1. Consider the function

\[ f(S, \tau) = \left( \frac{S}{B} \right)^{\lambda} c_E \left( \frac{B^2}{S}, \tau \right), \]

where \( c_E(S, \tau) \) is the price of a vanilla European call option, \( \lambda \) is a constant parameter. Show that \( f(S, \tau) \) satisfies the Black-Scholes equation

\[ \frac{\partial f}{\partial \tau} = \frac{\sigma^2}{2} S^2 \frac{\partial^2 f}{\partial S^2} + r S \frac{\partial f}{\partial S} - rf \]

when \( \lambda \) is chosen to be \( -\frac{2r}{\sigma^2} + 1 \).

*Hint:* Substitution of \( f(S, \tau) \) into the Black-Scholes equation gives

\[ \frac{\partial f}{\partial \tau} \left[ \frac{\sigma^2}{2} S^2 \frac{\partial^2 f}{\partial S^2} + r S \frac{\partial f}{\partial S} - rf \right] = \left( \frac{S}{B} \right)^{\lambda} \left[ \frac{\partial c_E}{\partial \tau} - \frac{\sigma^2}{2} \frac{\partial^2 c_E}{\partial \xi^2} \right. \]

\[ + (\lambda - 1)\sigma^2 \frac{\partial c_E}{\partial \xi} - \lambda(\lambda - 1)\frac{\sigma^2}{2} c_E - r \lambda c_E + r \xi \frac{\partial c_E}{\partial \xi} + rc_E \right], \]

where \( c_E = c_E(\xi, \tau), \xi = B^2 / S \).

2. By applying the following transformation on the dependent variable \( c \) in the Black-Scholes equation

\[ c = e^{\alpha y + \beta \tau} w, \]

where \( \alpha = \frac{1}{2} - \frac{r}{\sigma^2}, \beta = \frac{-\alpha^2 \sigma^2}{2} - r \), show that the convective diffusion equation

\[ \frac{\partial c}{\partial \tau} = \frac{\sigma^2}{2} \frac{\partial^2 c}{\partial y^2} + \left( r - \frac{\sigma^2}{2} \right) \frac{\partial c}{\partial y} - rc \]

is reduced to the prototype diffusion equation

\[ \frac{\partial w}{\partial \tau} = \frac{\sigma^2}{2} \frac{\partial^2 w}{\partial y^2}, \]

while the auxiliary conditions are transformed to become

\[ w(0, \tau) = e^{-\beta \tau} R(\tau) \text{ and } w(y, 0) = \max(e^{\alpha y}(e^y - X), 0). \]

Consider the following diffusion equation defined in a semi-infinite domain

\[ \frac{\partial v}{\partial t} = a^2 \frac{\partial^2 v}{\partial x^2}, \quad x > 0 \text{ and } t > 0, \quad a \text{ is a positive constant}, \]
with initial condition: \( v(x, 0) = f(x) \) and boundary condition: \( v(0, t) = g(t) \), the solution to the diffusion equation is given by

\[
v(x, t) = \frac{1}{2a\sqrt{\pi t}} \int_{0}^{\infty} f(\xi) \left[ e^{-(x-\xi)^2/4a^2t} - e^{-(x+\xi)^2/4a^2t} \right] d\xi + \frac{x}{2a\sqrt{\pi}} \int_{0}^{t} \frac{e^{-x^2/4a^2\xi}}{\omega^{3/2}} \left[ g(\xi) - g(t) \right] d\xi.
\]

Using the above form of solution, show that the price of the European down-and-out call option is given by

\[
c(y, \tau) = e^{\alpha y + \beta \tau} \left\{ \frac{1}{\sqrt{2\pi \tau}} \int_{0}^{\infty} \max(e^{-\alpha \xi}(e^{\xi} - X), 0) \left[ e^{-(y-x)^2/2\sigma^2\tau} - e^{-(y+x)^2/2\sigma^2\tau} \right] d\xi + \frac{y}{\sqrt{2\pi \sigma}} \int_{0}^{\tau} e^{-\beta(\tau-\omega)} e^{-y^2/2\sigma^2\omega} \frac{R(\tau - \omega)}{\omega^{3/2}} d\omega \right\}.
\]

Assuming \( B < X \), show that the price of the European down-and-out call option is given by

\[
c(S, \tau) = c_{E}(S, \tau) - \left( \frac{B}{S} \right)^{\delta-1} c_{E} \left( \frac{B^2}{S^2}, \tau \right)
+ \int_{0}^{\tau} e^{-r\omega} \frac{\ln \frac{S}{B} + (r - \frac{\mu^2}{2})\omega}{\sqrt{2\pi \sigma^2 \omega}} R(\tau - \omega) d\omega.
\]

The last term represents the additional option premium due to the rebate payment.

