Pricing Exotic Variance Swaps under 3/2-Stochastic Volatility Models

CHI HUNG YUEN & YUE KUEN KWOK

Department of Mathematics, Hong Kong University of Science and Technology, Hong Kong

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Abstract
We consider pricing of various types of exotic discrete variance swaps, like the gamma swaps and corridor swaps, under the 3/2-stochastic volatility models with jumps. The class of stochastic volatility models (SVM) that use a constant-elasticity-of-variance (CEV) process for the instantaneous variance exhibit nice analytical tractability when the CEV parameter takes just a few special values (namely, 0, 1/2, 1 and 3/2). The popular Heston model corresponds to the choice of the CEV parameter to be 1/2. However, the stochastic volatility dynamics derived from the Heston model fails to agree with empirical findings from actual market data. The choice of 3/2 for the CEV parameter in the SVM shows better agreement with empirical studies while it maintains a good level of analytical tractability. By using the partial integro-differential equation formulation, we manage to derive quasi-closed form pricing formulas for the fair strike values of various types of discrete variance swaps. Pricing properties of these exotic discrete variance swaps under different market conditions are explored.

Keywords: Variance swaps, gamma swaps, corridor swaps, 3/2-volatility model

1 Introduction

Variance and volatility derivatives become more popular in the financial market since their introduction in the late nineties. Since volatility is likely to grow when uncertainty and risk increase, hedge funds and private investors may use these derivatives to manage their exposure to the volatility risk associated with their trading positions. Speculators can also place their bids on the future movement of the underlying volatility via trading on these instruments. Stock options are less than ideal as the instruments to provide exposure to volatility since they have exposure to both volatility and direction of the stock price movement. One may argue that exposure to the stock price in an option can be hedged. However, delta hedging is at best inaccurate since volatility cannot be estimated accurately. On the other hand, writers of variance swaps can hedge their positions via replication by using a portfolio of options traded in the markets. The provision of pure exposure of volatility and effective replication by traded options provide the impetus for the growth of the markets for swaps and other derivatives on discrete realized variance. Reader may refer to Carr and Madan (1998) for an introduction to the theory of volatility trading.

Variance swaps are essentially forward contracts on discrete realized variance. In recent years, other variants of variance swaps that target more specific features of variance exposure, like the gamma swaps and corridor swaps (commonly called the third generation variance swaps), are also structured in the financial markets. The product specifications of these

1Correspondence author; email: maykwok@ust.hk; fax number: 852-2358-1643
variance swap products will be presented in Section 3. The potential uses in hedging and betting the various forms of volatility exposure can be found in numerous articles (Demeterfi et al., 1999; Carr and Lee, 2009; Bouzoubaa and Osseiran, 2010). The pricing of exotic discrete variance swaps under time-changed Lévy processes and stochastic volatility models have been studied by Itkin and Carr (2010), Crosby and Davis (2011) and Zhu and Lian (2011). Zheng and Kwok (2014) study the pricing of highly path dependent corridor and conditional swaps on discrete realized variance under the Heston stochastic model (Heston, 1993) with simultaneous jumps in the asset price and variance.

Itkin (2013) considers the class of stochastic volatility models (SVM) whose instantaneous variance is modeled by a constant-elasticity-of-variance (CEV) process. The number of analytically tractable models is rather limited, where the CEV parameter $\gamma$ takes only 4 specific values: 0, 1/2, 1 and 3/2. The Heston model (square root process) corresponds to the choice of $\gamma = 1/2$, and it enjoys the best analytical tractability. Indeed, the Heston model belongs to the class of affine models and the methodologies for finding the corresponding joint moment generating functions are very well developed (Duffie et al., 2000). These nice analytical tractability properties lead to successful derivation of succinct closed form pricing formulas for discrete variance swaps (Zheng and Kwok, 2014). Unfortunately, the Heston model has been shown to be inconsistent with observations in the variance markets. It leads to downward sloping volatility of variance smiles, contradicting with empirical findings from market data. On the other hand, the 3/2-model (choice of $\gamma = 3/2$) exhibits better agreement with empirical studies while maintains some level of analytical tractability. For example, based on S&P 100 implied volatilities, Jones (2003) and Bakshi et al. (2006) estimate that $\gamma$ should be around 1.3, which is close to 3/2 over 1/2 (Heston model). In addition, Jones concludes that jumps are needed in the underlying process for short maturity options. In the real world statistical measure, Javaheri (2004) analyzes the CEV type instantaneous variance process with the exponent of the volatility of variance process either be 1/2, 1 or 3/2 by using the time series data on S&P 500 daily returns and finds that the 3/2-power performs the best. Ishida and Engle (2002) estimate the power to be 1.71 for S&P 500 daily return for a 30-year period. Chacko and Viceira (2003) employ the technique of the Generalized Method of Moments on a 35-year period of weekly return and a 71-year period of monthly return. They estimate the power to be 1.10 and 1.65, respectively, over the two periods. By using S&P 500 index options over a period of 7 years, Poteshman (1998) concludes that the drift of the instantaneous variance is not of affine structure (like the Heston model).

There have been several recent works that use the 3/2-model for pricing variance and volatility derivatives. Goard and Mazur (2013) and Drimus (2012) compute the fair strike values for the continuously monitored variance and volatility derivatives under the 3/2-model without jumps. Instead of modeling the dynamics of the instantaneous volatility directly, Carr and Sun (2007) adopt a new approach by assuming continuous dynamics for the time-$T$ variance swap rate, taking advantage of the liquidity of the variance swap market. They manage to derive the analytic closed form formula for the joint conditional Fourier-Laplace transform of the terminal log-asset value and its quadratic variation for the 3/2-model. Chan and Platen (2012) derive exact pricing and hedging formulas of continuously monitored long dated variance swaps under the 3/2-model using the benchmark approach, a pricing concept that provides minimal fair strike values for variance swaps when an equivalent risk neutral probability measure does not exist. Itkin and Carr (2010) derive closed form pricing formulas for discrete variance swaps and options on quadratic variation under a class of time-changed Lévy models, which includes the 3/2-power clock change.

In this paper, we consider pricing of various types of discrete variance swaps, including the gamma swaps and corridor swaps, under the 3/2-stochastic volatility model with jumps. As a non-affine stochastic volatility model, the level of analytic tractability of the 3/2-model...
is lower than that of the affine Heston model. Since each term in the accumulated squared returns involves asset prices monitored at two successive monitoring instants, the analytic procedure requires a two-step solution of coupled partial integro-differential equations. A similar approach has been used by Rujivan and Zhu (2012) for pricing discrete variance swaps under the Heston stochastic volatility model. The resulting quasi-closed form formulas for the fair strikes of the generalized variance swaps are expressed in the form of integrals, the numerical valuation of them can be performed using standard quadrature to sufficiently high level of accuracy.

The later sections of this paper are organized as follows. In the next section, we present the model specification of the 3/2-stochastic volatility model with jumps. We then derive the partial integro-differential equations for pricing contingent claims with payoffs that are dependent on discrete realized variance. We illustrate how to use several skillful integral transform techniques to find the fundamental solutions of the partial integro-differential equations. In Section 3, we consider analytic pricing of the third generation swaps: gamma swaps and corridor swaps. The solution procedures involve an elegant combination of techniques developed by various earlier papers on pricing similar discrete swap products under the Heston model (Carr and Sun, 2007; Rujivan and Zhu, 2012; Zheng and Kwok, 2014). In particular, we adopt a two-step procedure that computes the expected discrete variance based on nested conditional expectation over two successive time intervals. In Section 4, we present the results of our numerical tests that were performed to illustrate the effective numerical valuation of the quasi-closed form pricing formulas of the variance swaps. Also, we performed detailed analysis on the pricing properties of the variance swaps with respect to different sets of parameters, like the correlation between asset price and its instantaneous variance, sampling frequency, volatility of variance, jump distribution. The conclusive remarks and summary of findings are presented in the last section.

2 3/2-stochastic volatility models and pricing formulation of variance derivatives

In this section, we first present the class of stochastic volatility models (SVM) that use a constant-elasticity-of-variance (CEV) process for the instantaneous variance and discuss briefly the analytical tractability of these SVM under various choices of the CEV parameter. We then provide the justification of the choice of 3/2-model as the underlying asset price process based on empirical studies. We also state a useful formula for the joint conditional Fourier-Laplace transform of the terminal asset price and their quadratic variation for the 3/2-model with jumps. Next, we derive the partial integro-differential equation formulation for pricing a contingent claim with payoff that is dependent on squared asset return. Since the squared asset return involves asset prices $S_{t_{i-1}}$ and $S_t$ at successive monitoring instants $t_{i-1}$ and $t_i$, and the fair strike value of the variance swap at $t_0$ is to be determined, the backward induction procedure involves a two-step solution over the successive time intervals $[t_{i-1}, t_i]$ and $[t_0, t_{i-1}]$. We show how to apply the appropriate jump condition at the time instant $t_{i-1}$. Once the fundamental solutions to the partial integro-differential equations are obtained, as an illustration of the pricing procedure, we show how to find the fair strike of a vanilla variance swap under the 3/2-model with jumps.

