Solution of the First HKUST Undergraduate Math Competition – Senior Level

1. Note $y = f(e^x) \Leftrightarrow y^y = e^x \Leftrightarrow y \ln y = x$. Then $dx = (\ln y + 1)dy$. So

$$\int_{x=0}^{x=e} f(e^x)\,dx = \int_{y=1}^{y=e} y(\ln y + 1)\,dy = \int_{y=1}^{y=e} (y \ln y + y)\,dy = \Big(\frac{y^2}{2}\ln y - \frac{y^2}{4}\Big)\Big|_{y=1}^{y=e} + \frac{y^2}{2}\Big|_{y=1}^{y=e} = \frac{3e^2-1}{4}.$$

2. (From linear algebra, the inequalities $\operatorname{rank}(XY) \le \operatorname{rank}(X)$ and $\operatorname{rank}(XYZ) \le \operatorname{rank}(Y)$ are useful.)

(*Solution 1*) Since the first two rows of AB are linearly independent, so $2 \le \operatorname{rank}(AB) \le \operatorname{rank}(A) \le 2$. Hence $\operatorname{rank}(AB) = 2$.

Next to get BA, we note $\operatorname{rank}(BA) \ge \operatorname{rank}(A(BA)B) = \operatorname{rank}((AB)^2)$. Now

$$(AB)^2 = \begin{pmatrix} 8 & 2 & -2 \\ 2 & 5 & 4 \\ -2 & 4 & 5 \end{pmatrix}^2 = \begin{pmatrix} 72 & 18 & -18 \\ 18 & 45 & 36 \\ -18 & 36 & 45 \end{pmatrix} = 9AB.$$

Since BA is a 2×2 matrix and $\operatorname{rank}(9AB) = 2$, so $\operatorname{rank}(BA) = 2$. Hence BA is invertible. Finally $(BA)^3 = B(ABAB)A = B(AB)^2A = 9BABA = 9(BA)^2$. Cancelling $(BA)^2$, we get $BA = 9I$.

(*Solution 2 due to Lau Lap Ming*) Since the first two rows of AB are linearly independent, so $2 \le \operatorname{rank}(AB) \le \operatorname{rank}(A) \le 2$. Then $\operatorname{rank}(AB) = \operatorname{rank}(A) = 2$.

Next $\det(AB - tI) = -t(t-9)^2$, so the eigenvalues of AB are 0 and 9. If λ is an eigenvalue of BA with eigenvector $v \ne 0$, then $AB(Av) = A(BAv) = A(\lambda v) = \lambda Av$. Since A is 3×2 and of rank 2, A is injective. Hence, $Av \ne 0$ and λ is an eigenvalue of AB. This implies the only possible eigenvalues of BA are 0 or 9. From row operations on the matrix of AB, we see the eigenspace of AB for the eigenvalue 0 is spanned by $\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ and the eigenspace V of AB for the eigenvalue 9 is spanned by $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$.

Restricting to V, $AB : V \to \mathbb{R}^2 \to V$ is bijective since $AB = 9I$ on V. So the linear maps $B : V \to \mathbb{R}^2$ and $A : \mathbb{R}^2 \to V$ must be bijective. In particular, $B(V) = \mathbb{R}^2$. Then for every $x \in \mathbb{R}^2$, there exists $v \in V$ such that $Bv = x$. So we have $BAx = B(ABv) = B(9v) = 9Bv = 9x$. Therefore, $BA = 9I$.

3. (This is an existence problem with solution to be found among continuous functions on $[0,1]$. In a course on metric spaces, a key theorem on existence problem is the contractive mapping theorem.) Define $T : C[0,1] \to C[0,1]$ by $(Tf)(x) = \int_0^x \int_0^y \frac{1}{2+t^{2\pi}}\,dt\,dy - \int_0^1 \frac{f(y)}{2+(xy)^\pi}\,dy$. Since $C[0,1]$ is a complete metric space with $d(f,g) = \|f-g\|_\infty$ and

$$|Tf(x) - Tg(x)| = \Big|\int_0^1 \frac{f(y)}{2+(xy)^\pi}\,dy - \int_0^1 \frac{g(y)}{2+(xy)^\pi}\,dy\Big| \le \frac{1}{2}\|f-g\|_\infty.$$

By the contractive mapping theorem, there exists $f \in C[0,1]$ such that $Tf = f$ and we are done.

