14 Modifications of the limit idea

We now mention some useful modifications of the limit idea.

- One-sided limits.
- $+\infty$ or $-\infty$ as limit.
- Limit as the input variable approaches $+\infty$ or $-\infty$.
- Infinite limit at infinty.

14.1 One-sided limits

For a usual (two-sided) limit, we look at points above and below the approach point.

Example. When we consider the limit $\lim_{x \to 0} \frac{|x|}{x}$, we allow $x > 0$ and $x < 0$.

If we are 'forced' to consider both, then there is no number L so that $|\ \frac{|x|}{x} - L\ |$ will be small when $|\ x - 0\ |$ is small; so the limit does not exists.

A one-sided limit is when we **restrict** inputs to either above or below the approach point.

Examples.

- For the function $\frac{|x|}{x}$, if we approach 0 from above 0, then $|\ \frac{|x|}{x} - 1\ |$ will be small (in fact zero). Similarly, if approach 0 from below 0, then $|\ \frac{|x|}{x} - (-1)\ |$ will be small (in fact zero). So, we have

$$\lim_{x \to 0^+} \frac{|x|}{x} = 1 , \quad \text{and} \quad \lim_{x \to 0^-} \frac{|x|}{x} = -1$$

  The notation $x \to 0^+$ is used to denote approach to 0 from above. Similarly, $x \to 0^-$ denotes approach to 0 from below.

- For the function $\sin(\frac{1}{x})$, when we limit ourselves to only positive values, there is still no L such that $|\ \sin(\frac{1}{x}) - L\ |$ is small when x is positive and small. The same is happens for $x < 0$; so,

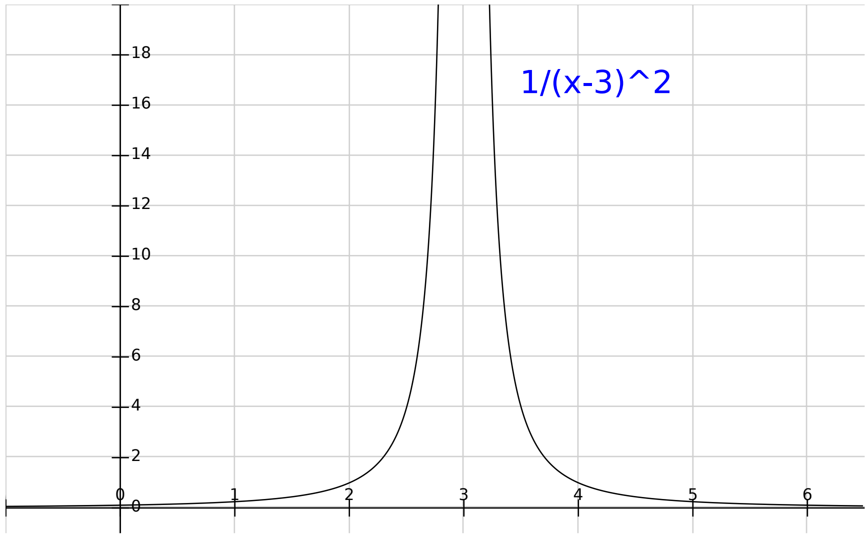
$$\lim_{x \to 0^+} \sin(\frac{1}{x}) , \quad \text{and} \quad \lim_{x \to 0^-} \sin(\frac{1}{x}) , \quad \text{do not exist.}$$

Observation: A function $f(x)$ has a limit L at point b precisely when

$$\lim_{x \to b^+} f(x) = L , \quad \textbf{and} \quad \lim_{x \to b^-} f(x) = L .$$

We begin with a motivational example of an infinite limit.

Example. $\lim_{x\to 3} \frac{1}{(x-3)^2} = +\infty$



Intuition: The intuition of an infinite (positive) limit as $x \to b$ is that outputs of a function (f) get large as x nears, but is not equal to, the point b.

