^T$:

$$ P[m_T^T \geq m] = N \left( \frac{-\ln \frac{m}{S} + \mu \tau}{\sigma \sqrt{t}} \right) - \left( \frac{S}{m} \right)^{1-2r} \frac{\ln \frac{m}{S} + \mu \tau}{\sigma \sqrt{t}} N \left( \frac{\ln \frac{m}{S} + \mu \tau}{\sigma \sqrt{t}} \right) $$

so that $P[m_T^T \geq m] = 0$ when $S = m$. 


Insensitivity of lookback option price to $m$ when $S = m$

We can argue that the value of the floating strike lookback call should be insensitive to infinitesimal changes in $m$ since the change in option value with respect to marginal changes in $m$ is proportional to the probability that $m$ will be the realized minimum at expiry

$$\left. \frac{\partial c_f^{\ell}}{\partial m} (S, m, \tau) \right|_{S=m} = 0.$$

Alternatively, we may argue that the future updating of the realized minimum value does not require the current realized minimum value $m$. Hence, the floating strike lookback call is insensitive to $m$ when $S = m$. 
5.3 Rollover strategy and strike bonus premium

- The sum of the first two terms in $c_{f\ell}$ can be seen as the price function of a European vanilla call with strike price $m$, while the third term can be interpreted as the strike bonus premium.

Rollover strategy

At any time, we hold a European vanilla call with the strike price set at the current realized minimum asset value. In order to replicate the payoff of the floating strike lookback call at expiry, whenever a new realized minimum value of the asset price is established at a later time, one should sell the original call option and buy a new call with the same expiration date but with the strike price set equal to the newly established minimum value.
**Strike bonus premium**

Since the call with a lower strike is always more expensive, an extra premium is required to adopt the rollover strategy. The sum of these expected costs of rollover is termed the strike bonus premium.

The strike bonus premium can be shown to be obtained by integrating a joint probability distribution function involving $m^T_t$ and $S_T$. Firstly, we observe

$$\text{strike bonus premium} = h(S, \tau; m) + S - me^{-r\tau} - c_E(S, \tau; m)$$

$$= h(S, \tau; m) - p_E(S, \tau; m),$$

where $c_E(S, \tau, m)$ and $p_E(S, \tau; m)$ are the price functions of European vanilla call and put, respectively. The last result is due to put-call parity relation.
Recall
\[ h(S, \tau; m) = e^{-r\tau} \int_0^m P[m_T \leq \xi] d\xi \]
and
\[ p_E(S, \tau; m) = e^{-r\tau} \int_0^\infty P[\max(m - S_T, 0) \geq x] dx \]
\[ = e^{-r\tau} \int_0^m P[S_T \leq \xi] d\xi. \]

Since the two stochastic state variables satisfies \( 0 \leq m_T \leq S_T \), we have
\[ P[m_T \leq \xi] - P[S_T \leq \xi] = P[m_T \leq \xi < S_T] \]
so that
\[ \text{strike bonus premium} = e^{-r\tau} \int_0^m P[m_T \leq \xi < S_T] d\xi. \]
Sub-replication and replenishing premium

Replenishing premium of a European put option

\[
\text{put value} = e^{-r\tau} \int_0^K P[S_T \leq \xi] \, d\xi
\]

We divide the interval \([0, K]\) into \(n\) subintervals, each of equal width \(\Delta \xi\) so that \(n \Delta \xi = K\). The put can be decomposed into the sum of \(n\) portfolios, the \(j^{th}\) portfolio consists of long holding a put with strike \(j \Delta \xi\) and short selling a put with strike \((j-1) \Delta \xi\), \(j = 1, 2, \ldots, n\), where all puts have the same maturity date \(T\).

Potential liabilities occur when \(S_T\) falls within \([0, K]\) (the put expires in-the-money). Hedge the exposure over successive \(n\) intervals:

\[n \Delta \xi = K \text{ and } \xi_j = j \Delta \xi\]
For the $j$th portfolio:

- hold one put with strike $j\Delta\xi$ (more expensive)
- short one put with strike $(j - 1)\Delta\xi$ (less expensive)

Present value of this $j$th portfolio

$$
e^{-r\tau}\{E[(j\Delta\xi - S_T)1_{\{S_T \leq \xi_j\}}] - E[((j - 1)\Delta\xi - S_T)1_{\{S_T \leq \xi_{j-1}\}}]\}
\approx e^{-r\tau}\Delta\xi P[S_T \leq \xi_j] \quad \text{(to leading order in } \Delta\xi).$$

In the limit $n \to \infty$, we obtain

$$\text{put value} = e^{-r\tau}\lim_{n \to \infty} \sum_{j=1}^{n} P[S_T \leq \xi_j] \Delta\xi
= e^{-r\tau} \int_{0}^{K} P[S_T \leq \xi] \, d\xi.$$ 

These $n$ portfolios can be visualized as the appropriate replenishment to the sub-replicating portfolio in order that the writer of the put option is immunized from possible loss at the maturity of the option. The sub-replicating portfolio is taken to be the null portfolio in the current example.
• With the addition of the $n$th portfolio [long a put with strike $K$ and short a put with strike $(K - \Delta \xi)$] into the sub-replicating portfolio, the writer faces a loss only when $S_T$ falls below $K - \Delta \xi$.

• Deductively, the protection over the interval $[(j - 1)\Delta \xi, j\Delta \xi]$ in the out-of-the-money region of the put is secured with the addition of the $j$th portfolio.

• One then proceeds one by one from the $n$th portfolio down to the 1st portfolio so that the protection over the whole interval $[0, K]$ is achieved.

• With the acquisition of all these replenishing portfolios, the writer of the put option is immunized from any possible loss at option’s maturity even the put expires out-of-the-money. The cost of acquiring all these $n$ portfolios is the replenishing premium.
Replenishing premium of a European call option

Potential liabilities occur when $S_T$ falls within $[K, \infty)$ (the call expires in-the-money). Hedge the exposure over successive $n$ intervals:

For the $j^{th}$ portfolio:

- hold one call with strike $K + (j - 1)\Delta \xi$ (more expensive)
- short one call with strike $K + j\Delta \xi$ (less expensive)
Present value of this $j^{th}$ portfolio
\[
= e^{-r\tau} \{ E[\{S_T - [K + (j - 1)\Delta\xi]\}] \mathbf{1}_{\{S_T \geq K + (j-1)\Delta\xi\}} \\
- E[[S_T - (K + j\Delta\xi)] \mathbf{1}_{\{S_T \geq K + j\Delta\xi\}}] \}
\approx e^{-r\tau} \Delta\xi P[S_T \geq K + j\Delta\xi] \quad (\text{to leading order in } \Delta\xi).
\]

In the limit $n \to \infty$, we obtain
\[
\text{call value} = e^{-r\tau} \lim_{n \to \infty} \sum_{j=1}^{n} P[S_T \geq K + j\Delta\xi] \Delta\xi \\
= e^{-r\tau} \int_{K}^{\infty} P[S_T \geq \xi] \, d\xi.
\]
Since the sub-replicating portfolio has been chosen to be the null portfolio, the call value is then equal to the replenishing premium.

