MATH304, Spring 2006

Solution to Test Two

1. (a) $z = \tanh^{-1} 2$ i.e. $\tanh z = 2$

$$\Rightarrow \quad \frac{e^z - e^{-z}}{e^z + e^{-z}} = 2 \quad \text{i.e.} \quad e^{2z} + 3 = 0$$

$$\Rightarrow \quad e^{2z} = -3 \Rightarrow z = \frac{1}{2}\log(-3) = \frac{1}{2}[\ln 3 + i(2k+1)\pi],\ k \text{ is any integer.}$$

(b) $z = i^i = e^{\log(i^i)} = e^{i \log i}$

$$= e^{i\left[i\left(2k\pi + \frac{\pi}{2}\right)\right]} = e^{-\left(2k\pi + \frac{\pi}{2}\right)},\ k \text{ is any integer.}$$

2. (a) The Riemann surface of $= z^{1/3}$ consists of 3 *sheets* superimposed over each other. They are joined together along the *branch cut* taken to be on the negative real axis. The *branch points* are $z = 0$ and $z = \infty$.

(b) $i = e^{i\pi/2} = e^{i(\pi/2+2\pi)} = e^{i(\pi/2+4\pi)}$. Hence, $i^{1/3} = \begin{cases} e^{i\pi/6} \\ e^{i5\pi/6} \\ e^{i3\pi/2} = -i \end{cases}$.

(c) $-2 = 2e^{i(\pi+2\pi)}$ if the branch $f(i) = e^{(5\pi/6)i}$ is taken. We then have $f(-2) = \sqrt[3]{2}e^{i3\pi/3} = -\sqrt[3]{2}$.

3. Since the length of $C_1(0)$ is 2π, it follows that $\left|\oint_{C_1(0)} e^{\frac{1}{z}}\, dz\right| \le 2\pi M$, where M is an upper bound of $\left|e^{\frac{1}{z}}\right|$ for z on the unit circle. For z on $C_1(0)$, write $z = e^{it}$ where $0 \le t \le 2\pi$; then

$$\frac{1}{z} = e^{-it} = \cos t - i \sin t,$$

and so

$$\left|e^{\frac{1}{z}}\right| = \left|e^{\cos t - i\sin t}\right| = \left|e^{\cos t}\right| \overbrace{\left|e^{-i\sin t}\right|}^{=1} = e^{\cos t} \le e^1 = e.$$

Thus $\left|\int_{C_1(0)} e^{\frac{1}{z}}\, dz\right| \le 2\pi e.$

Remark

We can evaluate the integral and obtain $\oint_{C_1(0)} e^{\frac{1}{z}}\, dz = 2\pi i$. Thus the bound that we obtained is correct but not best possible, since $|2\pi i| = 2\pi$ is the best upper bound.

4. Since the principal branch of $z^{1/2}$ is analytic in the complex plane excluding the origin and the negative real axis, the integral is path independent. We have

$$\begin{aligned} \int_{-i}^{i} z^{1/2}\, dz &= \left.\frac{2}{3} z^{3/2}\right]_{-i}^{i} \\ &= \frac{2}{3}\left[e^{i(\pi/2)(3/2)} - e^{-i(\pi/2)(3/2)}\right] \\ &= \frac{2}{3}\left[e^{i(3\pi/4)} - e^{-i(3\pi/4)}\right] = \frac{2\sqrt{2}}{3} i. \end{aligned}$$

5. Consider

$$\begin{aligned}
&\frac{1}{2\pi i}\oint_{|z|=1}\left[2\pm\left(z+\frac{1}{z}\right)\right]\frac{f(z)}{z}\,dz\\
=\;&\frac{2}{2\pi i}\oint_{|z|=1}\frac{f(z)}{z}\,dz\pm\frac{1}{2\pi i}\oint_{|z|=1}f(z)\,dz\pm\frac{1}{2\pi i}\oint_{|z|=1}\frac{f(z)}{z^2}\,dz\\
=\;&2f(0)\pm f'(0)=2\pm f'(0).
\end{aligned}$$

Using the polar coordinates: $z=e^{i\theta}, dz=ie^{i\theta}\,d\theta$

$$\begin{aligned}
&\frac{1}{2\pi i}\oint_{|z|=1}\left[2\pm\left(z+\frac{1}{z}\right)\right]\frac{f(z)}{z}\,dz\\
=\;&\frac{1}{2\pi}\int_0^{2\pi}f(e^{i\theta})(2\pm 2\cos\theta)\,d\theta.
\end{aligned}$$

Since $1+\cos\theta=2\cos^2\dfrac{\theta}{2}$ and $1-\cos\theta=2\sin^2\dfrac{\theta}{2}$, we have

$$\begin{aligned}
\frac{2}{\pi}\int_0^{2\pi}f(e^{i\theta})\cos^2\frac{\theta}{2}\,d\theta&=2+f'(0)\\
\frac{2}{\pi}\int_0^{2\pi}f(e^{i\theta})\sin^2\frac{\theta}{2}\,d\theta&=2-f'(0).
\end{aligned}$$