2.1 3/2-models with jumps

For pricing volatility products and derivatives on discrete realized variance, the first step is to identify an appropriate stochastic volatility model for the underlying price process. Goard
and Mazur (2013) consider the performance of the class of constant-elasticity-of-variance (CEV) process for the instantaneous variance \( v_t \) and examine their ability to capture the behavior of VIX, where the dynamics of \( v_t \) is governed by

\[
dv_t = \left(c_1 + \frac{c_2}{v_t} + c_3 v_t \ln v_t + c_4 v_t + c_5 v_t^2\right) dt + \epsilon v_t^{3/2} dZ_t. \tag{2.1}
\]

Here, the CEV parameter \( \gamma \), volatility of variance \( \epsilon \) and other parameters in the drift term are taken to be constants. To achieve analytic tractability, Itkin (2013) identifies the choice of \( \gamma \) to be limited to four values: 0, 1/2, 1 and 3/2. The choice of \( \gamma = 1/2 \) corresponds to the renowned Heston model, which has the highest level of analytical tractability due to its affine structure. Recently, various empirical studies on the variance exponent \( \gamma \) have shown that the 3/2-model with either linear drift or quadratic drift are the only acceptable models among the class of stochastic volatility models specified in eq. (2.1) for characterizing the VIX dynamics. Baldeaux and Badran (2012) demonstrate that the 3/2-model is capable to produce the market observed upward-sloping implied volatility for VIX options. They also show that jumps should be introduced in the asset price process in order to capture the volatility smile for short-maturity options. Fortunately, the resulting 3/2-model with jumps remains to be tractable, like its no-jump counterpart. In this paper, based on the above observations, we choose the 3/2-model with jumps as the underlying price process.

We assume that the dynamics of the stock price \( S_t \) and its instantaneous variance \( v_t \) under a risk-neutral measure \( Q \) is governed by

\[
\frac{dS_t}{S_t} = (r - d - \lambda m) dt + \sqrt{v_t} dW_t^1 + (e^J - 1) dN_t, \\
\]  
\[
\frac{dv_t}{v_t} = v_t[p(t) - qv_t] dt + \epsilon v_t^{3/2} dW_t^2, \tag{2.2}
\]

where \( r \) is the risk-free interest rate, \( d \) is the dividend yield, \( W_t^1 \) and \( W_t^2 \) are standard Brownian motions with \( dW_t^1 dW_t^2 = \rho dt \). Also, \( N_t \) is a compound Poisson process with constant arrival rate \( \lambda \) and it is independent of \( W_t^1 \) and \( W_t^2 \). For the jump process, \( J \) denotes the random jump size of the log stock price. It has a normal distribution with mean \( \nu \) and variance \( \zeta^2 \). Also, it is independent to the two Brownian motions and the Poisson process \( N_t \). The compensator parameter \( m \) is given by \( \mathbb{E}^Q[e^J - 1] = e^{\nu + \zeta^2/2} - 1 \). The parameters \( q \) in the drift term and the correlation coefficient \( \rho \) are constant. We may generalize the level parameter \( p(t) \) in the dynamics of \( v_t \) to be a deterministic continuous function of time that allows for flexibility for calibration.

The 3/2-dynamics of the variance process exhibits the mean-reverting feature. The mean-reverting rate depends on the current variance level, thus establishing a more volatile volatility structure than that of the Heston model. We let \( w_t \) be the reciprocal of the variance \( v_t \). By applying Ito’s lemma, \( w_t \) is seen to follow the time-inhomogeneous Heston model:

\[
dw_t = \left[q + \epsilon^2 - p(t) w_t\right] dt - \epsilon \sqrt{w_t} dW_t^2. \tag{2.3}
\]

There are certain technical conditions required in order to avoid anomalies in the 3/2-process. For a CIR process (the variance process in the Heston model), it is well known that the coefficients have to satisfy the Feller’s boundary condition in order to avoid reaching the zero value, causing \( v_t \) in the 3/2-process to explode. In our model, this condition is expressed as \( q \geq -\epsilon^2/2 \).

When pricing options under the share measure, Lewis (2000) shows that \( v_t \) has zero probability of explosion if and only if \( \rho < \epsilon/2 \). By imposing the two constraints: \( q > 0 \) and
ρ < 0 in our model, it becomes sufficient to avoid unboundedness in the variance process. Note that ρ < 0 is not that restrictive since negative correlation between \( S_t \) and \( v_t \) confirms quite well with the leverage effect commonly observed in the market: volatility goes up as stock price drops.

The success of analytical tractability of the 3/2-model relies on the availability of closed form representation of the joint conditional Fourier-Laplace transform of the terminal log asset price and the de-annualized realized variance \( \int_t^T v_s \, ds \). By assuming \( S_t \) to have no jump component in eq. (2.2), and letting \( X_t \) denote \( \ln(S_t e^{(r-d)(T-t)}) \), Carr and Sun (2007) obtain

\[
\mathbb{E}\left[e^{ikX_T - \mu \int_t^T v_s \, ds} \mid X_t, v_t\right] = e^{ikX_t} \frac{\Gamma(\gamma - \alpha)}{\Gamma(\gamma)} \left[\frac{2}{\epsilon^2 y(v_t, t)}\right]^\alpha M\left(\alpha, \gamma - \frac{2}{\epsilon^2 y(v_t, t)}\right), \tag{2.4}
\]

where

\[
y(v_t, t) = v_t \int_t^T e^{\int_u^t p(s) \, ds} \, du,
\]

\[
\alpha = -\left(\frac{1}{2} - \tilde{q}_k \frac{\epsilon^2}{\epsilon^2}\right) + \sqrt{\left(\frac{1}{2} - \frac{\tilde{q}_k}{\epsilon^2}\right)^2 + \frac{2c_k}{\epsilon^2}},
\]

\[
\gamma = 2\left(\alpha + 1 - \frac{\tilde{q}_k}{\epsilon^2}\right),
\]

\[
\tilde{q}_k = \rho \epsilon^2 - q,
\]

\[
c_k = \mu + \frac{k^2 + ik}{2}.
\]

The confluent hypergeometric function \( M(\alpha, \gamma, z) \) is defined as

\[
M(\alpha, \gamma, z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n z^n}{(\gamma)_n n!},
\]

where \((\alpha)_n = (\alpha)(\alpha - 1) \cdots (\alpha + n - 1)\) is the Pochhammer symbol. Later, we generalize the above analytic result to the 3/2-model with jumps.

2.2 Pricing formulation of derivatives on realized variance

There are two commonly adopted market conventions of computing the discrete realized variance, one is measured based on the actual rate of return while the other is based on the log return. We consider the tenor of the realized variance to be \([0, T]\) with monitoring dates \(0 = t_0 < t_1 < \cdots < t_N = T\), where \(T\) is the maturity date and \(N\) is the total number of monitoring dates. We use \( V_d^{(1)} \) to denote the discrete realized variance over \([t_0, t_N]\) in terms of actual return as defined by

\[
V_d^{(1)} = \frac{F_A}{N} \sum_{i=1}^{N} \left(\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}}\right)^2, \tag{2.5a}
\]

where \(F_A\) is the annualized factor. We take \(F_A = 252\) for daily monitoring and \(F_A = 52\) for weekly monitoring. Alternatively, the discrete realized variance over \([t_0, t_N]\) in terms of log return is given by

\[
V_d^{(2)} = \frac{F_A}{N} \sum_{i=1}^{N} \left(\frac{\ln S_{t_i}}{S_{t_{i-1}}}\right)^2. \tag{2.5b}
\]

Let \(\Delta t_i\) denote the time interval between \(t_{i-1}\) and \(t_i, i = 1, 2, \cdots, N\). We assume the time intervals to be uniform and use \(\Delta t\) to denote this common time interval. For the stochastic
volatility model with jumps defined in eq. (2.2), it is known that as $\Delta t \to 0$, we have

$$
\lim_{\Delta t \to 0} \sum_{i=1}^{N} \left( \ln \frac{S_{t_i}}{S_{t_{i-1}}} \right)^2 = \int_{0}^{t_N} v_t \, dt + \sum_{k=N_0}^{N_T} J_k^2,
$$

where $\int_{0}^{t_N} v_t \, dt$ is the continuous realized variance over $[t_0, t_N]$ and the last term sums all the square of jumps occurring within $[t_0, t_N]$. The fair strike of the vanilla swap on discrete realized variance (either actual return or log return) is given by

$$
K_n = \mathbb{E}_Q[V_d^{(n)}], \quad n = 1, 2,
$$

where $\mathbb{E}_Q$ denotes the expectation under the risk neutral measure $\mathbb{Q}$. By the linear property of expectation, the determination of the fair strike amounts to the evaluation of individual risk neutral expectation of squared return: $\mathbb{E}_Q[(\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}})^2]$ or $\mathbb{E}_Q[(\ln \frac{S_{t_i}}{S_{t_{i-1}}})^2]$ and summation over all monitoring instants.