The above result is distribution free.
Put-call parity relations of continuously monitored floating strike and fixed strike lookback options

We let $[T_0, T]$ be the continuously monitored period for the minimum value of the asset price process. The current time $t$ is within the monitoring period so that $T_0 < t < T$, and that the period of monitoring ends with the maturity of the lookback call option.

The terminal payoff of the continuously monitored floating strike lookback call option is given by

$$c_{f\ell}(S_T, T) = S_T - m^T_{T_0} = S_T - \min(m, m^T_t).$$

Here, $m^T_t$ is a stochastic state variable with dependence on $S_u, u \in [t, T]$. 
(i) forward as the sub-replicating instrument

Suppose we choose the sub-replicating instrument to be a forward with the same maturity and delivery price $m$. The terminal payoff of the sub-replicating instrument is below that of the forward only when $m_t^T < m$; otherwise, their terminal payoffs are equal. Here, $m_t^T$ is the random variable that determines the occurrence of under replication.

Potential liabilities occur when $m_t^T$ falls within $[0, m]$ with payoff $m - m_t^T$. Similar to the put with payoff $m - S_T$ when $S_T$ falls within $[0, m]$, the replenishing premium is (replacing $S_T$ in put by $m_t^T$)

$$e^{-rT} \int_0^m P[m_t^T \leq \xi] \, d\xi.$$  

The replenishing premium can be visualized as the value of a European fixed strike lookback put option with fixed strike $m$ and whose terminal payoff is

$$(m - m_t^T)^+ = (m - \min(m, m_t^T))^+ = (m - m_{T_0}^T)^+.$$
Let $p_{fix}(S, t; K)$ denote the value of a fixed strike lookback put with strike $K$, whose terminal payoff is $\max(K - m, 0)$. This gives the following put-call parity relation for lookback options:

$$c_{f\ell}(S, t; m) = S - e^{-r\tau}m + e^{-r\tau}\int_0^m P[m^T_t \leq \xi] \, d\xi$$

$$= S - e^{-r\tau}m + p_{fix}(S, t; m),$$

where $S$ is the current asset price and $S - e^{-r\tau}m$ is the present value of the forward with delivery price $m$ and maturity date $T$.

The probability distribution $P[m^T_t \leq \xi]$ is given by the distribution function for the restricted asset price process with the down barrier $\xi$ over the interval $[t, T]$. 
Suppose we change the sub-replication portfolio to be a European call option whose terminal payoff is \((S_T - m)^+\). Comparing \(S_T - m_t^T\) with \((S_T - m)^+\), there are two sources of risks. One is the realization of lower minimum value while the other is that \(S_T\) may stay below \(m\). The terminal payoff \(c_{f\ell}(S_T, T)\) is decomposed as

\[
S_T - m_t^T = (m - m_t^T) - (m - S_T)
\]

fixed strike vanilla put
lookback with strike \(m\)
put with strike \(m\)

When \(S_T \leq m\) and \(m_t^T \leq m\), both of the above puts are in-the-money.

Replenishing premium

\[
= e^{-r\tau} \int_0^m \{ P[m_t^T \leq \xi] - P[S_T \leq \xi] \} \, d\xi
\]

\[
= e^{-r\tau} \int_0^m P[m_t^T \leq \xi < S_T] \, d\xi
\]

= strike bonus premium = \(h(S, m, t) - p_E(S, t; m)\).
Remarks

There are 3 other possible cases of the relative position of $m^T_t$ and $S_T$ with respect to $m$.

1. Since $m^T_t \leq S_T$, we rule out $S_T < m$ and $m^T_t > m$.

2. The full replication is achieved when $S_T > m$ and $m^T_t > m$.

3. When $S_T > m$ and $m^T_t < m$, the amount of sub-replication is $m - m^T_t$. The corresponding replenishing premium is reduced to

$$e^{-r\tau} \int_0^m P[m^T_t \leq \xi] \, d\xi$$

since $P[S_T \leq \xi] = 0$ for $\xi \in [0, m]$.

The integral formulation of the replenishing premium on the last page covers all these 4 cases.
5.4 Partial differential equation formulation

We would like to illustrate how to derive the governing partial differential equation and the associated auxiliary conditions for the European floating strike lookback put option. First, we define the quantity

\[ M_n = \left[ \int_{T_0}^{t} (S_\xi)^n d\xi \right]^{1/n}, \quad t > T_0, \]

the derivative of which is given by

\[ dM_n = \frac{1}{n(M_n)^{n-1}} \int_{T_0}^{t} S^n d\xi dt \]

so that \( dM_n \) is deterministic. Taking the limit \( n \to \infty \), we obtain

\[ M = \lim_{n \to \infty} M_n = \max_{T_0 \leq \xi \leq t} S_\xi, \]

giving the realized maximum value of the asset price process over the lookback period \([T_0, t]\).
• We attempt to construct a hedged portfolio which contains one unit of a put option whose payoff depends on $M_n$ and $-\triangle$ units of the underlying asset. Again, we choose $\triangle$ so that the stochastic components associated with the option and the underlying asset cancel.

• Let $p(S, M_n, t)$ denote the value of the lookback put option and let $\Pi$ denote the value of the above portfolio. We then have

$$\Pi = p(S, M_n, t) - \triangle S.$$ 

• The dynamics of the portfolio value is given by

$$d\Pi = \frac{\partial p}{\partial t} dt + \frac{1}{n (M_n)^{n-1}} \frac{\partial p}{\partial M_n} dt + \frac{\partial p}{\partial S} dS + \frac{\sigma^2}{2} S^2 \frac{\partial^2 p}{\partial S^2} dt - \triangle dS$$

by virtue of Ito’s lemma. Again, we choose $\triangle = \frac{\partial p}{\partial S}$ so that the stochastic terms cancel.
Using the usual no-arbitrage argument, the non-stochastic portfolio should earn the riskless interest rate so that

\[ d\Pi = r\Pi dt, \]

where \( r \) is the riskless interest rate. Putting all equations together, we have

\[
\frac{\partial p}{\partial t} + \frac{1}{n} \frac{S^n}{(M_n)^{n-1}} \frac{\partial p}{\partial M_n} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 p}{\partial S^2} + rS \frac{\partial p}{\partial S} - rp = 0.
\]

Next, we take the limit \( n \to \infty \) and note that \( S \leq M \). When \( S < M \), \( \lim_{n \to \infty} \frac{1}{n} \frac{S^n}{(M_n)^{n-1}} = 0 \); and when \( S = M \), \( \frac{\partial p}{\partial M} = 0 \). Hence, the second term becomes zero as \( n \to \infty \).

