GEOMETRIC PROGRESSION

Examples The following are called geometric progressions:

1. $3, 6, 12, 24, \cdots$
2. $1, -1/3, 1/9, -1/27, \cdots$
3. $a, ar, ar^2, ar^3 \cdots$.

We note that the ratio between any two consecutive terms of each of the above sequences is always the same.

1. $6/3 = 12/6 = 24/12 = \cdots = 2$.
2. $\frac{-1/3}{1} = \frac{1/9}{-1/3} = \frac{-1/27}{1/9} = \cdots = -1/3$.
3. $ar/a = ar^2/ar = ar^3/ar^2 = \cdots = r$.

The ratios that appear in the above examples are called the *common ratio* of the geometric progression. It is usually denoted by r. The first term (e.g. 3, 1, a in the above examples) is called the *initial term*, which is usually denoted by the letter a.

Example Consider the geometric progression

$$a, ar, ar^2, ar^3, \cdots.$$

The third, sixth and twentieth terms of the progression are given by ar^2, ar^5 and ar^{19} respectively. The *pth* term is given by ar^{p-1}.

GEOMETRIC SERIES

The sum of the geometric progression

$$a, ar, ar^2, ar^3, \cdots,$$

denoted by

$$S_n = a + ar + ar^2 + \cdots ar^{n-1},$$

is called the *geometric series*.

Multiplying the above geometric series by r, and then subtracting the resulting series by the geometric series gives

$$S_n - rS_n = a - ar^n.$$

So we obtain

$$S_n = a\left(\frac{1-r^n}{1-r}\right).$$

Example Find the sum of the first seven terms of the sequence

$$2/3,\ -1,\ 3/2, \cdots.$$

The common ratio is $-3/2$. Thus

$$\begin{aligned} S_7 &= \frac{2}{3} - 1 + \frac{3}{2} - \left(\frac{3}{2}\right)^2 + \left(\frac{3}{2}\right)^3 - \left(\frac{3}{2}\right)^4 + \left(\frac{3}{2}\right)^5 \\ &= \frac{2/3(1-(-3/2))^7}{1-(-3/2))} \\ &= \frac{2}{5}\frac{\dot{2}}{3}\left(1 + \frac{2178}{128}\right) \\ &= \frac{1153}{240}. \end{aligned}$$

Example Find the sum of the first n terms of the sequence

$$1,\ \frac{1}{2},\ \frac{1}{2^2},\ \frac{1}{2^3},\ \cdots.$$

We note that the first term $a = 1$ and the common ratio is clearly $1/2$. So by the geometric series formula

$$S_n = \frac{1-(1/2)^n}{1-1/2} = 2\left(1 - \frac{1}{2^n}\right).$$

Remark Notice that the term $1/2^n$ can be made arbitrary small by choosing the n sufficiently large.

Exercises Find the sum of the first

1. seven terms of $\frac{1}{2},\ \frac{1}{3},\ \frac{2}{9}, \cdots.$ $\left(\frac{2059}{1458}\right)$
2. six terms of $-2,\ 2\frac{1}{2},\ -3\frac{1}{8}, \cdots.$ $\left(\frac{1281}{512}\right)$
3. eight terms of $\frac{3}{4},\ 1\frac{1}{2},\ 3, \cdots.$ $(191\frac{1}{4})$.
4. ten terms of $2,\ -4,\ 8,\ \cdots.$ (-682)
5. seven terms of $16.2,\ 5.4,\ 1.8, \cdots.$ $\left(\frac{1093}{45}\right)$
6. p terms of $1,\ 5,\ 25, \cdots.$ $\left(\frac{5^p-1}{4}\right)$
7. $2n$ terms of $3,\ -4,\ \frac{16}{3}, \cdots.$ $\left(\frac{9}{7}(1-(\frac{4}{3})^{2n}\right)$
8. twelve terms of $1,\ \sqrt{3},\ 3, \cdots.$ $(364(\sqrt{3}+1))$