We adopt the partial integro-differential equation approach to compute the above expectation of squared return over successive monitoring instants when the dynamics of the asset price $S_t$ follows the 3/2-model [see eq. (2.2)]. The challenge in the model formulation is that the risk neutral expectation is taken conditional on the filtration at time $t_0$ with respect to the two stochastic state variables: $S_{t_i}, I_{t_i}$. To resolve the path dependence on the asset price, we follow Little and Pant (2001) to introduce the path dependent state variable $I_t$, where

$$
I_t = \int_{0}^{t} \delta(u - t_{i-1}) S_u \, du = \left\{ \begin{array}{ll} S_{t_{i-1}} & t_{i-1} \leq t \\ 0 & 0 \leq t < t_{i-1}. \end{array} \right. \quad (2.6)
$$

Since $I_t$ takes different forms over $[0, t_{i-1})$ and $[t_{i-1}, t_i]$, we solve the governing equation in a two-step procedure. This is done by solving backward in time from $t_i$ to $t_{i-1}$, applying an appropriate jump condition at $t_{i-1}$, then solving backward from $t_{i-1}$ to $t_0$. Subsequently, we write $I_t$ as $I_{t_i}, i = 0, 1, \cdots, N$, for simplicity. Also, similar notations are used for $v_t$ and $I_t$.

Consider the pricing of a contingent claim with terminal payoff $F_i(S_i, v_i, I_i)$ at maturity date $t_i$, $i = 1, 2, \cdots, N$, the time-$t_0$ value of $U_i(S_0, v_0, I_0, t_0)$ under the risk neutral valuation framework is given by

$$
U_i(S_0, v_0, I_0, t_0) = e^{-r(t_i - t_0)} \mathbb{E}_Q[F_i(S_i, v_i, I_i)], \quad i = 1, 2, \cdots, N. \quad (2.7)
$$

By the Feynman-Kac theorem, the governing partial integro-differential equation (PIDE) for $U_i(S, v, I, t)$ is given by

$$
\frac{\partial U_i}{\partial t} + \frac{1}{2} v^2 S^2 \frac{\partial^2 U_i}{\partial S^2} + \rho v S \frac{\partial^2 U_i}{\partial S \partial v} + \frac{\epsilon^2}{2} v^2 \frac{\partial^2 U_i}{\partial v^2} + (r - d - \lambda m) \frac{\partial U_i}{\partial S} + [p(t)v - qv^2] \frac{\partial U_i}{\partial v} + \frac{\delta(t - t_{i-1}) S \frac{\partial U_i}{\partial I}}{I} - rU_i + \lambda \mathbb{E}_J[U_i(S e^J, v, I, t) - U_i(S, v, I, t)] = 0, \quad (2.8)
$$

where $\mathbb{E}_J$ is the expectation with respect to the jump process.

In our pricing problem of finding the fair strike value of a swap on discrete realized variance, the terminal payoff is independent of $v_i$. There is a jump in the value for $I$ across $t_{i-1}$ as shown in eq. (2.6), while $I$ assumes constant value across $[t_0, t_{i-1})$ and $[t_{i-1}, t_i]$ so that $\frac{\partial U}{\partial t} = 0$ at $t \neq t_{i-1}$. Note that there is no jump in the value for $U_i$ across $t_{i-1}$ since there
The two-step solution procedure for solving the PIDE can be summarized as follows:

(i) \( t \in (t_{i-1}, t_i) \)

\[
\frac{\partial U_i}{\partial t} + \frac{1}{2} v S^2 \frac{\partial^2 U_i}{\partial S^2} + \rho e v^2 S \frac{\partial U_i}{\partial S} + \frac{\epsilon^2}{2} v^3 \frac{\partial^2 U_i}{\partial v^2} + (r - d - \lambda m) S \frac{\partial U_i}{\partial S} + [p(t)v - q v^2] \frac{\partial U_i}{\partial v} - r U_i + \lambda \mathbb{E}[U_i(S e^{J_t}, v, I, t) - U_i(S, v, I, t)] = 0, \tag{2.10}
\]

with terminal condition: \( U_i(S, v, I, t_i) = F_i(S, S_{t-1}) \). Here, \( S_{t-1} \) is a known parameter and \( F_i \) has no dependence on \( v \).

(ii) \( t \in [t_0, t_{i-1}) \)

\[
\frac{\partial U_i}{\partial t} + \frac{1}{2} v S^2 \frac{\partial^2 U_i}{\partial S^2} + \rho e v^2 S \frac{\partial U_i}{\partial S} + \frac{\epsilon^2}{2} v^3 \frac{\partial^2 U_i}{\partial v^2} + (r - d - \lambda m) S \frac{\partial U_i}{\partial S} + [p(t)v - q v^2] \frac{\partial U_i}{\partial v} - r U_i + \lambda \mathbb{E}[U_i(S e^{J_t}, v, I, t) - U_i(S, v, I, t)] = 0, \tag{2.11}
\]

with terminal condition specified in eq. (2.9).

It is analytically tractable to solve for \( U_i \) over the time interval \([t_{i-1}, t_i]\) since the terminal payoff function \( F_i(S, S_{t-1}) \) is independent of \( v \) and \( S_{t-1} \) is a known parameter. However, since the solution \( U_i \) at \( t_{i-1} \) has dependence on \( v \), the solution of \( U_i \) at \( t_0 \) cannot be obtained in analytic closed form. We manage to express \( U_i \) at \( t_0 \) in a quasi-closed form in terms of an integral with the product of the transition density and the known solution \( U_i \) at \( t_{i-1} \) as the integrand.

To solve for \( U_i(x, v, t) \) over \([t_{i-1}, t_i]\) as governed by eq. (2.10), we take the Fourier transform of the governing equation with respect to \( x \), where \( x = \ln S \). We let \( \tau = t_i - t \) and define the Fourier transform of \( U_i(x, v, \tau) \) (dropping dependence on \( I \)) by

\[
\tilde{U}_i(k, v, \tau) = \int_{-\infty}^{\infty} e^{-ikx} U_i(x, v, \tau) \ dx. \tag{2.12}
\]

We define \( H_i(k, v, \tau) \) by

\[
H_i(k, v, \tau) = \exp(-s(k)\tau)\tilde{U}_i(k, v, \tau),
\]

where

\[
s(k) = ik(r - d - \lambda m) - (r + \lambda) + \lambda \exp(ik\nu - \frac{\epsilon^2}{2} k^2).
\]

The PIDE (2.10) is transformed into a partial differential equation in \( H_i \) as follows:

\[
\frac{\partial H_i}{\partial \tau} = -c_k v H_i + [p(t)v + \tilde{q} v^2] \frac{\partial H_i}{\partial v} + \frac{\epsilon^2}{2} v^3 \frac{\partial^2 H_i}{\partial v^2}, \tag{2.13}
\]

with initial condition: \( \mathcal{F}[F(e^z, \bar{S}_{t-1})] \), where \( c_k = (k^2 + ik)/2 \) and \( \tilde{q} = \rho e^{ik} - q \). If we can find the fundamental solution to eq. (2.13), then the solution \( U_i \) of the PDE (2.10) can be obtained easily by the following proposition.
Proposition 1 Let \( \hat{H}_i(k, v, \tau) \) denote the fundamental solution to eq. (2.13) with initial condition: \( \hat{H}_i(k, v, 0) = 1 \), then

\[
\hat{H}_i(k, v, \tau) = \frac{\Gamma(\gamma - \alpha)}{\Gamma(\gamma)} \left[ \frac{2}{e^{2y(v, \tau)}} \right]^\alpha \frac{M(\alpha, \gamma, -\frac{2}{e^{2y(v, \tau)}})}{\gamma - \alpha}.
\]

The solution for \( U_i(x, v, t) \) over \([t_{i-1}, t_i] \) is given by

\[
U_i(x, v, \tau) = \mathcal{F}^{-1} \left[ \exp(s(k)\tau)\hat{H}_i(k, v, \tau)\mathcal{F}[F_i(e^\delta, S_{i-1})] \right].
\]

The proof of Proposition 1 is presented in Appendix A.

Transition density function of the 3/2-variance process

Over the time interval \([t_0, t_{i-1}] \), it is cumbersome to use the fundamental solution \( \hat{H}_i \) in the Fourier domain to perform the expectation valuation with respect to the variance process. Instead, we employ the transition density function of the variance process \( v_t \) conditional on \( v_0 \) to simplify the expectation calculation. The reciprocal of \( v_t \) is a CIR process, whose transition density function is well known. By defining \( w_t = \frac{1}{v_t} \), their respective density functions \( p_{v_t} \) and \( \tilde{p}_{w_t} \) are related by (Jeanblanc et al., 2009)

\[
p_{v_t}(v_t, t|v_0, 0) = \tilde{p}_{w_t} \left( \frac{1}{v_t}, t \bigg| \frac{1}{v_0}, 0 \right) \frac{1}{v_t^2} = \frac{l(0, t)}{2l^*(0, t)v_t^2} \exp \left( \frac{-1}{v_0} - \frac{1}{v_t} \frac{l(0, t)}{v_t} \right) \left( \frac{v_0 l(0, t)}{v_t} \right)^{\nu/2} I_\nu \left( \frac{\sqrt{l(0, t)}}{\sqrt{v_0 v_t}} \right),
\]

where

\[
\nu = \frac{2}{\epsilon^2} (q + \epsilon^2) - 1, \quad l(s, t) = \exp \left( \int_s^t p(u) \, du \right), \quad l^*(s, t) = \frac{e^2}{2} \int_s^t l(s, u) \, du.
\]

Note that \( I_\nu \) is the modified Bessel function of the first kind of order \( \nu \) as defined by

\[
I_\nu (z) = \left( \frac{z}{2} \right)^\nu \sum_{k=0}^{\infty} \frac{\left( \frac{z}{2} \right)^k}{k! \Gamma(\nu + k + 1)}.
\]

2.3 Fair strike formulas for vanilla variance swaps

The generalized Fourier transform of the terminal payoff \( F_i(S, S_{i-1}) \) takes different forms, depending on whether actual return or log return is considered in the squared return.