The governing equation for the floating strike lookback put is given by

\[
\frac{\partial p}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 p}{\partial S^2} + rS \frac{\partial p}{\partial S} - rp = 0, \quad 0 < S < M, t > T_0.
\]
The domain of the pricing model has an upper bound $M$ on $S$. The variable $M$ does not appear in the equation, though $M$ appears as a parameter in the auxiliary conditions. The final condition is

$$p(S, M, T) = M - S.$$ 

In this European floating strike lookback put option, the boundary conditions are applied at $S = 0$ and $S = M$. Once $S$ becomes zero, it stays at the zero value at all subsequent times and the payoff at expiry is certain to be $M$.

Discounting at the riskless interest rate, the lookback put value at the current time $t$ is

$$p(0, M, t) = e^{-r(T-t)}M.$$ 

The boundary condition at the other end $S = M$ is given by

$$\frac{\partial p}{\partial M} = 0 \quad \text{at} \quad S = M.$$
5.5 Multistate lookback options

Two-asset semi-lookback option

The terminal payoff of a semi-lookback option depends on the extreme value of the price of one asset and the terminal values of the prices of other assets. Let $V^2_{semi}(S_1, S_2, t; \overline{S}_2[T_0, t])$ denote the value of the two-asset semi-lookback option whose terminal payoff is given by $(\overline{S}_2[T_0, T] - S_{1,T} - K)^+$. We write $\overline{S}_2[T_0, t] = M_2$ and $\overline{S}_2[t, T] = (M_2)^T_t$, and let the terminal payoff be expressed as

$$(\max(M_2 - K, (M_2)^T_t - K) - S_{1,T})^+. $$

We choose the sub-replicating instrument to be the put option on $S_1$ with strike $M_2 - K$, where the corresponding terminal payoff is

$$(((M_2 - K) - S_{1,T})^+. $$
Similarly, there are two sources of risks: (i) realization of higher maximum asset value, where $(M_2)_t^T > M_2$, (ii) $S_{1,T} > M_2 - K$ and $(M_2)_t^T - K > S_{1,T} \Leftrightarrow M_2 < S_{1,T} + K < (M_2)_t^T$.

- The first scenario leads to an increase in the strike of the semi-lookback option from $M_2 - K$ to $(M_2)_t^T - K$.

- The second scenario leads to positive terminal payoff in the semi-lookback option but zero terminal payoff in the vanilla put.

The two cases can be combined into $M_2 < S_{1,T} + K < (M_2)_t^T$. When such scenario occurs, the under replication at maturity is

$$(M_2)_t^T - K - S_{1,T} = [(M_2)_t^T - M_2] - [(S_{1,T} + K) - M_2].$$
The above terminal payoff is equivalent to the sum of those of the fixed strike lookback call on \((M_2)_t^T\) and vanilla call on \(S_{1,T} + K\), both with the same strike \(M_2\).

The required replenishing premium is given by

\[
e^{-r\tau} \left\{ \int_{M_2}^{\infty} P[(M_2)_t^T > \xi] \, d\xi - \int_{M_2}^{\infty} P[S_{1,T} + K > \xi] \, d\xi \right\} = e^{-r\tau} \int_{M_2}^{\infty} P[(M_2)_t^T > \xi \geq S_{1,T} + K] \, d\xi.
\]

The value of the two-asset semi-lookback option is given by

\[
V_{semi}^2(S_1, S_2, t; \overline{S}_2[T_0, t]) = p(S_1, t; M_2 - K) + e^{-r\tau} \int_{M_2}^{\infty} P[(M_2)_t^T > \xi \geq S_{1,T} + K] \, d\xi.
\]
Discretely monitored floating strike lookback call options

Suppose the monitoring of the minimum value of the asset price takes place only at discrete time instant \( t_j, j = 1, 2, \cdots, n \), where \( t_n \) is on or before the maturity date of the lookback call option. Suppose the current time is taken to be within \( [t_k, t_{k+1}) \). The terminal payoff of the discretely monitored floating strike lookback call option is given by

\[
c_{f\ell}^{\text{dis}}(S_T, T) = S_T - \min(S_{t_1}, S_{t_2}, \cdots, S_{t_n}).
\]

We use the notation \( S[i, j] \) to denote \( \min(S_{t_i}, S_{t_i+1}, \cdots, S_{t_j}), j > i \). At the current time, \( S[1, k] = \min(S_{t_1}, S_{t_2}, \cdots, S_{t_k}) \) is already known. Similar to the continuously monitored case, we choose the sub-replicating instrument to be a forward with the same maturity date \( T \) and delivery price \( S[1, k] \).
Potential liabilities occur when \( S[k+1, n] \) falls below \( S[1, k] \). This is similar to a put with the stochastic state variable \( S[k+1, n] \) and strike \( S[1, k] \).

The replenishing premium required to compensate for under replication is given by

\[
\text{replenishing premium} = e^{-r\tau} \int_0^{S[1,k]} P[S[k+1, n] \leq \xi] \, d\xi.
\]

The present value of the discretely monitored European floating strike lookback call option is then given by

\[
c_{\text{dis}}^\ell (S, t; S[1, k]) = S - e^{-r\tau} S[1, k] + e^{-r\tau} \int_0^{S[1,k]} P[S[k+1, n] \leq \xi] \, d\xi.
\]

Remark  
Following the strike bonus approach, we can also obtain

\[
c_{\text{dis}}^\ell (S, t; \underline{S}[1, k]) = c_E(S, t; \underline{S}[1, k]) + e^{-r\tau} \int_0^{S[1,k]} P[S[k+1, n] \leq \xi < S_T] \, d\xi.
\]
The distribution function $P[S[k + 1, n] \leq \xi]$ can be expressed as

$$P[S[k + 1, n] \leq \xi] = \sum_{j = k + 1}^{n} E \{ S_{t_j} \leq \xi, S_{t_j}/S_{t_i} \leq 1 \text{ for all } i \neq j, k + 1 \leq i \leq n \},$$

where the indicator function in the $j$th term corresponds to the event that $S_{t_j}$ is taken be the minima among $S_{t_{k+1}}, \ldots, S_{t_n};$ and $j$ runs from $k + 1$ to $n.$

Suppose the asset price follows the Geometric Brownian motion, then $S_{t_j}$ and $S_{t_j}/S_{t_i}, i \neq j, k + 1 \leq i \leq n,$ are all lognormally distributed. The expectation values in above equation can be expressed in terms of multi-variate cumulative normal distribution functions.
One-asset lookback spread option

The terminal payoff of an one-asset lookback spread option is given by

\[ c_{sp}(S_T, T; K) = \left( M_{T_0}^T - m_{T_0}^T - K \right)^+. \]

Choice of sub-replicating portfolio:

- long holding of one unit of European lookback call and one unit of lookback put, both of floating strike;
- short holding of a riskless bond of par value \( K \).

