1. As an alternative approach to derive the value of a European floating strike lookback call, we consider

\[ c_f(S, m, \tau) = e^{-r\tau} E_Q[S_T - \min(m, m_T^\tau)] \]

\[ = S - e^{-r\tau} E_Q[\min(m, m_T^\tau)], \]

where \( S_t = S, m_{T_0}^t = m \) and \( \tau = T - t \). We may decompose the above expectation calculation into two terms:

\[ E_Q[\min(m, m_T^\tau)] = m P(\ln \frac{m_T^\tau}{S} \geq \ln \frac{m}{S}) \]

\[ = m \left[ N \left( -\frac{\ln \frac{m}{S} + \mu \tau}{\sigma \sqrt{\tau}} \right) - \left( \frac{S}{m} \right)^{1 - \frac{2y}{\sigma^2}} N \left( \frac{\ln \frac{m}{S} + \mu \tau}{\sigma \sqrt{\tau}} \right) \right] . \]

The second term can be expressed as

\[ E_Q[m_T^\tau 1_{\{m_T^\tau > m\}}] = \int_{-\infty}^{\ln \frac{m_T^\tau}{S}} S e^y f_{\min}(y) \, dy. \]

(a) Show that the density function of \( y_T = \ln \frac{m_T^\tau}{S} \) is given by

\[ f_{\min}(y) = \frac{1}{\sigma \sqrt{\tau}} e^{\frac{2y \mu}{\sigma^2}} N \left( \frac{y + \mu \tau}{\sigma \sqrt{\tau}} \right) + \frac{2e^{\frac{2y \mu}{\sigma^2}}}{\sigma^2 \tau} N \left( \frac{y + \mu \tau}{\sigma \sqrt{\tau}} \right). \]

(b) Find the price function \( c_f(S, m, \tau) \) of the European floating strike lookback call option.

2. Using the following form of the distribution function of \( m_T^\tau \)

\[ P(m \leq m_T^\tau) = N \left( -\frac{\ln \frac{m}{S} + \mu \tau}{\sigma \sqrt{\tau}} \right) - \left( \frac{S}{m} \right)^{1 - \frac{2y}{\sigma^2}} N \left( \frac{\ln \frac{m}{S} + \mu \tau}{\sigma \sqrt{\tau}} \right) , \]

show that \( P(m \leq m_T^\tau) \) becomes zero when \( S = m \).

3. Suppose we use a straddle (combination of a call and a put with the same strike \( m \)) in the rollover strategy for hedging the floating strike lookback call and write

\[ c_f(S, m, \tau) = c_E(S, \tau; m) + p_E(S, \tau; m) + \text{strike bonus premium}. \]

Find an integral representation of the strike bonus premium in terms of the distribution functions of \( S_T \) and \( m_T^\tau \). Compare the strike bonus premium as given by the following alternative representation:

\[ \text{strike bonus premium} = e^{-r\tau} \int_0^m P(m_T^\tau \leq \xi \leq S_T) \, d\xi. \]
4. Prove the following put-call parity relation between the prices of the fixed strike lookback call and floating strike lookback put:

\[ c_{fix}(S, M, \tau; X) = p_{f\ell}(S, \max(M, X), \tau) + S - X e^{-\tau r}. \]

Deduce that

\[ \frac{\partial c_{fix}}{\partial M} = 0 \quad \text{for} \quad M < X. \]

Give a financial interpretation why \( c_{fix} \) is insensitive to \( M \) when \( M < X \).

5. Derive the following partial differential equation for the floating strike lookback put option

\[ \frac{\partial V}{\partial \tau} = \frac{\sigma^2}{2} \frac{\partial^2 V}{\partial \xi^2} - \left( r + \frac{\sigma^2}{2} \right) \frac{\partial V}{\partial \xi}, \quad 0 < \xi < \infty, \tau > 0, \]

where \( V(\xi, \tau) = p_{f\ell}(S, M, t)/S \) and \( \tau = T - t, \xi = \ln \frac{M}{S} \). The auxiliary conditions are

\[ V(\xi, 0) = e^\xi - 1 \quad \text{and} \quad \frac{\partial V}{\partial \xi}(0, \tau) = 0. \]

Solve the above Neumann boundary value problem to obtain the put price formula.

**Hint:** Define \( W = \frac{\partial V}{\partial \xi} \) so that \( W \) satisfies the same governing differential equation but the boundary condition becomes \( W(0, \tau) = 0 \). Solve for \( W(\xi, \tau) \), then integrate \( W \) with respect to \( \xi \) to obtain \( V \). Be aware that an arbitrary function \( \phi(t) \) is generated upon integration with respect to \( \xi \). Obtain an ordinary differential equation for \( \phi(t) \) by substituting the solution for \( V \) into the original differential equation.