(i) Actual return

\[
\mathcal{F} \left[ F_i^{(1)}(S, S_{i-1}) \right] = \mathcal{F} \left[ \left( \frac{S - S_{i-1}}{S_{i-1}} \right)^2 \right] = \mathcal{F} \left[ \left( \frac{e^\delta}{S_{i-1}} - 1 \right)^2 \right] = 2\pi \left[ \frac{\delta(k + 2i)}{S_{i-1}^2} - \frac{2\delta(k + i)}{S_{i-1}} + \delta(k) \right].
\]

(2.15a)
(ii) Log return

\[
\mathcal{F}[F_i^{(2)}(S, S_{i-1})] = \mathcal{F}\left[\left(\ln \frac{S}{S_{i-1}}\right)^2\right] = \mathcal{F}[(x - \ln S_{i-1})^2] = 2\pi [-\delta^{(2)}(k) - 2i \delta^{(1)}(k) \ln S_{i-1} + \delta(k)(\ln S_{i-1})^2],
\]

where \(\delta^{(n)}(k)\) denotes the \(n\)th-order derivative of the Dirac function.

Let \(U_i^{(n)}(x, v, \tau)\) denote the solution to eq. (2.10) corresponding to the terminal condition: \(F_i^{(n)}(e^x, S_{i-1}), n = 1, 2\). Using Proposition 1, we derive the solutions for \(U_i^{(1)}\) and \(U_i^{(2)}\) at \(\tau = \Delta t\ (t = t_{i-1})\) as follows:

\[
U_i^{(1)}(x, v, \Delta t) = \mathcal{F}^{-1}\left[\exp(s(k)\Delta t)\hat{H}_i(k, v, \Delta t)\mathcal{F}\left[\left(e^x\right)^2\right]\right] = \int_{-\infty}^{\infty} e^{ikx} \exp(s(k)\Delta t)\hat{H}_i(k, v, \Delta t) \left[\frac{\delta(k + 2i) - 2\delta(k + i) + \delta(k)}{S_{i-1}^2} - 2\delta(k + i) + \delta(k)\right] dk = e^{s(-2i)\Delta t}\hat{H}_i(-2i, v, \Delta t) - 2e^{s(-i)\Delta t} + e^{-r\Delta t},
\]

(2.16a)

and

\[
U_i^{(2)}(x, v, \Delta t) = \mathcal{F}^{-1}\left[\exp(s(k)\Delta t)\hat{H}_i(k, v, \Delta t)\mathcal{F}\left[(x - \ln S_{i-1})^2\right]\right] = \int_{-\infty}^{\infty} e^{ikx} \exp(s(k)\Delta t)\hat{H}_i(k, v, \Delta t) \left[-\delta^{(2)}(k) - 2i \delta^{(1)}(k) \ln S_{i-1} + \delta(k)(\ln S_{i-1})^2\right] dk = -f^{(2)}(0) + 2if^{(1)}(0) \ln S_{i-1} + f(0)(\ln S_{i-1})^2 = -g^{(2)}(0),
\]

(2.16b)

where

\[
f(k) = g(k)e^{ikx}, \quad g(k) = \exp(s(k)\Delta t)\hat{H}_i(k, v, \Delta t),
\]

\[
f^{(n)}(0) = \frac{\partial^n f}{\partial k^n}\bigg|_{k=0}, \quad g^{(n)}(0) = \frac{\partial^n g}{\partial k^n}\bigg|_{k=0}.
\]

The realized discrete variance consists of \(N\) terms [see eqs. (2.5a,b)]. The risk neutral expectation of the first term, \(i = 1\), does not require the two-step expectation calculation since \(S_0\) is known. For \(i \geq 2\), it is necessary to implement the two-step backward induction from \(t_i\) to \(t_{i-1}\), then \(t_{i-1}\) to \(t_0\). Given the value of the variance swap at time \(t_{i-1}\), the value at time \(t_0\) can be obtained by taking the expectation of \(v_{i-1}\) conditional on \(v_0\) since \(U_i^{(n)}(x, v, \Delta t)\) does not depend on \(x\). The fair strike can be expressed as a sum of expectation integrals with respect to the conditional transition density of \(v_{i-1}, i = 2, 3, \ldots, N\) and \(\frac{F_A}{N}e^{r\Delta t}U_i^{(n)}(x, v_0, \Delta t)\), the details of which are summarized in the following proposition.

**Proposition 2** The fair strike of a variance swap with discrete sampling on dates: \(0 = t_0 < t_1 < \cdots < t_N = T\) is given by

\[
K_n = \frac{F_A}{N}e^{r\Delta t}\left[U_i^{(n)}(x_0, v_0, \Delta t) + \sum_{i=2}^{N} \int_{0}^{\infty} U_i^{(n)}(x, v, \Delta t)p_v(v, t_{i-1}|v_0, 0) dv\right], \quad n = 1, 2.
\]

(2.17)
3 Third generation variance swaps: Gamma swaps and corridor swaps

The third generation (generalized) variance swaps are structured so as to provide more specific exposure to equity variance by adding weights at different monitoring instants in the evaluation of the accumulated realized variance. The weight adjusted discrete realized variance assumes the form

\[ \frac{F_{AN}}{N} \sum_{i=1}^{N} w_i \left( \frac{S_i - S_{i-1}}{S_{i-1}} \right)^2 \quad \text{or} \quad \frac{F_{AN}}{N} \sum_{i=1}^{N} w_i \ln \left( \frac{S_i}{S_{i-1}} \right)^2 \]

based on actual return or log return, respectively. In this section, we consider the gamma swaps and corridor swaps, where \( w_i \) takes the forms

\[ w_i = \frac{S_i}{S_0} \quad \text{and} \quad w_i = 1_{\{L < S_i \leq U\}} \]

respectively. The motivation for these two generalized variance swaps and their uses have been discussed in Lee (2010) and Zheng and Kwok (2014).

3.1 Fair strike formulas for gamma swaps

The derivation of the fair strike formula for the gamma swap, either in actual return or log return, amounts to the following respective risk neutral expectation calculation:

\[
E_Q \left[ S_i \left( \frac{S_i - S_{i-1}}{S_{i-1}} \right)^2 \right] \quad \text{and} \quad E_Q \left[ S_i \left( \ln \frac{S_i}{S_{i-1}} \right)^2 \right].
\]

Similar to the derivation of the fair strike formula for the vanilla variance swap, we compute the generalized Fourier transform of the terminal payoff of the gamma swap as follows:

(i) actual return

\[
\mathcal{F}[G^{(1)}_i(S, S_{i-1})] = \mathcal{F} \left[ S \left( \frac{S - S_{i-1}}{S_{i-1}} \right)^2 \right] = 2\pi \left[ \frac{\delta(k + 3i)}{S^2 - 2\delta(k + 2i) S_i - S_{i-1}} + \delta(k + i) \right]. \quad (3.1a)
\]

(ii) log return

\[
\mathcal{F}[G^{(2)}_i(S, S_{i-1})] = \mathcal{F} \left[ S \left( \ln \frac{S}{S_{i-1}} \right)^2 \right] = 2\pi \left[ -\delta^{(2)}(k + i) - 2i\delta^{(1)}(k + i) \ln S_{i-1} + \delta(k + i)(\ln S_{i-1})^2 \right]. \quad (3.1b)
\]

We follow a similar two-step solution procedure, where the risk neutral expectation calculations are performed over the successive time intervals \([t_{i-1}, t_i]\) and \([t_0, t_{i-1}]\). Unlike the vanilla variance swaps, the time-\(t_{i-1}\) value of \(U_i^{(n)}\) for a generalized variance swap has dependence on both \(S_{i-1}\) and \(v_{i-1}\). Consequently, this poses the additional technical challenge of finding the joint transition density of \(\ln S_i\) and \(v_t\) in the risk neutral expectation calculation over \([t_0, t_{i-1}]\). We write \(x_t = \ln S_t\), and let \(p_{x,v}(x_t, v_t, t|x_s, v_s, s)\) denote the joint transition density function of \(x_t\) and \(v_t\) from time \(s\) to \(t\). For both the gamma swaps and corridor
swaps, there is an extra $e^z$ term at time $t_{i-1}$ after the first step evaluation. Therefore, it is more convenient to have a closed form formula for the $x_t$-transformed density function rather than the transition density itself.