All these instruments have the same maturity as that of the lookback spread option.

Terminal payoff of this sub-replicating portfolio

\[ = (M_{T_0}^T - S_T) + (S_T - m_{T_0}^T) - K = M_{T_0}^T - m_{T_0}^T - K. \]
Note that
\[
M_{T_0}^T - m_{T_0}^T - K = \max(M, M_t^T) - \min(m, m_t^T) - K
\geq M - m - K,
\]
so if the lookback spread is currently in-the-money, then it will expire in-the-money. The sub-replication is a full replication when the lookback spread option is currently in-the-money.

On the other hand, if the lookback spread option is currently out-of-the-money, the terminal payoff of the sub-replicating portfolio would be less than that of the lookback spread option if the lookback spread option expires out-of-the-money. That is,
\[
\max(M, M_t^T) - \min(m, m_t^T) - K < 0.
\]

The sources of risks include (i) \( M_t^T > M \) and \( m_t^T < m \), (ii) \( M_t^T - m_t^T - K < 0 \Leftrightarrow m_t^T + K > M_t^T \). These cases can be combined into
\[
M < M_t^T < m_t^T + K < m + K.
\]
We hedge the exposure over successive subintervals within $[M, m + K]$: $(M + (j - 1)\Delta \xi, M + j\Delta \xi)$ and write $\xi_j = M + j\Delta \xi$.

The potential liability of $\Delta \xi$ has to be replenished when both of the following 2 events occur together:

$$M_t^T < \xi_j \quad \text{and} \quad m_t^T + K > \xi_j.$$ 

The expected replenishing premium over the $j^{\text{th}}$ interval is

$$e^{-r\tau} \Delta \xi \ P[M_t^T < \xi_j < m_t^T + K].$$

Summing over all intervals, we obtain

$$e^{-r\tau} \int_M^{m+K} P[M_t^T < \xi < m_t^T + K] \, d\xi.$$
In summary

(i) \( M - m - K \geq 0 \) (currently in-the-money or at-the-money)

\[
c_{sp}(S, t; M, m) = c_{f\ell}(S, t; m) + p_{f\ell}(S, t; M) - Ke^{-r\tau};
\]

(ii) \( M - m - K < 0 \) (currently out-of-the-money)

\[
c_{sp}(S, t; M, m) = c_{f\ell}(S, t; m) + p_{f\ell}(S, t; M) - Ke^{-r\tau} \\
+ e^{-r\tau} \int_{M}^{m+K} P[M^T_{t} < \xi < m^T_{t} + K] \ d\xi.
\]
The distribution function $P[M < \xi \leq m + K]$ can be deduced from

$$
P[X \geq x, \overline{X} \leq y] = \sum_{n=-\infty}^{\infty} e^{[2n\alpha(y-x)]/\sigma^2} \left\{ N \left( \frac{y - \alpha t - 2n(y-x)}{\sigma \sqrt{t}} \right) \right. \\
- N \left( \frac{x - \alpha t - 2n(y-x)}{\sigma_1 \sqrt{t}} \right) \\
- e^{2\alpha x/\sigma^2} \left[ N \left( \frac{y - \alpha t - 2n(y-x) - 2x}{\sigma \sqrt{t}} \right) \right. \\
- N \left( \frac{x - \alpha t - 2n(y-x) - 2x}{\sigma \sqrt{t}} \right) \left. \right\}. $$
Two-asset lookback spread option

Let \( S_{1,u} \) and \( S_{2,u} \) denote the price process of asset 1 and asset 2, respectively. Similarly, we write \( \mathcal{S}_1[t_1,t_2] \) and \( \mathcal{S}_2[t_1,t_2] \) as the realized maximum value of \( S_{1,u} \) and realized minimum value of \( S_{2,u} \) over the period \([t_1,t_2]\), respectively. The terminal payoff of a two-asset lookback spread option is given by

\[
c_{sp}(S_{1,T}, S_{2,T}, T; K) = (\mathcal{S}_1[T_0, T] - \mathcal{S}_2[T_0, T] - K)^+,
\]

where \( K \) is the strike price.
Replicating portfolio

Since we can express $\bar{S}_1[T_0, T] - S_2[T_0, T] - K$ as

$$(\bar{S}_1[T_0, T] - S_{1,T}) + (S_{2,T} - \bar{S}_2[T_0, T]) + S_{1,T} - S_{2,T} - K,$$

a natural choice of the sub-replicating portfolio would consist of long holding of one European floating strike lookback put on asset 1, one European floating strike lookback call on asset 2, one unit of asset 1 and short holding of one unit of asset 2 and a riskless bond of par value $K$. All instruments in the portfolio have the same maturity as that of the two-asset lookback spread option.

Similar to the one-asset counterpart, the two-asset lookback spread option is guaranteed to expire in-the-money if it is currently in-the-money.

The sub-replicating portfolio will expire with a terminal payoff below that of the lookback spread option if the lookback spread option expires ont-of-the-money.
The current value of the two-asset European lookback spread option is given by

(i) \( \overline{S}_1[T_0, t] - \overline{S}_2[T_0, t] - K \geq 0 \)

\[
c_{sp}(S_1, S_2, t; \overline{S}_1[T_0, t], \overline{S}_2[T_0, t]) = p_{f\ell}(S_1, t; \overline{S}_1[T_0, t]) + c_{f\ell}(S_2, t; \overline{S}_2[T_0, t]) + S_1 - S_2 - Ke^{-r\tau};
\]

(ii) \( \overline{S}_1[T_0, t] - \overline{S}_2[T_0, t] - K < 0 \)

\[
c_{sp}(S_1, S_2, t; \overline{S}_1[T_0, t], \overline{S}_2[T_0, t]) = p_{f\ell}(S_1, t; \overline{S}_1[T_0, t]) + c_{f\ell}(S_2, t; \overline{S}_2[T_0, t]) + S_1 - S_2 - Ke^{-r\tau} + e^{-r\tau} \int_{\overline{S}_1[T_0, t]}^{S_2[T_0, t] + K} P(\overline{S}_1[t, T] < \xi \leq \overline{S}_2[t, T] + K) \, d\xi.
\]

- The joint distribution of maximum of one asset and minimum of another asset cannot be deduced using the method of images. One has to resort to eigenfunction expansion technique, and the resulting expression involves the modified Bessel functions.
5.6 Dynamic fund protection

Provides an investor with a floor level of protection during the investment period.

- This feature generalizes the concept of a put option, which provides only a floor value at a particular time.