Quantitative formulation of infinite limit:

- Given a **challenge** to make the quantity $f(x)$ large, say larger than some (big) **tolerance** T,
- we can find a **'tolerance-reply'** positive number R with the property that

$$0 < |x-b| < R \overset{\text{implies}}{\Longrightarrow} f(x) > T\,.$$

Example. To see $\lim_{x\to 3} \frac{1}{(x-3)^2} = +\infty$, suppose we have a challenge to make $f(x) = \frac{1}{(x-3)^2} > T$. How close to 3 do we need to take x? We have

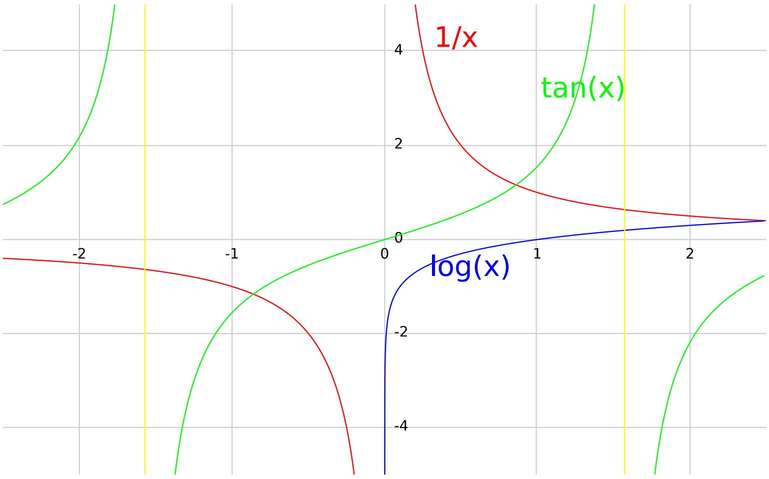
$$\frac{1}{(x-3)^2} > T \iff (x-3)^2 < \frac{1}{T}$$

$$\iff |\,x-3\,| < R = \sqrt{\frac{1}{T}}\,.$$

We can also talk of one-sided infinite limits.

Examples.

- $\lim_{x \to 0^-} \frac{1}{x} = -\infty$, and $\lim_{x \to 0^+} \frac{1}{x} = +\infty$
- $\lim_{x \to \frac{\pi}{2}^-} \tan(x) = +\infty$, and $\lim_{x \to \frac{\pi}{2}^+} \tan(x) = -\infty$
- $\lim_{x \to 0^+} \log_{10}(x) = -\infty$



Vertical asymptote

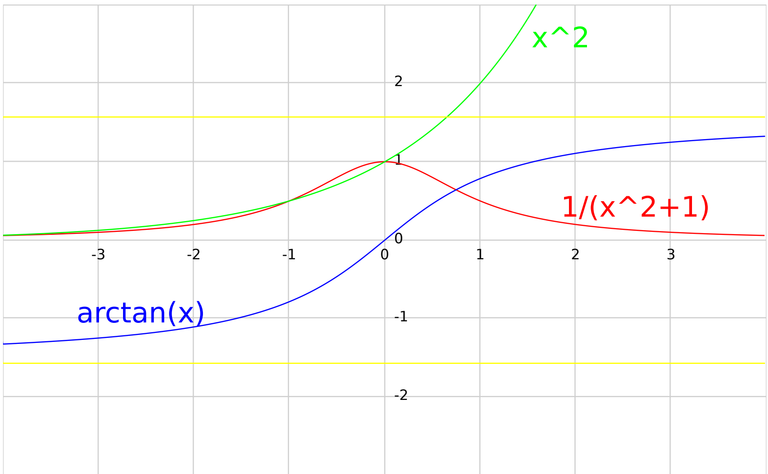
If a function has an two-sided or one-sided **infinite** limit at b, we say the line $x = b$ is a vertical asymptote. Graphically, the graph 'approaches' the vertical line $x = b$. In the above examples:

- The vertical line $x = 0$ is a vertical asymptote of the function $\frac{1}{x}$.
- The lines $x = -\frac{\pi}{2}$, and $x = \frac{\pi}{2}$ are vertical asymptotes of the function $\tan(x)$.
- The line $x = 0$ is a vertical asymptote of $\log_{10}(x)$.