To proceed further for the derivation of the analytic fair strike formulas of the gamma swaps, we assume $p(t)$ to be a constant value $p$ in all subsequent calculations in order to achieve analytic closed form representation of the transformed density function. Let $\tilde{G}(\tau; -z, v_t|x_s, v_s)$ be the $x_t$-transformed density, we obtain

$$
\tilde{G}(\tau; -z, v_t|x_s, v_s) = \int_{-\infty}^{\infty} e^{izy} p_{x,v}(y, v_t, t|x_s, v_s, s) \, dy = G(\tau; -z, v_t, v_s) e^{izr} \exp(h(z)\tau),
$$

(3.2)

where

$$
G(\tau; -z, v_t, v_s) = \frac{e^{(1+\mu_z)p\tau}}{e^{p\tau} - 1} \exp \left( -\frac{2p}{(e^{p\tau} - 1)v_t^2} - \frac{2p}{(e^{p\tau} - 1)v_t^2} \right)
$$

$$
\left( \frac{2p}{e^{p\tau} - 1} \right) \left( \frac{\nu_s}{v_t} \right)^{\mu_s} I_{\nu_s}^1 \frac{p\tau/2}{(e^{p\tau} - 1)^{1/2}}
$$

and

$$
h(z) = i(r - d - \lambda m)z + [\exp(iz - \zeta z^2/2) - 1]\lambda,
$$

$$
\mu_z = \frac{1}{2}(1 + \hat{\theta}_z), \quad \hat{\theta}_z = \frac{b(q + iz\epsilon)}{e^2}, \quad c_z = \frac{(z^2 - i\zeta)}{2},
$$

$$
\tilde{c}_z = \frac{2c_z}{e^2}, \quad \nu_z = 2\sqrt{\mu_z + \tilde{c}_z}, \quad \tau = t - s.
$$

Note that $I_{\nu_s}$ is the modified Bessel function of the first kind of order $\nu_z$. The derivation of eq. (3.2) is presented in Appendix B.

Let $U_1^{(n)}(x, v, \tau)$ denote the solution to eq. (2.10) corresponding to the terminal condition: $G_1^{(n)}(x^*, S_{t-1})$, $n = 1, 2$. Using Proposition 1, we derive $U_1^{(1)}(x, v, \Delta t)$ and $U_1^{(2)}(x, v, \Delta t)$ as follows:

$$
U_1^{(1)}(x, v, \Delta t) = \mathcal{F}^{-1} \left[ \exp(s(k)\Delta t)\hat{H}_i(k, v, \Delta t) \left[ \frac{\delta(k + 3i)}{S_{t-1}^2} - \frac{2\delta(k + 2i)}{S_{t-1}} + \delta(k + i) \right] \right]
$$

$$
e^x[e^{s(-3)i}\hat{H}_i(-3i, v, \Delta) - 2e^{s(-2)i}\hat{H}_i(-2i, v, \Delta t) + e^{s(-i)\Delta t}],
$$

(3.3a)

and

$$
U_1^{(2)}(x, v, \Delta t) = \mathcal{F}^{-1} \left[ \exp(s(k)\Delta t)\hat{H}_i(k, v, \Delta t) \left[ -\delta^{(2)}(k + i) - 2i\delta^{(1)}(k + i) \ln S_{t-1} + \delta(k + i)(\ln S_{t-1})^2 \right] \right]
$$

$$
= -f^{(2)}(-i) + 2if^{(1)}(-i) \ln S_{t-1} + f(-i)(\ln S_{t-1})^2
$$

$$
e^x[-g^{(2)}(-i)],
$$

(3.3b)

where

$$
f^{(n)}(\alpha) = \left. \frac{\partial^n f}{\partial k^n} \right|_{k=\alpha}, \quad g^{(n)}(\alpha) = \left. \frac{\partial^n g}{\partial k^n} \right|_{k=\alpha}.
$$

In a similar manner as shown in the computation of the fair strike of a variance swap, the risk neutral expectation calculation of the first term in the realized discrete variance
In a corridor swap, the realized squared return monitored at the time $t_i$ becomes

The corresponding risk neutral expectation terms become

Alternatively, one may use $S_{i-1}$ as the monitoring asset price of the corridor feature on the $i$th sampling date for the calculation of the accumulated realized variance. The corresponding risk neutral expectation terms become

The formula for the fair strike of the gamma swap is presented in the following proposition.

**Proposition 3** The fair strike of a gamma swap with discrete sampling on dates: $0 = t_0 < t_1 < \cdots < t_N = T$ is given by

$$K_n = \frac{e^{\Delta t}}{S_0} \frac{F_A}{N} \left[ U_i^{(n)}(x_0, v_0, \Delta t) + \sum_{i=2}^{N} \int_0^\infty \Psi_i^{(n)}(v, \Delta t|x_0, v_0) \, dv \right], \quad n = 1, 2. \quad (3.5)$$

### 3.2 Fair strike formulas for corridor swaps

In a corridor swap, the realized squared return monitored at the time $t_i$ is added to the accumulated variance only when $S_{i-1}$ falls within the corridor $(L, U)$. The calculation of the fair strike amounts to the following risk neutral expectation calculations:

$$\mathbb{E}_Q\left[ \left( \frac{S_i - S_{i-1}}{S_{i-1}} \right)^2 1_{\{L < S_{i-1} \leq U\}} \right] \quad \text{and} \quad \mathbb{E}_Q\left[ \left( \ln \frac{S_i}{S_{i-1}} \right)^2 1_{\{L < S_{i-1} \leq U\}} \right].$$

Alternatively, one may use $S_i$ rather than $S_{i-1}$ as the monitoring asset price of the corridor feature on the $i$th sampling date for the calculation of the accumulated realized variance. The corresponding risk neutral expectation terms become

$$\mathbb{E}_Q\left[ \left( \frac{S_i - S_{i-1}}{S_{i-1}} \right)^2 1_{\{L < S_i \leq U\}} \right] \quad \text{and} \quad \mathbb{E}_Q\left[ \left( \ln \frac{S_i}{S_{i-1}} \right)^2 1_{\{L < S_i \leq U\}} \right].$$
Carr and Lewis (2004) develop an approximate pricing and hedging method for the discrete corridor variance swaps by using a portfolio of European style options with a continuum of strikes. Zheng and Kwok (2014) present closed form formula for the fair strike of the corridor variance swap under the 3/2-model. Here, we derive fair strike formula for the corridor swap under the 3/2-model. The use of the following relation

\[ 1_{\{L < S_{i-1} \leq U\}} = 1_{\{S_{i-1} < L\}} - 1_{\{S_{i-1} < L\}} \]

simplifies pricing of the corridor variance swap into that of the downside variance swap with one-sided upper boundary in the monitoring corridor. Therefore, it suffices to compute the respective downside conditional expectation as follows:

\[ \mathbb{E}_Q \left[ \ln \left( \frac{S_i - S_{i-1}}{S_{i-1}} \right)^2 1_{\{S_{i-1} \leq U\}} \right] \quad \text{and} \quad \mathbb{E}_Q \left[ \ln \left( \frac{S_i}{S_{i-1}} \right)^2 1_{\{S_{i-1} \leq U\}} \right] . \]

The indicator function admits the following Fourier integral representation over the real line

\[ 1_{\{S_{i-1} \leq U\}} = 1_{\{x_{i-1} \leq u\}} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iux} e^{-iwx_{i-1}} d\omega, \]

where \( u = \ln U, \omega = \omega_t + i\omega_1 \) and \( \omega_1 \in (-\infty, 0) \). This Fourier integral representation essentially modifies the terminal payoff function by multiplying a factor of \( e^{-i(\omega_t + i\omega)x} \). Since the indicator function is conditioning on time \( t_{i-1} \) rather than \( t_i \), in the backward induction from time \( t_i \) to \( t_{i-1} \), we obtain the same \( U_i^{(n)} \) as those in the vanilla variance swaps (with no dependence on \( x \) at time \( t_{i-1} \)). For the terms corresponding to \( i \geq 2 \) in the discrete variance formulas (2.5a,b), by changing the order of integration of \( x \) with \( \omega_t \), the expectation calculation coupled with the Fourier transform representation of the indicator function can be expressed as

\[
\mathbb{E}_Q \left[ 1_{\{x \leq u\}} e^{rT_i} U_i^{(n)}(x, v, \Delta t) \right] \\
= e^{rT_1} \int_0^\infty \int_0^\infty \left[ \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{iux} e^{-iwx} U_i^{(n)}(x, v, \Delta t) p_{x,v}(x, v, t_{i-1}|x_0, v_0, 0) d\omega \right] dx \ dv \\
= \frac{e^{rT_1}}{2\pi} \int_0^\infty \int_0^\infty e^{iux} U_i^{(n)}(x, v, \Delta t) \tilde{G}(t_{i-1}; \omega, v|x_0, v_0) d\omega \ dv. \quad (3.6)
\]

For \( i \geq 2 \) and \( \tau = \Delta t \), we define

\[
\Phi_i^{(1)}(\omega, v, \Delta t|x_0, v_0) = U_i^{(1)}(x, v, \Delta t) \tilde{G}(t_{i-1}; \omega, v|x_0, v_0) \\
= [e^{s(-2)i\Delta t} \tilde{H}_i(-2i, v, \Delta t) - 2e^{s(-i)\Delta t} + e^{s\Delta t}] \tilde{G}(t_{i-1}; \omega, v|x_0, v_0), \quad (3.7a)
\]

and

\[
\Phi_i^{(2)}(\omega, v, \Delta t|x_0, v_0) = U_i^{(2)}(x, v, \Delta t) \tilde{G}(t_{i-1}; \omega, v|x_0, v_0) = \left[ -g^{(2)}(0) \right] \tilde{G}(t_{i-1}; \omega, v|x_0, v_0). \quad (3.7b)
\]

Combining the above results, the analytic formula for the fair strike of the downside variance swap is summarized in the following proposition.