- The dynamic fund protection ensures that the fund value is upgraded if it ever falls below a certain threshold level.
Typical sample path of the fund unit values

$F(t)$

$\tilde{F}(t)$

$K$

upgraded fund

primary fund
1. \( \tilde{F}(0) = F(0) \)

2. \( \tilde{F}(t) = F(t) \max \left\{ 1, \max_{0 \leq s \leq t} \frac{K}{F(s)} \right\} \)

- Whenever \( \tilde{F}(t) \) drops below \( K \), just enough money will be added so that the upgraded fund unit value does not fall below \( K \).

- Write \( M(t) = \max \left\{ 1, \frac{K}{\min_{0 \leq s \leq t} F(s)} \right\} \), then

\[
\tilde{F}(T) = F(T) \max \left\{ M(t), \frac{K}{\min_{t \leq s \leq T} F(s)} \right\}.
\]

Here, \( M(t) \) is the number of units of primary fund acquired at time \( t \). The lookback feature of the dynamic protection is revealed by \( M(t) \).
Protection with reference to a stock index under finite number of resets

\( F(t) \): value of the primary fund

\( I(t) \): value of the reference stock index

\( \tilde{F}(t) \): value of the protected fund

When the investor makes its reset decision, the sponsor of the protected fund has to purchase additional units of the primary fund so that the protected fund value is upgraded to that of the reference index.
$F$ as the numeraire

At the reset instant $\xi_i$, $i = 1, 2, \ldots$, we have

$$\tilde{F}(\xi_i) = I(\xi_i) = n(\xi_i)F(\xi_i),$$

where $n(\xi_i) = I(\xi_i)/F(\xi_i) > 1$ is the new number of units of the primary fund in the investment fund.

It is obvious that $n(\xi_1) < n(\xi_2) < \cdots$. The normalized upgraded fund value $\tilde{F}/F$ is related to $I/F$. This motivates us to use $F$ as the numeraire.
Fund value dynamics

Under the risk neutral valuation framework, we assume that the primary fund value $F(t)$ and the reference index value $I(t)$ follow the Geometric Brownian motions:

$$\frac{dF}{F} = (r - q_p) dt + \sigma_p \, dZ_p,$$

$$\frac{dI}{I} = (r - q_i) dt + \sigma_i \, dZ_i,$$

where $r$ is the riskless interest rate, $q_p$ and $q_i$ are the dividend yield of the primary fund and stock index, respectively, $\sigma_p$ and $\sigma_i$ are the volatility of the primary fund value and reference index value, respectively, and $dZ_p dZ_i = \rho \, dt$. Here, $\rho$ is the correlation coefficient between the primary fund process and reference index process.
Pricing formulation of the protected fund with $n$ resets

Let $V_n(F, I, t)$ denote the value of the investment fund with dynamic protection with respect to a reference stock index, where the investor has $n$ reset rights outstanding. We first consider the simpler case, where there has been no prior reset. That is, the number of units of the primary fund is equal to one at current time $t$.

The dimension of the pricing model can be reduced by one if $F$ is chosen as the numeraire. We define the stochastic state variable

$$x = \frac{I}{F}$$

where $F$ is the fund value at the grant date (same as the primary fund value).
Note that $x$ follows the Geometric Brownian motion

$$\frac{dx}{x} = (q_p - q_i)dt + \sigma \ dZ,$$

where $\sigma^2 = \sigma_p^2 - 2\rho \sigma_p \sigma_i + \sigma_i^2$. Accordingly, we define the normalized fund value function with $F$ as the numeraire by

$$W_n(x, t) = \frac{V_n(F, I, t)}{F}.$$

Whenever a reset has occurred, the upgraded fund value $\tilde{F}$ will be used as the numeraire so that $x = \frac{I}{\tilde{F}}$. Note that $F(t)$ and $\tilde{F}(t)$ have the same dynamics equation since $\tilde{F}(t)$ is scalar multiple of $F(t)$.

- The investor should never reset when $F(t)$ stays above $I(t)$.

- With only a finite number of reset rights, he does not reset immediately when $F(t)$ just hits the level of $I(t)$.
With reference to the variable $x$, the investor resets when $x$ reaches some sufficiently high threshold value (denoted by $x^*_n$). The value of $x^*_n$ is not known in advance, but has to be solved as part of the solution to the pricing model.

Upon reset at $x = x^*_n$, the sponsor has to increase the number of units of the primary fund so that the new value of the investment fund equals $I$.

The corresponding number of units should then be $x^*_n$, which is the ratio of the reference index value to the primary fund value right before the reset moment.
After the reset, we have

- the number of resets outstanding is reduced by one,

- the value of $x$ becomes one since the ratio of the reference index value to the newly upgraded fund value is one.

We then have

$$W_n(x^*, t) = x^*W_{n-1}(1, t).$$

To show the claim, note that

$$W_{n-1}(1, t^+) = \frac{V_{n-1}(\tilde{F}, \tilde{F}, t^+)}{\tilde{F}(t^+)} = \frac{V_n(\tilde{F}(t^-), x_n^*(t^-)\tilde{F}(t^-), t^-)}{x_n^*(t^-)\tilde{F}(t^-)}$$

$$= \frac{1}{x_n^*(t^-)}W_n(x, t^-).$$
Smooth pasting condition

We should have the smooth pasting condition at $x = x_n^*$, based on optimality consideration:

$$W'_n(x^*_n, t) = W_{n-1}(1, t).$$

This extra smooth pasting condition determines the value of $x_n^*$ such that the investment fund value is maximized.

The terminal payoff of the investment fund is simply equal to $F$, if no reset has occurred throughout the life of the fund. This gives $W_n(x, T) = 1$. 
**Governing equation**

In the continuation region, inside which the investor chooses not to exercise the reset right, the value function $V_n(F,I,t)$ satisfies the Black-Scholes equation with the two state variables, $F$ and $I$.

In terms of $x$, the governing equation and the associated auxiliary conditions for $W_n(x,t)$ are given by

\[
\frac{\partial W_n}{\partial t} + \frac{\sigma^2}{2}x^2 \frac{\partial^2 W_n}{\partial x^2} + (q_p - q_i)x \frac{\partial W_n}{\partial x} - q_p W_n = 0, \quad t < T, \quad x < x^*_n(t),
\]

\[
W_n(x^*_n,t) = x^*_n W_{n-1}(1,t) \quad \text{and} \quad W'_n(x^*_n,t) = W_{n-1}(1,t), \quad W_n(x,T) = 1,
\]

where $x^*_n(t)$ is the time-dependent threshold value at which the investor optimally exercises the reset right.
Free boundary value problem

The pricing model leads to a free boundary value problem with the free boundary $x^*_n(t)$ separating the continuation region \{$(x, t) : x < x^*_n(t), t < T$\} and the stopping region \{$(x, t) : x \geq x^*_n(t), t < T$\}. The free boundary is not known in advance but has to be determined as part of the solution of the pricing model. This is a *multiple* optimal stopping problem.

