14 Modifications of the limit idea

We now mention some useful modifications of the limit idea.

- One-sided limits.
- $+\infty$ or $-\infty$ as limit.
- Limit as the input variable approaches $+\infty$ or $-\infty$.
- Infinite limit at infinity.

14.1 One-sided limits

For a usual (two-sided) limit, we look at points above and below the approach point.

Example. When we consider the limit $\lim_{x \to 0} \frac{|x|}{x}$, we allow $x > 0$ and $x < 0$.

If we are ‘forced’ to consider both, then there is no number $L$ so that $\frac{|x|}{x} - L$ will be small when $|x - 0|$ is small; so the limit does not exist.

A one-sided limit is when we restrict inputs to either above or below the approach point.

Examples.

- For the function $\frac{|x|}{x}$, if we approach 0 from above 0, then $\frac{|x|}{x} - 1$ will be small (in fact zero). Similarly, if approach 0 from below 0, then $\frac{|x|}{x} - (-1)$ will be small (in fact zero).
  
So, we have

$$\lim_{x \to 0^+} \frac{|x|}{x} = 1, \quad \text{and} \quad \lim_{x \to 0^-} \frac{|x|}{x} = -1$$

The notation $x \to 0^+$ is used to denote approach to 0 from above. Similarly, $x \to 0^-$ denotes approach to 0 from below.

- For the function $\sin\left(\frac{1}{x}\right)$, when we limit ourselves to only positive values, there is still no $L$ such that $\sin\left(\frac{1}{x}\right) - L$ is small when $x$ is positive and small. The same is happens for $x < 0$; so,

$$\lim_{x \to 0^+} \sin\left(\frac{1}{x}\right), \quad \text{and} \quad \lim_{x \to 0^-} \sin\left(\frac{1}{x}\right), \quad \text{do not exist.}$$

Observation: A function $f(x)$ has a limit $L$ at point $b$ precisely when

$$\lim_{x \to b^+} f(x) = L, \quad \text{and} \quad \lim_{x \to b^-} f(x) = L.$$
We begin with a motivational example of an infinite limit.

Example. \( \lim_{x \to 3} \frac{1}{(x-3)^2} = +\infty \)

Intuition: The intuition of an infinite (positive) limit as \( x \to b \) is that outputs of a function \( f \) get large as \( x \) nears, but is not equal to, the point \( b \).

Quantitative formulation of infinite limit:

- Given a **challenge** to make the quantity \( f(x) \) large, say larger than some (big) tolerance \( T \),
- we can find a ‘**tolerance-reply**’ positive number \( R \) with the property that

\[
0 < |x - b| < R \implies f(x) > T.
\]

Example. To see \( \lim_{x \to 3} \frac{1}{(x-3)^2} = +\infty \), suppose we have a challenge to make \( f(x) = \frac{1}{(x-3)^2} > T \).

How close to 3 do we need to take \( x \)? We have

\[
\frac{1}{(x-3)^2} > T \iff (x-3)^2 < \frac{1}{T} \iff |x-3| < R = \sqrt{\frac{1}{T}}.
\]
We can also talk of one-sided infinite limits.

Examples.

- \( \lim_{x \to 0^-} \frac{1}{x} = -\infty \), and \( \lim_{x \to 0^+} \frac{1}{x} = +\infty \)

- \( \lim_{x \to \frac{\pi}{2}^-} \tan(x) = +\infty \), and \( \lim_{x \to \frac{\pi}{2}^+} \tan(x) = -\infty \)

- \( \lim_{x \to 0^+} \log_{10}(x) = -\infty \)

**Vertical asymptote**

If a function has an two-sided or one-sided infinite limit at \( b \), we say the line \( x = b \) is a vertical asymptote. Graphically, the graph ‘approaches’ the vertical line \( x = b \). In the above examples:

- The vertical line \( x = 0 \) is a vertical asymptote of the function \( \frac{1}{x} \).
- The lines \( x = -\frac{\pi}{2} \), and \( x = \frac{\pi}{2} \) are vertical asymptotes of the function \( \tan(x) \).
- The line \( x = 0 \) is a vertical asymptote of \( \log_{10}(x) \).
14.4 Limit at $\infty$

The limit idea can also be modified to become one which tells us the behavior as the input variable ‘approaches’ $\infty$.

Examples.

- $\lim_{x \to +\infty} \frac{1}{x^2+1} = 0$, and $\lim_{x \to -\infty} \frac{1}{x^2+1} = 0$.
- $\lim_{x \to -\infty} 2^x = 0$.
- $\lim_{x \to -\infty} \arctan(x) = -\frac{\pi}{2}$, and $\lim_{x \to +\infty} \arctan(x) = \frac{\pi}{2}$.

Some non-examples of limits at infinity.

- $\lim_{x \to +\infty} \sin(x) = \text{Does Not Exists}$, $\lim_{x \to +\infty} x \sin(x) = \text{Does Not Exists}$.

**Horizontal asymptote**

If a function has limit $L$ at either $-\infty$ or $\infty$, we say the line $y = L$ is a horizontal asymptote. Graphically, the graph ‘approaches’ the horizontal line $y = L$. In the above examples:

Examples.

- $\lim_{x \to +\infty} \frac{1}{x^2+1} = 0$, and $\lim_{x \to -\infty} \frac{1}{x^2+1} = 0$; so, the line $y = 0$ is a horizontal asymptote.
- $\lim_{x \to -\infty} 2^x = 0$; so, the line $y = 0$ is a horizontal asymptote.
- $\lim_{x \to -\infty} \arctan(x) = -\frac{\pi}{2}$, and $\lim_{x \to +\infty} \arctan(x) = \frac{\pi}{2}$; so, the lines $y = -\frac{\pi}{2}$ and $y = \frac{\pi}{2}$ are horizontal asymptotes.
14.5 Infinite limit at ∞

Another modification of the limit idea is to quantify a function having infinite limit at infinity.