**Proposition 4** For a discretely sampled downside variance swap on dates: \( 0 = t_0 < t_1 < \cdots < t_N = T \), with \( S_{i-1} \) as the monitoring asset price on the \( i \)th monitoring date, the fair strike is given by

\[
K_n = \frac{F_A}{N} e^{rT} \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{iux}}{i\omega} \left( \int_0^\infty \sum_{i=2}^N \Phi_i^{(n)}(\omega, v, \Delta t|x_0, v_0) \ dv \right) d\omega \\
+ 1_{\{x \leq u\}} U_i^{(1)}(x_0, v_0, \Delta t) \right], \ n = 1, 2. \quad (3.8)
\]
where \( u = \ln U \), and \( U_1^{(n)}(x_0, v_0, \Delta t) \) are given by eqs. (2.16a,b).

By following a similar approach, we can derive the analytic fair strike formula for the downside variance swap with \( S_t \) as the monitoring asset price on the \( i \)th monitoring date. Consider the vanilla variance swap with payoff at maturity \( t_i \) multiplied by \( e^{-i\omega x} \), the value at time \( t_{i-\tau} (\tau = \Delta t) \) can be obtained as follows:

(i) actual return

\[
U_i^{(3)}(\omega, x, v, \Delta t) = \mathcal{F}^{-1} \left[ \exp(s(k)\Delta t)\tilde{H}_i(k, v, \Delta t)\mathcal{F} \left[ e^{-i\omega x}(e^{x} - 1)^2 \right] \right] = e^{-i\omega x} \left[ e^{s(-2i-\omega)\Delta t}\tilde{H}_i(-2i - \omega, v, \Delta t) - 2e^{s(-i-\omega)\Delta t}\tilde{H}_i(-i - \omega, v, \Delta t) \right. \\
+ \left. e^{s(-\omega)\Delta t}\tilde{H}_i(-\omega, v, \Delta t) \right];
\]

(3.9a)

(ii) log return

\[
U_i^{(4)}(\omega, x, v, \Delta t) = \mathcal{F}^{-1} \left[ \exp(s(k)\Delta t)\tilde{H}_i(k, v, \Delta t)\mathcal{F} \left[ e^{-i\omega x}(x - \ln I)^2 \right] \right] = e^{-i\omega x} \left[ -g^{(2)}(-\omega) \right].
\]

(3.9b)

For the terms corresponding to \( i \geq 2 \) in the discrete variance formulas (2.5a,b), we denote the integral of \( U_i^{(n)} \) and \( p_{x,v} \) over the \( x \) variable as

\[
\Phi_i^{(3)}(\omega, v, \Delta t|x_0, v_0) = \int_{-\infty}^{\infty} U_i^{(3)}(\omega, x, v, \Delta t) p_{x,v}(x, v, t_{i-1}|x_0, v_0) \, dx
\]

\[
= \left[ e^{s(-2i-\omega)\Delta t}\tilde{H}_i(-2i - \omega, v, \Delta t) - 2e^{s(-i-\omega)\Delta t}\tilde{H}_i(-i - \omega, v, \Delta t) \right. \\
+ \left. e^{s(-\omega)\Delta t}\tilde{H}_i(-\omega, v, \Delta t) \right] G(t_{i-1}; \omega, v|x_0, v_0),
\]

(3.10a)

and

\[
\Phi_i^{(4)}(\omega, v, \Delta t|x_0, v_0) = \int_{-\infty}^{\infty} U_i^{(4)}(\omega, x, v, \Delta t) p_{x,v}(x, v, t_{i-1}|x_0, v_0) \, dx
\]

\[
= \left[ -g^{(2)}(-\omega) \right] G(t_{i-1}; \omega, v|x_0, v_0).
\]

(3.10b)

Combining the above results, we obtain the analytic fair strike formula of the alternative downside variance swap as summarized in Proposition 5. Recall that in order to recover the original fair strike of the downside variance swap, it is necessary to perform integration with respect to \( \omega_r \) by using the Fourier integral representation formula of the indicator function.

**Proposition 5** For a discretely sampled downside variance swap on dates: \( 0 = t_0 < t_1 < \cdots < t_N = T \) with \( S_t \) as the monitoring asset price on the \( i \)th monitoring date, the fair strike is given by

\[
K_n = \frac{F_A}{N} \left[ \frac{e^{t\Delta t}}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\omega x}}{i\omega} \left( U_1^{(n)}(\omega, x_0, v_0, \Delta t) + \int_{0}^{\infty} \sum_{i=2}^{N} \Phi_i^{(n)}(\omega, v, \Delta t|x_0, v_0) \, dv \right) \, d\omega_r \right], \quad n = 3, 4.
\]

(3.11)
4 Numerical results

In this section, we present the results of our numerical tests on the fair strike formulas and investigate the impact of different parameters on the fair strikes of the various types of variance swaps. We show the comparison of the values obtained in our method with the benchmark results obtained from Monte Carlo simulation for assessing the level of accuracy of our analytic formulas. We also show the relation between the sampling frequency $N$ and the fair strikes for different types of variance swaps. The impact of different model parameters, including correlation coefficient $\rho$, volatility of variance $\epsilon$ and jump intensity $\lambda$, on the fair strikes of the different types of variance swaps are examined.

In our numerical calculations, we adopted the same set of model parameter values from Drimus (2012). These parameter values are obtained through simultaneous calibration of the 3-month and 6-month S&P 500 implied volatilities on July 31, 2009 for the 3/2-model (see Table 1). We assume $d=0$, $S_0=1$ and $T=1$. In order to produce comparable results with the no-jump model in Drimus (2012), we also assume $\lambda=0$ when we investigate the impact of $N$, $\rho$ and $\epsilon$ on the fair strike values of the variance swaps. When we explore the impact of jump intensity in Figure 4, $\lambda$ is set to assume varying values. Besides, we take the interest rate to be flat at $r=0.48\%$ with reference to the Treasury 1-Year Yield Curve on the calibrated date. Furthermore, we assume $U=1$ for the upper barrier in the corridor of the downside variance swaps and there are 252 trading days in one year.

<table>
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<tr>
<th>Parameter</th>
<th>$v_0$</th>
<th>$\rho$</th>
<th>$q$</th>
<th>$\epsilon$</th>
<th>$\rho$</th>
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<tbody>
<tr>
<td>Value</td>
<td>0.060025</td>
<td>4.9790</td>
<td>22.84</td>
<td>8.56</td>
<td>-0.99</td>
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</table>

Table 1: Model parameters of the 3/2-model.

Assessment of numerical accuracy in evaluation of analytic formulas

One may be concerned with accuracy in numerical evaluation of the confluent hypergeometric functions in our analytic formulas for the fair strike prices of variance swaps under the 3/2-model. We performed Monte Carlo simulation to obtain benchmark values for the fair strike prices of the different types of variance swaps and make assessment on accuracy in numerical evaluation of the analytic fair strike formulas. We applied the Euler scheme in the Monte Carlo simulation procedure with 100,000 simulated paths. As pointed out by Drimus (2012), the 3/2-model shows more radical behavior of the volatility dynamics compared to the Heston model. When he calibrated the same set of option data to both the 3/2-model and Heston model parameters, the 3/2-variance process is seen to lead to more volatile sample paths under a comparable set of parameters. We also notice that exceedingly small time step is required in order to stabilize the Monte Carlo simulated calculations.

On the other hand, our experience in the computation suggests that the range of $v_{i-1}$ for integration should be taken to be $[0,10]$ and that of $\omega_r$ to be $[-100,100]$ while $\omega_i=-0.02$ would be sufficient for convergence of the generalized Fourier transform. The calculations were performed on an Intel i7 PC. By taking advantage of the multi-cores CPU in parallel computing, we have designed parallel codes for the Monte Carlo simulation and numerical integration in the expectation calculations. The Monte Carlo simulation was coded in Matlab for its simplicity in programming. Our numerical integration calculations were performed in Mathematica to take advantage of its built-in confluent hypergeometric functions and Bessel functions.
In Table 2, we list the numerical results obtained by the Monte Carlo simulation and numerical integration of the analytic fair strike formulas with the number of sampling dates \( N = 52 \). The numerical evaluation of the analytic formulas is seen to provide reasonable accuracy with significantly less computation time when compared to that of the Monte Carlo simulation. If we increase the number of the sample paths in the Monte Carlo simulation to 1,000,000, we can obtain more accurate results at the expense of significantly higher computational costs. For the downside variance swaps, our analytic formulas require numerical evaluation of a double integral. As a result, the computational times become more demanding.