At times close to expiry, the investor should choose to reset even when $F(t)$ is only slightly below $I(t)$, so we deduce that $x^*_n(T^-) = 1$.

- The free boundary $x^*_n(t)$ is a monotonically decreasing function of $t$ since the holder should reset at a lower threshold fund value as time is approaching maturity. At fixed value of $t$, $x^*_n(t)$ is a monotonically decreasing function of $n$ since the investor should become more conservative with less number of resets outstanding.
The exercise payoff is the value of the dynamic protected fund with outstanding reset rights reduced by one. When \( n \to \infty \), \( x_n^*(\infty) \) tends to 1 since the investor chooses to reset whenever \( \tilde{F}(t) \) falls to \( I(t) \). As \( x_n^*(\infty) > x_n^*(t) > 1 \), so \( x_\infty^*(t) = 1 \).
Simplified pricing model under limit of infinite number of resets - automatic reset

With $x^*_\infty(t) = 1$ for all $t < T$. This gives $W'_\infty(x^*_\infty, t) = W_\infty(1, t)$. For convenience, we define

$$W_\infty(y, t) = \frac{V_\infty(\tilde{F}, I, t)}{\tilde{F}}, \text{ where } y = \ln x = \ln \frac{I}{\tilde{F}}, \tau = T - t.$$  

Note that the free boundary $x^*_n(t)$ becomes the fixed boundary $y = \ln x^*_\infty(t) = \ln 1 = 0$.

The governing equation and auxiliary conditions for $W_\infty(y, \tau)$ are reduced to

$$\frac{\partial W_\infty}{\partial \tau} = \frac{\sigma^2}{2} \frac{\partial^2 W_\infty}{\partial y^2} + \mu \frac{\partial W_\infty}{\partial y} - q_p W_\infty, \quad \tau > 0, \quad y < 0;$$

$$\frac{\partial W_\infty}{\partial y}(0, \tau) = W_\infty(0, \tau), \quad W_\infty(y, 0) = 1,$$

where $\mu = q_p - q_i - \frac{\sigma^2}{2}$ [Note that $y < 0 \iff F(t) > I(t)$].
The Robin boundary condition: \( \frac{\partial W_\infty}{\partial y}(0, \tau) = W_\infty(0, \tau) \) can be expressed as \( \frac{\partial V_\infty}{\partial F}(I, \tilde{F}, t) = 0 \) at \( \tilde{F} = I \). If the index value is taken to be the constant value \( K \), then the Neumann boundary condition: \( \frac{\partial V_\infty}{\partial F}(\tilde{F}, t) \bigg|_{\tilde{F}=K} = 0 \) is equivalent to the reflecting boundary condition with respect to the upgraded fund \( \tilde{F} \) placed at the guarantee level \( K \).

The Robin boundary condition at \( y = 0 \) leads to a slight complication in the solution procedure. The analytic representation of the solution \( W_\infty(y, \tau) \) admits different forms, depending on \( q_p \neq q_i \) or \( q_p = q_i \).
Let $\alpha = 2(q_i - q_p)/\sigma^2$ and $\tilde{\mu} = \mu + \sigma^2$, the price function $V_\infty(F, I, t)$ is found to be

(i) $q_p \neq q_i$

$$
V_\infty(F, I, t) = I e^{-q_i \tau} \left( 1 - \frac{1}{\alpha} \right) N \left( \frac{\ln(I/F) + \tilde{\mu} \tau}{\sigma \sqrt{\tau}} \right) + \frac{I}{\alpha} \left( \frac{I}{F} \right)^\alpha e^{-q_p \tau} N \left( \frac{\ln(I/F') - \mu \tau}{\sigma \sqrt{\tau}} \right) + Fe^{-q_p \tau} N \left( \frac{-\ln(I/F') - \mu \tau}{\sigma \sqrt{\tau}} \right), \ F > I.
$$

(ii) $q_p = q_i$ (write the common dividend yield as $q$)

$$
V_\infty(F, I, t) = I e^{-q \tau} \sigma \sqrt{\tau} n \left( \frac{\ln(I/F') + (\sigma^2 \tau/2)}{\sigma \sqrt{\tau}} \right) + I e^{-q \tau} \left( \frac{\ln I}{F} + 1 + \frac{\sigma^2 \tau}{2} \right) N \left( \frac{\ln(I/F) + (\sigma^2 \tau/2)}{\sigma \sqrt{\tau}} \right) + Fe^{-q \tau} N \left( \frac{-\ln(I/F') + (\sigma^2 \tau/2)}{\sigma \sqrt{\tau}} \right), \ F > I.
$$
Derivation of $W_\infty(y, \tau)$

We perform the continuation of the initial condition to the whole domain $(-\infty, \infty)$, where

$$W_\infty(y, 0) = \begin{cases} 
1 & \text{if } y < 0, \\
\psi(y) & \text{if } y \geq 0.
\end{cases}$$

no upgrading occurs until maturity

$\psi(y)$ to be determined so as to satisfy the Robin condition
The function $\Psi(y)$ is determined such that the Robin boundary condition is satisfied. Let $g(y, \tau; \xi)$ denote the fundamental solution to the governing differential equation, where

$$g(y, \tau; \xi) = \frac{e^{-q_p \tau}}{\sqrt{2\pi \sigma^2 \tau}} \exp \left( -\frac{(\xi - y - \mu \tau)^2}{2 \sigma^2 \tau} \right).$$

The following relations are useful in later derivation procedure.

$$\int_{-\infty}^{0} g(y, \tau; \xi) \, d\xi = e^{-q_p \tau} \left[ 1 - N \left( \frac{y + \mu \tau}{\sigma \sqrt{\tau}} \right) \right]$$

and

$$\frac{\partial}{\partial y} \int_{-\infty}^{0} g(y, \tau; \xi) \, d\xi = -e^{-q_p \tau} n \left( \frac{y + \mu \tau}{\sigma \sqrt{\tau}} \right) = -g(y, \tau; 0).$$
The solution to $W_\infty(y, \tau)$ can be formally expressed as

$$ W_\infty(y, \tau) = \int_{-\infty}^{\infty} W_\infty(\xi, 0) g(y, \tau; \xi) \, d\xi $$

$$ = e^{-qp\tau} \left[ 1 - N\left(\frac{y + \mu\tau}{\sigma\sqrt{\tau}}\right) \right] + \int_0^{\infty} \Psi(\xi) g(y, \tau; \xi) \, d\xi. $$

Performing the differentiation with respect to $y$ on both sides of the above equation and integration by parts, we have

$$ \frac{\partial W_\infty}{\partial y}(y, \tau) = -g(y, \tau; 0) - \int_0^{\infty} \Psi(\xi) \frac{\partial g}{\partial \xi}(y, \tau; \xi) \, d\xi $$

$$ = \int_0^{\infty} \Psi'(\xi) g(y, \tau; \xi) \, d\xi. $$
Note that

\[ W_\infty(0, \tau) = \int_0^\infty \Psi(\xi) g(0, \tau; \xi) \, d\xi + \int_0^\infty g(0, \tau; -\xi) \, d\xi. \]