Examples.

\[
\lim_{x \to +\infty} x = +\infty , \quad \lim_{x \to +\infty} \sqrt{x} = +\infty , \quad \lim_{x \to +\infty} -x^3 + x^2 = -\infty , \\
\lim_{x \to +\infty} 2^x = +\infty , \quad \lim_{x \to +\infty} \log_{10}(x) = +\infty ,
\]

The intuition is that as the input \( x \) becomes large so will the output.

15 Continuity

The common functions such as linear, polynomial, exponential, sin, cos, absolute-value have an important mathematics property called **continuity**.

The intuition is the graph of continuous functions do not have jumps.

15.1 Continuity at a point:

Suppose an interval \( \mathcal{D} \) is part of the domain of a function \( f \), and \( b \in \mathcal{D} \) is an interior point. The function \( f \) is said to be **continuous at the point** \( b \) if:

- The limit \( \lim_{x \to b} f(x) \) exists.
- The limit value equals \( f(b) \).

If \( b \) is an endpoint of \( \mathcal{D} \) we require the one-sided limit exists and its value is equal to \( f(b) \).
15.2 Continuity on an interval: 

A function is said to be **continuous on an entire interval** \( D \) if it is continuous at all points in the interior as well as the endpoints.

**Examples**

- If \( p(x) = c_r x^r + c_{r-1} x^{(r-1)} + \cdots + c_1 x + c_0 \) is a polynomial, we use the limit rules to deduce
  \[
  \lim_{x \to b} p(x) = c_r b^r + c_{r-1} b^{(r-1)} + \cdots + c_1 b + c_0 = p(b) .
  \]
  Therefore, a polynomial is continuous at any point \( b \), and it is continuous on any interval.

- By the limit quotient rule, a rational function \( f(x) = \frac{p(x)}{q(x)} = \frac{c_r x^r + c_{r-1} x^{(r-1)} + \cdots + c_1 x + c_0}{d_s x^s + d_{s-1} x^{(s-1)} + \cdots + d_1 x + d_0} \) will, as \( x \to b \) have limit
  \[
  L = \frac{c_r b^r + c_{r-1} b^{(r-1)} + \cdots + c_1 b + c_0}{d_s b^s + d_{s-1} b^{(s-1)} + \cdots + d_1 b + d_0} = f(b) \text{ whenever } q(b) \neq 0 .
  \]
  Therefore, the rational function is continuous at any point \( b \) for which the bottom (denominator) \( q(b) \neq 0 \).
  The rational function is continuous on any interval not containing a zero of the polynomial \( q(x) \).

- The absolute-value function \(|x|\) satisfies \( \lim_{x \to b} |x| = |b| \) for any \( b \). It is continuous at any point \( b \), and continuous on any interval.

A point where a function is not continuous is called a **point of discontinuity**.

**Example**

- **The floor function.** For any (real) number \( x \), we set
  \[
  \lfloor x \rfloor = \text{the largest integer less than or equal to } x
  \]
  For instance, some stores use the floor function in rounding purchases to the nearest dollar.
  The function \( f(x) = \frac{1}{10} \lfloor 10x \rfloor \) rounds a number to the largest multiple of 0.10 less than or equal to \( x \). The floor function satisfies:
  \[
  \begin{align*}
  \cdot & \quad \text{When } b \text{ is not an integer, we have } \lim_{x \to b} \lfloor x \rfloor = \lfloor b \rfloor . \\
  \cdot & \quad \text{When } b \text{ is an integer, we have } \\
  & \quad \lim_{x \to b^-} \lfloor x \rfloor = \lfloor b \rfloor - 1 \text{ and } \lim_{x \to b^+} \lfloor x \rfloor = \lfloor b \rfloor .
  \end{align*}
  \]
  The floor function is continuous at any non-integer \( b \), and discontinuous at any integer.
15.3 Rules related to continuous functions.

- **Sum rule:** If the functions $f$ and $g$ are continuous at $b$, then so is their sum. If they are continuous on an interval $\mathcal{D}$, then so is their sum.

- **Product rule:** If the functions $f$ and $g$ are continuous at $b$, then so is their product. If they are continuous on an interval $\mathcal{D}$, then so is their product.

- **Reciprocal rule:** If a function $f$ is continuous at $b$, and $f(b) \neq 0$, then the reciprocal function $\frac{1}{f}$ is continuous at $b$. If $f$ is continuous and non-zero on an interval $\mathcal{D}$, then $\frac{1}{f}$ is continuous too.

- **Composition rule:** If $f$ and $g$ are two functions whose composition $f \circ g$ makes sense, and $g$ is continuous at $b$, and $f$ is continuous at $g(b)$, then $f \circ g$ is continuous at $b$. 
Two useful alternate ways to say a function $f$ is continuous at a point $b$ are:

- A function $f$ is continuous at $b$ if
  \[
  \lim_{x \to b} (f(x) - f(b)) = 0
  \]

- A function $f$ is continuous at $b$ if
  \[
  \lim_{h \to 0} (f(b + h) - f(b)) = 0
  \]

The term $(f(b + h) - f(b))$ came up in our introductory discussion of secant slopes and tangent slopes. We shall see later that if a function $f$ has a tangent slope at the graph point $(b, f(b))$, then $f$ is continuous at $b$. 