<table>
<thead>
<tr>
<th>swap type</th>
<th>variance swap</th>
<th>gamma swap</th>
</tr>
</thead>
<tbody>
<tr>
<td>convention</td>
<td>actual return</td>
<td>log return</td>
</tr>
<tr>
<td>analytic</td>
<td>0.080939</td>
<td>0.083874</td>
</tr>
<tr>
<td>time (s)</td>
<td>(0.742)</td>
<td>(5.742)</td>
</tr>
<tr>
<td>MC (0.1M)</td>
<td>0.080853</td>
<td>0.083827</td>
</tr>
<tr>
<td>time (s)</td>
<td>(211.571)</td>
<td>(211.571)</td>
</tr>
<tr>
<td>SE (%)</td>
<td>0.014228</td>
<td>0.017914</td>
</tr>
<tr>
<td>MC (1M)</td>
<td>0.080937</td>
<td>0.083891</td>
</tr>
<tr>
<td>time (s)</td>
<td>(2859.911)</td>
<td>(2859.911)</td>
</tr>
</tbody>
</table>

Table 2: Comparison of the fair strike values obtained from the numerical integration of the analytic strike formulas and Monte Carlo simulation. The computational times are measured in units of second. Here, “analytic” stands for numerical evaluation of the analytic formulas via numerical integration, MC (0.1M) and MC (1M) for Monte Carlo simulation using 0.1 million paths and 1 million paths, respectively, and SE for the standard error in the simulation. For Monte Carlo simulation, computational times quoted are valid for both actual return and log return since the discrete realized variance under both conventions can be computed by using the same set of simulation paths.

**Pricing properties of discrete variance swaps under 3/2-model**

We would like to explore the impact of sampling frequency on the fair strike values of various types of variance swaps. When pricing under the Heston model, Zhu and Lian (2011) and Zheng and Kwok (2014) show that the fair strike values of variance swap and gamma swap decrease with the sampling frequency. In Figure 1, we show the plots of the fair strike values of various types of variance swaps against the sampling frequency under the 3/2-model. We found that the fair strike values of the vanilla variance swap and gamma swap using the convention of log return are always higher than those of actual return, a similar phenomenon that has been observed by Zhu and Lian (2012) for pricing discrete variance swaps under the Heston model. For the variance swap based on log return, the fair strike decreases when the sampling frequency increases, consistent with similar results obtained under the Heston model. However, when the convention of actual return is used, we note that the fair strike values of vanilla variance swap and gamma swap increase steadily when \( N \) increases. For the less frequently sampled downside variance swaps, the difference in the fair strike values can be substantial under different conventions of calculating the discrete realized return. We also note that the fair strike values of the two types of downside variance swaps with either \( S_t \) or \( S_{t-1} \) as the monitoring asset price of the corridor feature are quite close. When the sampling frequency increases, the fair strike values of the various variance products under the two different conventions of realized return converge to the same set of common values.
For simplicity, we only show the plots of the fair strike values of the variance swaps based on the convention of log return in Figures 2, 3 and 4. Figure 2 shows the fair strike values against correlation coefficient, \( \rho \). The fair strike of the vanilla variance swap is less sensitive to \( \rho \) when compared to the gamma swap. The fair strike of the variance swap decreases slowly with increasing value of \( \rho \) while for the gamma swap, it shows a moderate increasing trend. When \( \rho \) is close to zero, the leverage effect is minimal, thus the effectiveness of avoiding the extreme price movement by adding the weights in the gamma swap is reduced. Therefore, the fair strike values of the gamma swap and vanilla swap almost equal to each other at zero value of \( \rho \). For the downside variance swap, the fair strike is decreasing when \( \rho \) increases.

A higher value of the volatility of variance \( \epsilon \) normally indicates more volatile behavior of the volatility dynamics in the Heston model. However, the reverse relation is found in the 3/2-model. Figure 3 demonstrates the relation of the fair strike values with varying values of \( \epsilon \). As \( \epsilon \) increases, the fair strike values of all the variance swap products decrease and the rate of decrease is more rapid at small value of \( \epsilon \). In actual market conditions, large values of \( \epsilon \) are commonly found when calibration is performed for real market data [see Drimus (2012) and Baldeaux and Badran (2012)].

To investigate the effect of jump intensity \( \lambda \) and jump size distribution on the fair strike values of different variance swap products, we take \( \mu = -0.3 \) and \( \zeta = 0.4 \) to represent the scenario of large jump, while \( \mu = -0.03 \) and \( \zeta = 0.04 \) for small jump. Figure 4 shows that the fair strike values of all variance swaps are very sensitive (almost insensitive) to the jump intensity parameter for large (small) jump size. When the jump intensity increases, jumps occur more frequently with a higher probability. A large mean of jump size gives a more acute rate of increase, in particular for the vanilla variance swap; while a small one shows less obvious impact on the fair strike values. The mean of jump size and its standard deviation are seen to be critical in determining the fair strike other than the jump intensity parameter.

5 Conclusion

In this paper, we illustrate the analytic-numerical procedure for finding the fair strike values for various types of discretely sampled variance products under the 3/2-model with jumps. As revealed by various empirical studies, the 3/2-model is a better choice than the Heston model for fitting the volatility structure. The variance swap products considered in this paper include the third generation swaps like the gamma swaps and corridor swaps. The payoff structures in these discrete variance swaps consist of summation of terms that are dependent on two time instants \( t_{i-1} \) and \( t_i \) over successive monitoring dates. We first compute the conditional risk neutral expectation of the payoffs at time \( t_{i-1} \) by solving for the fundamental solution of the governing partial integro-differential equation in the Fourier domain. The analytic formula for the fair strike value at initiation is derived by summing terms that are obtained by integrating the conditional value at time \( t_{i-1} \) with respect to the transition density of the 3/2-model. The numerical integration can be effectively performed

\[ \int_{0}^{T} \]
via numerical quadrature. The same methodology can be applied to find the fair strike values of other higher moment swaps under the 3/2-model, though the pricing procedure becomes notoriously tedious.

We performed numerical valuation of our fair strike formulas of various types of variance swaps and compared the numerical results with the strike values obtained from the Monte Carlo simulation. The computational times for the vanilla variance swaps and gamma swaps are significantly less than those required by the Monte Carlo simulation runs to achieve a similar level of numerical accuracy. We analyze the impact of different model parameters on the fair strike values of various variance swaps. We show that for the less frequently sampled variance products, the difference of fair strike values between continuously monitored and discretely monitored, or between different conventions of calculating the discrete realized variance (actual return or log return) cannot be ignored. The effect of jump intensity in the asset price process on the fair strike values depends sensibly on the underlying jump size and its standard deviation. Finally, we also show that the two types of downside variance swaps have very close fair strike values and both converge to the same value when the monitoring frequency becomes high.

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REFERENCES


Appendix A. Proof of Proposition 1

At $\tau = 0$, we have

$$U_i(x, v, 0) = \mathcal{F}^{-1} \left[ \exp(s(k)0)\hat{H}_i(x, v, 0)\mathcal{F}[F_i(e^x, S_{i-1})] \right] = F_i(e^x, S_{i-1}),$$

so the initial condition is easily seen to be satisfied. Next, we take

$$H_i(k, v, \tau) = \hat{H}_i(k, v, \tau)\mathcal{F}[F_i(e^x, S_{i-1})],$$

and substitute into the two sides of eq. (2.13). By canceling out the term $\mathcal{F}[F_i(e^x, S_{i-1})]$ on both sides, $H_i$ is seen to satisfy eq. (2.13). The Fourier transform of $U_i$ is obtained by the relation:

$$\tilde{U}_i(k, v, \tau) = \exp(s(k)\tau)H_i(k, v, \tau) = \exp(s(k)\tau)\hat{H}_i(x, v, \tau)\mathcal{F}[F_i(e^x, S_{i-1})].$$

Subsequently, $U_i$ is obtained by taking the inverse Fourier transform of $\tilde{U}_i$. In fact, the fundamental solution is closely related to the marginal Fourier-Laplace transform of $x$ given in eq. (2.4) with $\mu = 0$. One can show this relationship by using the Parseval identity.

We decompose the governing stochastic differential equation of $x$ into the continuous part $x^C$ and the jump part $x^J$ as follow:

$$x^C = \int_0^t (r - d - \lambda m - \frac{1}{2} v) du + \int_0^t \sqrt{v_u} dW_u^1 \quad \text{and} \quad x^J = \sum_{i=N_i+1}^{N_t} J_i.$$

Let $p_{x_i}(x_i, t_i|x, v, t)$ denote the transition density of $x_i$ at time $t_i$ given $v$ and $x$ at time $t$. Suppose we treat $e^{ikx}\hat{H}_i(k, v, \tau)$ to be the marginal characteristic function given in eq. (2.4) with $\mu = 0$, then

$$\hat{H}_i(k, v, \tau) = \frac{\Gamma(\gamma - \alpha)}{\Gamma(\gamma)} \left[ \frac{2}{e^2 y(v, t)} \right]^\alpha M \left( \alpha, \gamma, -\frac{2}{e^2 y(v, t)} \right),$$

where

$$y(v, t) = v \int_t^{t_i} e^{\int_0^s (r(s) - c_2) ds} du,$$

$$\alpha = -\left( \frac{1}{2} - \frac{\tilde{q}_k}{e^2} \right) + \sqrt{\left( \frac{1}{2} - \frac{\tilde{q}_k}{e^2} \right)^2 + \frac{2c_k}{e^2}},$$

$$\gamma = 2 \left( \alpha + 1 - \frac{\tilde{q}_k}{e^2} \right), \quad \tilde{q}_k = \rho e^{ik} - q, \quad c_k = \frac{k^2 + i k}{2}.$$