We apply the Robin boundary condition: \( \frac{\partial W_\infty}{\partial y}(0, \tau) = W_\infty(0, \tau) \) to obtain

\[ \int_0^\infty \{(\psi'(\xi) - \psi(\xi))g(0, \tau; \xi) - g(0, \tau; -\xi)\} \, d\xi = 0 \]

This relation works for any choice of \( g \). In order that the Robin boundary condition is satisfied, \( \psi(\xi) \) and \( g(0, \tau; \xi) \) have to observe the following relation:

\[ \psi'(\xi) - \psi(\xi) = \frac{g(0, \tau; -\xi)}{g(0, \tau; \xi)} = e^{(\alpha+1)\xi}, \text{ where } \alpha = \frac{2(q_i - q_p)}{\sigma^2}. \]

Interestingly, \( g(0, \tau; -\xi)/g(0, \tau; \xi) \) has no dependence on \( \tau \). If otherwise, this procedure does not work.
The auxiliary condition for $\Psi(\xi)$ is obtained by observing continuity of $W(y,0)$ at $y = 0$, giving $\Psi(0) = 1$. The solution of $\Psi(\xi)$ depends on $\alpha \neq 0$ or $\alpha = 0$, namely,

(i) when $\alpha \neq 0$,

$$\Psi(\xi) = e^{\xi} \left(\frac{e^{\alpha \xi}}{\alpha} + 1 - \frac{1}{\alpha}\right);$$

(ii) when $\alpha = 0$, $\Psi(\xi) = e^{\xi} \left[1 + \lim_{\alpha \to 0} \frac{e^{\alpha \xi} - 1}{\alpha}\right] = (1 + \xi)e^{\xi}.$
By substituting the known solution of $\Psi(\xi)$, we obtain

(i) when $\alpha \neq 0$,

$$W_\infty(y, \tau) = e^{y-q_i \tau} \left( 1 - \frac{1}{\alpha} \right) N \left( \frac{y + \bar{\mu} \tau}{\sigma \sqrt{\tau}} \right) + \frac{1}{\alpha} e^{(1+\alpha) y-q_p \tau} N \left( \frac{y - \mu \tau}{\sigma \sqrt{\tau}} \right) + e^{-q_p \tau} N \left( \frac{-y - \mu \tau}{\sigma \sqrt{\tau}} \right), \quad y < 0;$$

(ii) when $\alpha = 0$ (write $q$ as the common dividend yield),

$$W_\infty(y, \tau) = e^{-q \tau} N \left( \frac{-y + (\sigma^2 \tau/2)}{\sigma \sqrt{\tau}} \right) + e^{y-q \tau} \sigma \sqrt{\tau} n \left( \frac{y + (\sigma^2 \tau/2)}{\sigma \sqrt{\tau}} \right) + e^{y-q \tau} \left( y + 1 + \frac{\sigma^2 \tau}{2} \right) N \left( \frac{y + (\sigma^2 \tau/2)}{\sigma \sqrt{\tau}} \right), \quad y < 0.$$

**Remark**

When $q_p = 0$, the primary fund does not pay dividend. The dynamic protection has no value, so we expect

$$\lim_{F \to \infty} V_\infty(F, I, t) = F.$$
Mid-contract valuation

Let $M$ denote the path dependent state variable which represents the realized maximum value of the state variable $x$ from the grant-date to the mid-contract time $t$, that is,

$$M = \max_{0 \leq u \leq t} \frac{I(u)}{F(u)}.$$

At any mid-contract time, the number of units of primary fund held in the investment fund is given by

$$n(t) = \begin{cases} 
1 & \text{if } M \leq 1, \\
M & \text{if } M > 1.
\end{cases}$$

If the primary fund value has been staying above or at the reference stock index so far (corresponding to $M \leq 1$), then upgrade has never occurred, so the number of units of primary fund remains at one. Otherwise, the number of units is upgraded to $M$. 
Governing differential equation

Let $V_{\text{mid}}(F, I, M, t)$ denote the mid-contract investment fund value at time $t$, with dependence on the state variable $M$. Like the governing equation for the lookback option price function, $M$ only enters into the auxiliary conditions but not the governing differential equation. Both $V_{\text{mid}}(F, I, M, t)$ and $V_{\infty}(F, I, t)$ satisfy the same two-state Black-Scholes equation, namely,

$$
\left( \frac{\partial}{\partial t} + \mathcal{L}_{F,I} \right) V_{\text{mid}}(F, I, M, t) = 0 \quad \text{and} \quad \left( \frac{\partial}{\partial t} + \mathcal{L}_{F,I} \right) V_{\infty}(F, I, t) = 0.
$$

(A.1)

where

$$
\mathcal{L}_{F,I} = \frac{\sigma_p^2}{2} F^2 \frac{\partial^2}{\partial F^2} + \rho \sigma_p \sigma_i F I \frac{\partial^2}{\partial F \partial I} + \frac{\sigma_i^2}{2} I^2 \frac{\partial^2}{\partial I^2} + (r - q_p) F \frac{\partial}{\partial F} + (r - q_i) I \frac{\partial}{\partial I} - r.
$$
Terminal payoff

The terminal payoff of the investment fund value at maturity $T$ is given by $F \max(M, 1)$, a payoff structure that involves both $F$ and $M$. The valuation of the mid-contract value may seem to be quite involved, but economic intuition gives the mid-contract value $V_{\text{mid}}(F, I, M, t)$ in terms of the grant-date value $V_{\infty}(F, I, t)$.

Relationship between $V_{\text{mid}}(F, I, M, t)$ and $V_{\infty}(F, I, t)$

When $M > 1$, the number of units of primary fund is increased to $M$ so that the investment fund is equivalent to one unit of “new” primary fund having fund value $MF$. When $M \leq 1$, $V_{\text{mid}}$ is insensitive to $M$ since the terminal payoff value will not be dependent on the current realized maximum value $M$. That is, $V_{\text{mid}}$ remains constant at different values of $M$, for all $M \leq 1$. By continuity of the price function with respect to the variable $M$, $V_{\text{mid}}$ at $M = 1$ is equal to the limiting value of $V_{\text{mid}}$ (corresponding to the regime: $M > 1$) as $M \to 1^+$. 
Mid-contract value function

\[
V_{\text{mid}}(F, I, M, t) = V_{\infty}(\max(M, 1)F, I, t)
\]

\[
= \begin{cases} 
V_{\infty}(F, I, t) & M \leq 1, \\
V_{\infty}(MF, I, t) & M > 1. 
\end{cases}
\]  \hspace{1cm} (A.2)

• For \( M \leq 1 \), \( V_{\text{mid}}(F, I, M, t) = V_{\infty}(F, I, t) \) is obvious since there has been no upgrading of fund value occurs. It is only necessary to show

\[
V_{\text{mid}}(F, I, M, t) = V_{\infty}(MF, I, t) \text{ for } M > 1.
\]
Proof of the formula

When the current index value $I$ equals $MF$, then $V_{\text{mid}}$ is insensitive to $M$. We impose the usual auxiliary condition for a lookback option:

$$\frac{\partial V_{\text{mid}}}{\partial M} \bigg|_{M=I/F} = 0.$$ 

One can check that $V_{\text{mid}}(F, I, M, t)$ given in Eq.(A.2) satisfies the governing differential equation (A.1) since the multiplier $M$ is canceled in the differential equation when $F$ is multiplied by $M$. It suffices to show that $V_{\text{mid}}(F, I, M, t)$ satisfies the terminal payoff condition and the above auxiliary condition.