3. Let the exit time density \( q^+(t; x_0, t_0) \) to the upper barrier \( \ell \) have dependence on the initial state \( X(t_0) = x_0 \), \( 0 < x_0 < \ell \). We write \( \tau = t - t_0 \) so that \( q^+(t; x_0, t_0) \) is visualized as \( q^+(x_0, \tau) \). Show that the partial differential equation formulation is given by

\[
\frac{\partial q^+}{\partial \tau} = \mu \frac{\partial q^+}{\partial x_0} + \frac{\sigma^2}{2} \frac{\partial^2 q^+}{\partial x_0^2}, \quad 0 < x_0 < \ell, \quad \tau > 0,
\]

with auxiliary conditions:

\[
q^+(0, \tau) = 0, \quad q^+(\ell, \tau) = \delta(\tau) \quad \text{and} \quad q^+(x_0, 0) = \delta(\ell - x_0).
\]

By solving the above partial differential equation, show that

\[
q^+(t; x_0, t_0) = e^{\frac{\mu t}{\ell^2} - x_0} \frac{\sigma^2}{\ell^2} \sum_{k=1}^{\infty} e^{-\lambda_k(t - t_0)} k\pi \sin \frac{k\pi(\ell - x_0)}{\ell},
\]

where

\[
\lambda_k = \frac{1}{2} \left( \frac{\mu^2}{\sigma^2} + \frac{k^2\pi^2\sigma^2}{\ell^2} \right), \quad k = 1, 2, \ldots.
\]
4. Suppose the dynamics of the logarithm of the stock price $S_t$ is governed by

$$d \ln S_t = \left( r - \frac{\sigma^2}{2} \right) dt + \sigma dW_t,$$

the density function of $\ln S_T$ conditional on $\ln S_0$ at time 0 is given by

$$f(S_T; S_0) = \frac{1}{\sqrt{2\pi \sigma^2 T}} \exp\left( -\frac{\left( \frac{S_T}{S_0} - \lambda T \right)^2}{2\sigma^2 T} \right), \quad \lambda = r - \frac{\sigma^2}{2}.$$

(a) Let $\tau_L$ denote the first passage time of the stock price to the lower barrier $L$, where $L < S_0$. Using the reflection principle, show that

$$f(S_T; S_0 | \tau_L < T) = f \left( S_T; L \frac{2}{S_0} \right) \left( \frac{L}{S_0} \right)^{2\lambda/\sigma^2}.$$

(b) Similarly, let $\tau_U$ denote the first passage time of the stock price to the upper barrier $U$, where $S_0 < U$. Show that

$$f(S_T; S_0 | \tau_U < T) = f \left( S_T; U \frac{2}{S_0} \right) \left( \frac{U}{S_0} \right)^{2\lambda/\sigma^2}.$$

(c) Let $\tau_{U/L}$ ($\tau_{L/U}$) be the first time that the stock price process hits the upper barrier $U$ (lower barrier $L$) after hitting the lower barrier $L$ (upper barrier $U$). That is,

$$\tau_{U/L} = \inf \{ t | S(t) = U, t > \tau_L \}$$

$$\tau_{L/U} = \inf \{ t | S(t) = L, t > \tau_U \}.$$

Show that

$$f_{U/L}(S_T; S_0) = f \left( S_T; S_0 \left( \frac{U}{L} \right)^2 \right) \left( \frac{U}{L} \right)^{2\lambda/\sigma^2}$$

$$f_{L/U}(S_T; S_0) = f \left( S_T; S_0 \left( \frac{L}{U} \right)^2 \right) \left( \frac{L}{U} \right)^{2\lambda/\sigma^2}.$$
Double reflection of a sample stock price path

(d) Use the density function $f_{L/U}(S_T; S_0)$ to find the price formula of the call option with strike price $X$, where $L < X < U$, and subject to knock-out upon sequential breaching of up-barrier $U$ first and down-barrier $L$ afterwards.

(e) Lastly, deduce that the density function of $\ln S_T$, conditional on the stock price hitting neither the lower barrier $L$ nor the upper barrier $U$ before time $T$, is given by

$$f(S_T; S_0|\min(\tau_L, \tau_U) > T) = \sum_{n=-\infty}^{\infty} f\left(S_T; S_0 \left(\frac{U}{L}\right)^{2n}\right) \left(\frac{U}{L}\right)^{2n\lambda/\sigma^2} - f\left(S_T; \frac{U^2}{S_0} \left(\frac{U}{L}\right)^{2n}\right) \left[\left(\frac{U}{S_0}\right) \left(\frac{U}{L}\right)^n\right]^{2\lambda/\sigma^2}. $$