By performing the risk neutral valuation of the discounted payoff, we obtain

$$U_i(x, v, \tau) = \int_{-\infty}^{\infty} e^{-rt} F_i(e^{x_i}, S_{i-1})p_{x_i}(x_i, t_i|x, v, t) dx_i$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-rt} \mathcal{F}[F_i(e^{x_i}, S_{i-1})]\mathcal{F}[p_{x_i}(x_i, t_i|x, v, t)] dk$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-rt} \mathcal{F}[F_i(e^{x_i}, S_{i-1})] \int_{-\infty}^{\infty} e^{ikx}p_{x_i}(x_i, t_i|x, v, t) dx_i dk$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-rt} \mathcal{F}[F_i(e^{x_i}, S_{i-1})] \mathcal{E}[e^{ikx+ikx'}|x, v] dk$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{F}[F_i(e^{x_i}, S_{i-1})] e^{ikx}\hat{H}_i(k, v, \tau) \exp(s(k)\tau) dk$$

$$= \mathcal{F}^{-1} \left[ \mathcal{F}[F_i(e^{x_i}, S_{i-1})] e^{ikx}\hat{H}_i(k, v, \tau) \exp(s(k)\tau) \right].$$

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Appendix B. Deviation of eq. (3.2)

Let $p_{x,v}(x_t,v_t,t|x_s,v_s,s)$ denote the joint density of $x_t$ and $v_t$ given $x_s$ and $v_s$ at time $s \leq t$.

By observing

$$x_t = x_s + \int_s^t \left( r - d - \lambda m - \frac{1}{2}v \right) du + \int_s^t \sqrt{v_u} dW_u^1 + \sum_{i=N_0+1}^{N_t} J_i,$$

we obtain

$$p_{x,v}(x_t,v_t,t|x_s,v_s,s) = p_{x,v}(x_t-x_s,v_t,0,v_s,s).$$

With the above translation invariant relation, we may assume $x_s = 0$ for simplicity since the joint transition density $p_{x,v}$ can be obtained easily given any value of $x_s$. Unfortunately, we cannot find an explicit formula for the joint transition density. However, the $x_t$-transformed density $\tilde{G}(\tau; -z,v_t|x,s)$ of $p_{x,v}(x_t,v_t,t|x_s,v_s,s)$ is available in closed form, where $\tau = t - s$ and $z$ is the transform variable.

As the first step in our analytic derivation, we consider a pure diffusion process $x_t$ by setting the jump component $x^j = 0$. Let $p_d(y,\eta,t|0,v,s)$ be the joint transition density of the pure diffusion process, where $y$ and $\eta$ are assumed to be known at time $t$, then $p_d$ satisfies the following Kolmogorov backward equation

$$\frac{\partial p_d}{\partial s} = \frac{1}{2} v \frac{\partial^2 p_d}{\partial y^2} + \rho c v^2 \frac{\partial^2 p_d}{\partial y \partial v} + \frac{\rho^2}{2} v^3 \frac{\partial^2 p_d}{\partial v^2} + (r - d - \frac{1}{2}v) \frac{\partial p_d}{\partial y} + (pv - qv^2) \frac{\partial p_d}{\partial v},$$

with initial condition: $p_d(y,\eta,t|0,v,t) = \delta(y)\delta(v-\eta)$. The above partial differential equation can be solved analytically by performing the Fourier transform with respect to the variable $y$. We define the $y$-transformed density $G(\tau; -z,\eta, v)$ as follows:

$$G(\tau; -z,\eta, v) = \left[ \int_{-\infty}^{\infty} e^{izy} p_d(y,\eta,t|0,v,s) \, dy \right] e^{-i(r-d)\tau}.$$

The above backward equation can be simplified into a one-dimensional problem:

$$\frac{\partial G}{\partial \tau} = \frac{c^2}{2} v^2 \frac{\partial^2 G}{\partial v^2} + [pv - (q + iz \rho) v] \frac{\partial G}{\partial v} - c_z v G,$$

where

$$c_z = (z^2 - iz)/2,$$

with initial condition: $G(0; z,\eta, v) = \delta(v-\eta)$. By using the following set of transformed variables:

$$\tilde{y} = \frac{\tilde{p}}{v}, \quad Y = \frac{\tilde{p}}{\eta}, \quad \tilde{t} = \rho \tau, \quad \tilde{p} = \frac{2p}{c^2}, \quad \tilde{c} = \frac{2c_z}{c^2}, \quad \tilde{q}_z = \frac{2(q + iz \rho)}{c^2},$$

and assume that $G(\tau; z,\eta, v)$ takes the form (Lewis, 2000)

$$G(\tau; z,\eta, v) = \frac{Y^2}{\tilde{p}} \left( \frac{\tilde{y}}{Y} \right)^{R(z)} e^{[1-R(z)]\tilde{t}} g(\tilde{t}, \tilde{y},Y,z),$$

where

$$R(z) = -\mu_z - \delta_z, \quad \mu_z = \frac{1}{2} (1 + \tilde{q}_z), \quad \delta_z = \sqrt{\mu_z^2 + \tilde{c}_z},$$

and

$$\tilde{c}_z = (z^2 - iz)/2.$$
the partial differential equation for $G(\tau; z, \eta, v)$ can be further transformed to an ordinary differential equation in terms of $g(\tilde{t}, \tilde{y}, Y, z)$ with the independent spatial variable $\tilde{y}$. The solution of this ordinary differential equation is readily available by taking the Laplace transform in the variable $\tilde{y}$. The solution procedure is outlined in an unpublished monograph of Alan Lewis. By combining all the above results, we obtain the closed form solution for $G(\tau; -z, \eta, v)$ as shown in eq. (3.19).

The log asset price process $x_t$ [see eq. (2.2)] in our model can be decomposed into the continuous component $x^C$ and the jump component $x^J$ as shown in Appendix A. By applying the above result for the continuous process on $x^C$ and evaluating the compound Poisson process $x^J$ separately, we obtain the $x_t$-transformed density function $\tilde{G}(\tau; -z, v_t|x_s, v_s)$ of $p_{x,v}(x_t, v_t, t|x_s, v_s, s)$ as follows:

$$\tilde{G}(\tau; -z, v_t|x_s, v_s) = \int_{-\infty}^{\infty} e^{izy} p_{x,v}(y, v_t, t|x_s, v_s, s) \, dy$$

$$= \mathbb{E}[e^{izx_t}|x_s, v_t, v_s]$$

$$= \mathbb{E}[e^{izx_t + izx^C + izx^J}|x_s, v_t, v_s]$$

$$= e^{izx_s} \mathbb{E}[e^{izx^C}|x_s = 0, v_t, v_s] \mathbb{E}[e^{izx^J}]$$

$$= e^{izx_s} \left[ \int_{-\infty}^{\infty} e^{izx^C} p_d(x^C, v_t, t|0, v_s, s) \, dx^C \right] e^{\exp(\nu z - \zeta^2 z^2/2) - 1} \lambda \tau$$

$$= G(\tau; -z, v_t, v_s) e^{i(r - d - \lambda m) \tau} e^{izx_s} e^{\exp(\nu z - \zeta^2 z^2/2) - 1} \lambda \tau$$

$$= G(\tau; -z, v_t, v_s) e^{izx_s} \exp(h(z) \tau),$$

where

$$h(z) = i(r - d - \lambda m)z + [\exp(\nu z - \zeta^2 z^2/2) - 1] \lambda.$$
Figure 1: Plots of the fair strike values of the vanilla variance swaps, gamma swaps and downside variance swaps against sampling frequency, $N$. The label VSA stands for the vanilla variance swap defined by actual return, VSL for log return. Similarly, VGA stands for the gamma swap defined by actual return, and VGL for log return. Also, DSA1 and DSL1 stand for the downside variance swaps with $S_{i-1}$ as the monitoring asset price on the $i^{th}$ monitoring date based on actual and log return, respectively, while DSA2 and DSL2 for $S_i$ as the monitoring asset price.

Figure 2: Plots of the fair strike values of the vanilla variance swap, gamma swap and downside variance swap against correlation coefficient, $\rho$. The label VSL stands for the vanilla variance swap, VGL for the gamma swap. Also, DSL1 stands for the downside variance swap with $S_{i-1}$ as the monitoring asset price on the $i^{th}$ monitoring date. All discrete variance calculations are based on the convention of log return.
Figure 3: Plots of the fair strike values of the vanilla variance swap, gamma swap and downside variance swap against volatility of variance, $\epsilon$. The label VSL stands for the vanilla variance swap, VGL for the gamma swap. Also, DSL1 stands for the downside variance swap with $S_{i-1}$ as the monitoring asset price on the $i^{th}$ monitoring date. All discrete variance calculations are based on the convention of log return.

Figure 4: Plots of the fair strike values of the vanilla variance swap, gamma swap and downside variance swap against jump intensity, $\lambda$. The label VSL stands for the vanilla variance swap, VGL for the gamma swap. Also, DSL1 stands for the downside variance swap with $S_{i-1}$ as the monitoring asset price on the $i^{th}$ monitoring date. Furthermore, LJ stands for large jump with $\nu = -0.3$ and $\zeta = 0.4$ while SJ stands for small jump with $\nu = -0.03$ and $\zeta = 0.04$. All discrete variance calculations are based on the convention of log return.