The limit idea can also be modified to become one which tells us the behavior as the input variable 'approaches' ∞.

Examples.

- $\lim_{x \to +\infty} \frac{1}{x^2+1} = 0$, and $\lim_{x \to -\infty} \frac{1}{x^2+1} = 0$.
- $\lim_{x \to -\infty} 2^x = 0$.
- $\lim_{x \to -\infty} \arctan(x) = -\frac{\pi}{2}$, and $\lim_{x \to +\infty} \arctan(x) = \frac{\pi}{2}$.



Some non-examples of limits at infinity.

$$\lim_{x \to +\infty} \sin(x) = \text{Does Not Exists}, \quad \lim_{x \to +\infty} x\sin(x) = \text{Does Not Exists},$$

Horizontal asymptote

If a function has limit L at either $-\infty$ or ∞, we say the line $y = L$ is a horizontal asymptote. Graphically, the graph 'approaches' the horizontal line $y = L$. In the above examples:

Examples.

- $\lim_{x \to +\infty} \frac{1}{x^2+1} = 0$, and $\lim_{x \to -\infty} \frac{1}{x^2+1} = 0$; so, the line $y = 0$ is a horizontal asymptote.
- $\lim_{x \to -\infty} 2^x = 0$; so, the line $y = 0$ is a horizontal asymptote.
- $\lim_{x \to -\infty} \arctan(x) = -\frac{\pi}{2}$, and $\lim_{x \to +\infty} \arctan(x) = \frac{\pi}{2}$; so, the lines $y = -\frac{\pi}{2}$ and $y = \frac{\pi}{2}$ are horizontal asymptotes.

14.5 Infinite limit at ∞

Another modification of the limit idea is to quantify a function having infinite limit at infinity.

Examples.

$$\lim_{x\to+\infty} x = +\infty\ , \quad \lim_{x\to+\infty} \sqrt{x} = +\infty\ , \quad \lim_{x\to+\infty} -x^3+x^2 = -\infty\ ,$$

$$\lim_{x\to+\infty} 2^x = +\infty\ , \quad \lim_{x\to+\infty} \log_{10}(x) = +\infty\ ,$$

The intuition is that as the input x becomes large so will the output.

15 Continuity

The common functions such as linear, polynomial, exponential, sin, cos, abosulte-value have an important mathematics property called **continuity**.

The intuition is the graph of continuous functions do not have jumps.

15.1 Continuity at a point:

Suppose an interval $\mathcal{D}$ is part of the domain of a function f, and $b \in \mathcal{D}$ is an interior point. The function f is said to be **continuous at the point** b if:

- The limit $\lim_{x\to b} f(x)$ exists.
- The limit value equals $f(b)$.

If b is an endpoint of $\mathcal{D}$ we require the one-sided limit exists and its value is equal to $f(b)$.

15.2 Continuity on an interval:

f is said to be **continuous on an entire interval** $\mathcal{D}$ if it is continuous at all points in the interior as well as the endpoints.

Examples

- If $p(x) = c_r x^r + c_{r-1}x^{(r-1)} + \cdots + c_1 x + c_0$ is a polynomial, we use the limit rules to deduce

$$\lim_{x\to b} p(x) \;=\; c_r b^r + c_{r-1}b^{(r-1)} + \cdots + c_1 b + c_0 \;=\; p(b)\;.$$

  Therefore, a polynomial is continuous at any point b, and it is continuous on any interval.

