Let f be a function whose domain is an interval $\mathcal{D}$. Take b to be an interior point of $\mathcal{D}$. The slope of the secant line determined by the two graph points $P = (b, f(b))$ and $Q = (x, f(x))$ is

$$\frac{f(x) - f(b)}{x - b}.$$

This fraction is called the **difference quotient** – the y-change divided by the x-change. If the limit

$$\lim_{x \to b} \frac{f(x) - f(b)}{x - b} \quad \text{exists,}$$

it is called the **derivative of the function f at the point** b. The value of the limit is denoted $f'(b)$. The physical interpretation of the derivative is that it is the slope of the tangent line at the graph point $(b, f(b))$.

As mentioned before, we can equivalently write x as $b + h$, and the existence of the secant limit becomes:

$$\lim_{h \to 0} \frac{f(b+h) - f(b)}{h} \quad \text{exists,}$$

The **derivative function** f' associated to f is the function which gives the derivative value of f at b whenever it exists.

Examples:

- Absolute value function. Take $f(x) = |x|$. The domain is all numbers $(-\infty, \infty)$. We saw before whether the difference quotient $\frac{|b+h|-|b|}{h}$ has a limit as $h \to 0$ depends on whether b is negative, zero or positive. The answers were:

$$\lim_{h\to 0} \frac{|b+h|-|b|}{h} = \begin{cases} -1 & \text{when } b < 0 \\ \text{does not exists} & \text{for } b = 0 \\ 1 & \text{when } b > 0 \end{cases}$$

The domain of the derivative function f' is $x \neq 0$.

- Power functions. Suppose n is a positive integer, and we take the function $f(x) = x^n$. The domain is $(-\infty, \infty)$. The difference quotient is

$$\begin{aligned} \frac{x^n - b^n}{x - b} &= \frac{(x-b)(x^{n-1} + x^{n-2}b + x^{n-3}b^2 + \cdots + xb^{n-2} + b^{n-1})}{x-b} \\ &= (x^{n-1} + x^{n-2}b + x^{n-3}b^2 + \cdots + x\,b^{n-2} + b^{n-1}) \end{aligned}$$

Now, for $k = 1, 2, \ldots, (n-1)$, the function $x^k\, b^{n-k-1}$ has limit b^{n-1} as $x \to b$. Therefore,

$$\lim_{x\to b} \frac{x^n - b^n}{x - b} = n\, b^{n-1}\ ;$$

so, the power function x^n is differentiable. We have $f'(x) = n\, x^{n-1}$.

The derivative of a function f is the limit of the difference quotient

$$\frac{f(x+h)-f(x)}{h}\ .$$

It is useful to write the horizontal change h as Δx, and the vertical change $f(x+h)-f(x)$ as Δy, and restate there being a derivative as the limit

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \quad \text{exists.}$$

A **very useful intuition** is to imagine changing x by an **'infinitesimal amount' dx**. This infinitesimal change in the input x will cause the output $y = f(x)$ will change an 'infinitesimal amount' dy, and the ratio

$$\frac{dy}{dx} \;=\; \text{derivative at input } x$$

is the tangent slope. The notation $\frac{dy}{dx}$, and $\frac{df}{dx}$ is used to denote the derivative function of the function $y = f(x)$. The notation

$$\frac{dy}{dx}\Big|_{x=b} \qquad \text{is used to denote the derivative value at } b$$

Example:

$$\frac{d(\sin(x))}{dx} \;=\; \cos(x)\ , \quad \frac{d(\sin(x))}{dx}\Big|_{x=0} \;=\; \cos(x)\Big|_{x=0} \;=\; \cos(0) \;=\; 1\ .$$

It is a tautology (saying the same thing) that if $y = f(x)$, then $\frac{dy}{dx} = f'(x)$. Both $\frac{dy}{dx}$ and $f'(x)$ are notations for the same thing. But, it is useful to write:

$$\begin{aligned} dy &= f'(x)\,dx && \text{exact equality for infinitesimals} \\ \Delta y &\doteq f'(x)\,\Delta x && \text{approximately equal for } \Delta x \neq 0 \end{aligned}$$

In particular, the derivative $f'(b)$ at b tells us approximately how much the function will change (Δy) if we make a change of Δx.

18 Derivative at input b means continuous at input b.

We compare the notions of continuity at input b and differentiable at input b. The assumption is we have a function f with domain an interval, and b is an interior point.

- f is **continuous at** b if the limit

$$\lim_{h\to 0} \left(f(b+h) - f(b) \right) = 0$$

- f is **differentiable at** b if the limit

$$\lim_{h\to 0} \frac{f(b+h) - f(b)}{h} = L \quad \text{exists.}$$

Now, it is easy that $\lim_{h\to 0} h = 0$. But, then, using the product rule for limits we have:

$$\lim_{h\to 0} \frac{f(b+h) - f(b)}{h} = L \quad \text{and} \quad \lim_{h\to 0} h = 0$$

$$\Longrightarrow$$

$$\lim_{h\to 0} f(b+h) - f(b) = \lim_{h\to 0} \left(\left(\frac{f(b+h) - f(b)}{h}\right) h \right)$$

$$= \lim_{h\to 0} \left(\frac{f(b+h) - f(b)}{h}\right) \lim_{h\to 0} h = L\,0 = 0$$

Therefore,

$$\text{differentiable at } b \quad \Longrightarrow \quad \text{continuous at } b.$$

19 Differentiation rules.

19.1 Basic Differentiation rules.

Recall the limit of a sum is the sum of the limits and the limit of a function multiplied by a constant c is c times the limit of the function. Applied to derivatives we get:

- Sum rule: If the functions f and g are differentiable at b, then so is their sum. If they are differentiable on an interval $\mathcal{D}$, then so is their sum.

$$(f + g)'(b) = f'(b) + g'(b)\,, \quad \text{also written as}$$
$$\frac{d(f + g)}{dx} = \frac{df}{dx} + \frac{dg}{dx}$$

- Scalar rule: For a constant c

$$(c\,f)'(b) = c\,f'(b)\,, \quad \text{also written as}$$
$$\frac{d(c\,f)}{dx} = c\,\frac{d\,f}{dx}$$

19.2 Rule for derivative of product of two functions.

- Product rule: If the functions f and g are differentiable at b or an interval $\mathcal{D}$, then so is their product.

$$(f\,g)'(b) = f'(b)\,g(b) + f(b)\,g'(b)\,, \quad \text{also written as}$$
$$\frac{d(f\,g)}{dx} = \frac{df}{dx}g + f\frac{dg}{dx}$$

The intuition for the product rule is the following:

- A change of of the input from b to $b + \Delta x$ results in

$$f(b+\Delta x) \doteq f(b) + f'(b)\,\Delta x \quad \text{and} \quad g(b+\Delta x) \doteq g(b) + g'(b)\,\Delta x$$

- Therefore, the product at input $b + \Delta x$ becomes

$$\begin{aligned} f(b+\Delta x)\;g(b+\Delta x) &\doteq \big(\,f(b) + f'(b)\,\Delta x\,\big)\,\big(\,g(b) + g'(b)\,\Delta x\,\big) \\ &\doteq f(b)\;g(b) + f'(b)\,\Delta x\,g(b) + f(b)\,g(b)\,\Delta x + f'(b)\,g'(b)\;(\Delta x)^2 \end{aligned}$$

- The difference quotient (change in vertical divided by change in horiztontal) is:

$$\frac{f(b+\Delta x)\,g(b+\Delta x) - f(b)\,g(b)}{\Delta x} \doteq f'(b)\,g(b) + f(b)\,g(b) + f'(b)\,g'(b)\;(\Delta x)$$

- The above is approximate, as we let $\Delta x \to 0$, we get differentials equality:

$$(f\, g)' \;=\; f'\, g \;+\; f\, g' \,.$$

Example. We find the derivative of the function $x^3 \sin(x)$ using the product rule:

$$\begin{aligned}(x^3 \sin(x))' &= (x^3)' \sin(x) \;+\; x^3 \,(\sin(x))' \\ &= 3\,x^2 \;\sin(x) \;+ x^3 \;\cos(x)\end{aligned}$$

19.3 Rule for reciprocal.

- Reciprocal rule: If a function f has non-zero value and is differentiable at b or an interval $\mathcal{D}$, then so is it reciprocal.

$$(\frac{1}{f})' (b) \;=\; -\frac{f'(b)}{(f(b))^2}\,, \quad \text{also written as} \quad \frac{d}{dx}\,(\frac{1}{f}) \;=\; \frac{df}{dx}\,\frac{-1}{f^2}$$

Reason the reciprocal rule is true:
We take the difference quotient $\frac{\frac{1}{f(b+h)} - \frac{1}{f(b)}}{h}$ and do some algebraic manipulation.

$$\frac{\frac{1}{f(b+h)} \;-\; \frac{1}{f(b)}}{h} \;=\; \frac{f(b) \;-\; f(b+h)}{f(b+h)\; f(b)\; h} \;=\; \frac{f(b+h) \;-\; f(b)}{h}\;(\frac{-1}{f(b)\; f(b+h)})$$

$$\begin{aligned}\lim_{h\to 0} \frac{\frac{1}{f(b+h)} \;-\; \frac{1}{f(b)}}{h} &= \lim_{h\to 0} \Big(\frac{f(b+h) \;-\; f(b)}{h}\;(\frac{-1}{f(b)\; f(b+h)}) \Big) \\ &= \lim_{h\to 0} \frac{f(b+h) \;-\; f(b)}{h} \quad \lim_{h\to 0}(\frac{-1}{f(b)\; f(b+h)}) \;=\; f'(b)\;(\frac{-1}{(f(b))^2})\end{aligned}$$

19.4 Rule for quotients.

The product and reciprocal rules can be combined into a quotient rule.

- Quotient rule: If a functions f and g are differentiable at b and $g(b) \neq 0$, then so is the quotient $\frac{f}{g}$.

$$\left(\frac{f}{g}\right)'(b) = \frac{f'(b)\,g(b) - f(b)\,g'(b)}{(g(b))^2}\,, \quad \text{also written as}$$

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{\frac{df}{dx}\,g - f\,\frac{dg}{dx}}{g^2}$$

Example. We find the derivative of the tangent function $\tan(x) = \frac{\sin(x)}{\cos(x)}$ using the quotient rule:

$$\begin{aligned}\frac{d\left(\frac{\sin(x)}{\cos(x)}\right)}{dx} &= \frac{\frac{d\,(\sin(x))}{dx}\,\cos(x) - \sin(x)\,\frac{d\,(\cos(x))}{dx}}{(\,\cos(x)\,)^2}\\ &= \frac{(\cos(x))\,\cos(x) - \sin(x)\,(-\sin(x))}{(\,\cos(x)\,)^2} = \frac{(\cos(x))^2 + (\sin(x))^2}{(\,\cos(x)\,)^2}\\ &= \frac{1}{(\,\cos(x)\,)^2} = (\,\sec(x)\,)^2\end{aligned}$$