Firstly, when $M > 1$, it is guaranteed that $M$ at maturity must be greater than one so the terminal payoff becomes $F \max(1, M) = MF$. Since $V_\infty(F, I, T) = F$ so that $V_\infty(MF, I, T) = MF$, hence the terminal payoff condition is satisfied.
Recall: \( W_\infty \left( \ln \frac{I}{F}, \tau \right) = \frac{V_\infty (F, I, t)}{F} \) so that

\[
V_\infty (MF, I, t) = MF W_\infty \left( \ln \frac{I}{MF}, \tau \right).
\]

We perform differentiation with respect to \( M \) to obtain

\[
\frac{\partial V_\infty}{\partial M} (MF, I, t) = F \left[ W_\infty (y, \tau) - \frac{\partial W_\infty}{\partial y} (y, \tau) \right],
\]

where \( y = \ln(I/MF) \). When \( M = I/F \), we have \( y = 0 \) so that

\[
\left. \frac{\partial V_{\text{mid}}}{\partial M} (F, I, M, t) \right|_{M=I/F} = \left. \frac{\partial V_\infty}{\partial M} (MF, I, t) \right|_{M=I/F}
\]

\[
= F \left[ W_\infty (0, \tau) - \frac{\partial W_\infty}{\partial y} (0, \tau) \right] = 0
\]

by virtue of the Robin boundary condition.
Cost to the sponsor

\[ U_{\text{grant}}(F, I, t) = \text{cost to the sponsor at the grant date} \]

\[ U_{\text{mid}}(F, I, M, t) = \text{cost to the sponsor at mid-contract time} \]

Note that the terminal payoff \( U_{\text{mid}}(F, I, M, T) \) is given by

\[ U_{\text{mid}}(F, I, M, T) = V_{\text{mid}}(F, I, M, T) - F = \max(M - 1, 0)F. \]

Relation between \( U_{\text{mid}} \) and \( V_{\text{mid}} \)

\[ U_{\text{mid}}(F, I, M, t) = V_{\text{mid}}(F, I, M, t) - Fe^{-q_p(T-t)} \]

since both \( V_{\text{mid}}(F, I, M, t) \) and \( Fe^{-q_p(T-t)} \) satisfy the Black-Scholes equation and the terminal payoff condition.

The factor \( e^{-q_p(T-t)} \) appears in front of \( F \) since the holder of the protected fund does not receive the dividends paid by the primary fund.
At the grant-date, we have $F \geq I$ so that $M = I/F \leq 1$. We then have

$$U_{grant}(F, I, t) = U_{mid}(F, I, M, t) = V_\infty(F, I, t) - Fe^{-qp(T-t)}.$$  

The two cost functions $U_{mid}(F, I, M, t)$ and $U_{grant}(F, I, t)$ are related by

$$U_{mid}(F, I, M, t) = U_{grant}(\max(M, 1)F, I, t) + \max(M - 1, 0)Fe^{-qp(T-t)}.$$  

The last term in the above equation gives the present value of additional units of primary fund supplied by the sponsor due to the protection clause. The sponsor has to add $M - 1$ units of primary fund when $M > 1$, but supplements nothing when $M \leq 1$. 
Integral representation of the price function $V_{mid}(F, I, M, t)$

Let $\widehat{M}_t^{t'} = \max \left( \max_{t \leq u \leq t'} \frac{I(u)}{F(u)}, 1 \right)$

and at the current time $t$, the following quantity

$$\widehat{M}_0^{t} = \max \left( \max_{0 \leq u \leq t} \frac{I(u)}{F(u)}, 1 \right) = \max(M, 1)$$

is known. The terminal payoff of the protected fund can be expressed as $\max(\widehat{M}_0^{t}, \widehat{M}_t^{T'})F_T$, and $F$ is the current value of the primary fund.
Rollover hedging strategy

We increase the number of units of fund to $\hat{M}_t^u e^{-q_p(T-u)}$ whenever a higher realized maximum value of $\hat{M}_t^u$ occurs at time $u$, where $t \leq u \leq T$. This rollover strategy would guarantee that the number of units of fund at maturity $T$ is $\max(\hat{M}_t^0, \hat{M}_T^T)$. The corresponding present value of the replenishment premium is

$$e^{-q_p(T-t)} F E[\max(\hat{M}_T^T - \hat{M}_t^0, 0)] = e^{-q_p(T-t)} F \int_{\hat{M}_0^t}^{\infty} P[\hat{M}_T^T \geq \xi] \, d\xi.$$ 

The value of the protected fund is the sum of the value of the sub-replicating portfolio and the replenishment premium. An alternative analytic representation of the mid-contract price function is given by

$$V_{\text{mid}}(F, I, M, t) = \max(M, 1)e^{-q_p(T-t)} F + e^{-q_p(T-t)} F \int_{\max(M,1)}^{\infty} P[\hat{M}_T^T \geq \xi] \, d\xi.$$
Note that $\max(M, 1)$ is the number of units of primary fund to be held at time $t$, given the current value of $M$. In the remaining life of the contract, the holder is not entitled to receive the dividend yield, so the discount factor $e^{-qp(T-t)}$ should be appended.

**Distribution function**

For $\xi \geq 1$, we have the following distribution function under the Geometric Brownian motion:

\[
P[\widetilde{M}_t^T \geq \xi] = P \left[ \max_{t \leq u \leq T} \frac{I(u)}{F(u)} \geq \xi \right] = e^{2\mu \xi/\sigma^2} N \left( -\frac{\xi + \mu \tau}{\sigma \sqrt{\tau}} \right) + N \left( -\frac{\xi - \mu \tau}{\sigma \sqrt{\tau}} \right),
\]

where $\mu = qp - qi - \frac{\sigma^2}{2}$ and $\sigma^2 = \sigma_p^2 - 2\rho \sigma_p \sigma_i + \sigma_i^2$. 