- By the limit quotient rule, a rational function $f(x) = \frac{p(x)}{q(x)} = \frac{c_r x^r + c_{r-1}x^{(r-1)} + \cdots + c_1 x + c_0}{d_s x^s + d_{s-1}x^{(s-1)} + \cdots + d_1 x + d_0}$ will, as $x \to b$ have limit $L = \frac{c_r b^r + c_{r-1}b^{(r-1)} + \cdots + c_1 b + c_0}{d_s b^s + d_{s-1}b^{(s-1)} + \cdots + d_1 b + d_0} = f(b)$ whenever $q(b) \neq 0$. Therefore, the rational function is continuous at any point b for which the bottom (denominator) $q(b) \neq 0$. The rational function is continuous on any interval not containing a zero of the polynomial $q(x)$.

- The absolute-value function $|x|$ satisfies $\lim_{x\to b} |x| \;=\; |b|$ for any b. It is continuous at any point b, and continuous on any interval.

A point where a function is not continuous is called **a point of discontinuity**.

Example

- **The floor function.** For any (real) number x, we set

$$\lfloor\, x \,\rfloor \;=\; \text{the largest integer less than or equal to x}$$

  For instance, some stores use the floor function in rounding purchases to the nearest dollar. The function $f(x) = \frac{1}{10}\lfloor\, 10\, x \,\rfloor$ rounds a number to the largest multiple of 0.10 less than or equal to x. The floor function satisfies:

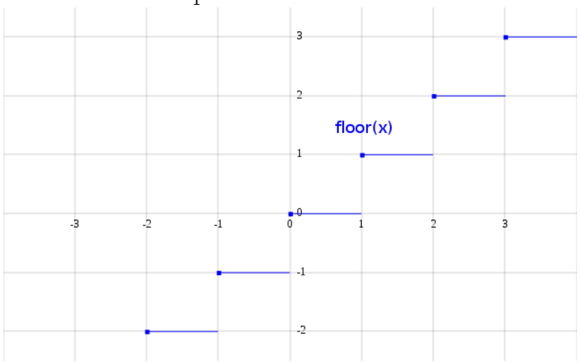
  · When b is not an integer, we have $\lim_{x\to b}\lfloor x\rfloor \;=\; \lfloor b\rfloor$.

  · When b is an integer, we have

$$\lim_{x\to b^-}\lfloor x\rfloor \;=\; \lfloor b\rfloor - 1 \quad \text{and} \;\; \lim_{x\to b^+}\lfloor x\rfloor \;=\; \lfloor b\rfloor\;.$$

  The floor function is continuous at any non-integer b, and discontinuous at any integer.

Graph of floor function



15.3 Rules related to continuous functions.

- Sum rule: If the functions f and g are continuous at b, then so is their sum. If they are continuous on an interval $\mathcal{D}$, then so is their sum.
- Product rule: If the functions f and g are continuous at b, then so is their product. If they are continuous on an interval $\mathcal{D}$, then so is their product.
- Reciprocal rule: If a function f is continuous at b, and $f(b) \neq 0$, then the reciprocal function $\frac{1}{f}$ is continuous at b. If f is continuous and non-zero on an interval $\mathcal{D}$, then $\frac{1}{f}$ is continuous too.
- Composition rule: If f and g are two functions whose composition $f \circ g$ makes sense, and g is continuous at b, and f is continuous at $g(b)$, then $f \circ g$ is continuous at b.

15.4 Useful alternate ways to say continuous.

Two useful alternate ways to say a function f is continuous at a point b are:

- A function f is continuous at b if

$$\lim_{x \to b} \; (\; f(x) \; - \; f(b) \;) \; = \; 0$$

- A function f is continuous at b if

$$\lim_{h \to 0} \; (\; f(b+h) \; - \; f(b) \;) \; = \; 0$$

The term ($f(b+h) - f(b)$) came up in our introductory discussion of secant slopes and tangent slopes. We shall see later that if a function f has a tangent slope at the graph point $(b, f(b))$, then f is continuous at b.