4. (For a binomial coefficient problem, we should think about the binomial expansion of $(1+x)^n$.) Observe the sum $I = \sum_{j=0}^{p} \binom{p}{j}\binom{p+j}{j} = \sum_{j=0}^{p} \binom{p}{j}\binom{p+j}{p}$ is the coefficient of x^p in the expansion of

$$\sum_{j=0}^{p} \binom{p}{j}(1+x)^{p+j} = \Big(\sum_{j=0}^{p} \binom{p}{j}(1+x)^j\Big)(1+x)^p = (2+x)^p(1+x)^p.$$

Expanding $(2+x)^p(1+x)^p$, we get $I = \sum_{k=0}^{p} \binom{p}{k}\binom{p}{p-k}2^k$. Since p divides $\binom{p}{k}$ for $0 < k < p$, we have

$$I \equiv \binom{p}{0}\binom{p}{p}2^0 + \binom{p}{p}\binom{p}{0}2^p = 2^p + 1 (\text{mod } p^2).$$

5. Note $\operatorname{Re}\frac{1}{n^{1+it}} = \operatorname{Re} e^{-\ln n - it\ln n} = \frac{\cos(t\ln n)}{n}$ and $x + x^2 = \left(x+\frac{1}{2}\right)^2 - \frac{1}{4} \geq -\frac{1}{4}$. Let $w = t\ln 2$. Then

$$\begin{aligned}
\operatorname{Re} h(1+it) &= 1 + \frac{\cos w}{2} + \frac{\cos(t\ln 3)}{3} + \frac{\cos 2w}{4} + \frac{\cos(t\ln 5)}{5} \\
&\geq 1 + \frac{\cos w}{2} - \frac{1}{3} + \frac{2\cos^2 w - 1}{4} - \frac{1}{5} \\
&= 1 - \frac{1}{3} - \frac{1}{4} - \frac{1}{5} + \frac{\cos w + \cos^2 w}{2} \geq \frac{13}{60} - \frac{1}{8} > 0.
\end{aligned}$$

6. Since K is obtained by adjoining finitely many algebraic elements to F, inductively, we may suppose $K = F(\alpha)$ for some algebraic $\alpha \in \mathbb{C}$ over F. Let

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in F[x]$$

be the minimal polynomial of α over F. Clearly, $f(x) \in F(\zeta)[x]$ annihilates α. It is enough to show $f(x) \in F(\zeta)[x]$ is also the minimal polynomial of α over $F(\zeta)$ because then

$$[K(\zeta) : F(\zeta)] = [F(\zeta)(\alpha) : F(\zeta)] = n = [K : F].$$

Suppose $g(x) \overset{(*)}{=} x^m + g_{m-1}(\zeta)x^{m-1} + \cdots + g_0(\zeta) \in F(\zeta)[x]$ is another polynomial such that $g(\alpha) = 0$, where $g_i(\zeta) \in F(\zeta)$. Since $F \subseteq \mathbb{C}$, F is an infinite field, one can find $u \in F$ such that the product $p(\zeta)$ of the denominators of $g_i(\zeta) \in F(\zeta)$ do not annihilate u when ζ is replaced by u. Since $p(\zeta)(\alpha^m + g_{m-1}(\zeta)\alpha^{m-1} + \cdots + g_0(\zeta)) = 0$, the polynomial $p(x)(\alpha^m + g_{m-1}(x)\alpha^{m-1} + \cdots + g_0(x))$ is the zero polynomial in $F(\alpha)[x] = K[x]$. Since $p(u) \neq 0$, we get

$$\alpha^m + g_{m-1}(u)\alpha^{m-1} + \cdots + g_0(u) = 0,$$

where $g_i(u) \in F$. Since f is the minimal polynomial of α over F, this implies $m \geq n$. Therefore, $f(x) \in F(\zeta)[x]$ is the minimal polynomial of α over $F(\zeta)$.